# Matching with single-peaked preferences.

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**Abstract**

The crawler is a new efficient, strategyproof, and individually rational mechanism for house matching markets with single-peaked preferences. In a house matching market each agent is endowed with exactly one house. These houses are ordered - by their size for example - and all agents preferences are singlepeaked with respect to that order. The crawler screens agents in order of their houses' sizes, starting with the smallest. The first agent who does not want to move to a larger house is matched with his most preferred house. Agents who currently occupy houses sized between this agent's original and chosen houses "crawl" to the next largest unmatched house. This process is repeated until all agents are matched. The crawler is easier to understand than Gale's top trading cycles and can be extended to allow for indifferences. A variant of the crawler can be used in environments where countably many agents gradually enter and exit the market.

## 1 Introduction

Consider a house matching problem in which each agent $i$ in a set $\{1, \ldots, n\}$ is endowed with a house, also called $i$. Suppose there is some objective linear

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order on all houses. House could be ordered by their location, so that $i < j$ means that house $i$ lies to the South of house $j$. Alternatively houses could be ordered by their sizes, their energy efficiency, etc. All agents preferences are single-peaked with respect to the objective order on houses. If preferences are single-peaked with respect to the north-south ordering then agents hope to live as close as possible to their preferred latitude. If size is the relevant objective order, then each agent has an ideal house size. Such an agent prefers a house that is a bit smaller (larger) than his ideal house to any other house that is yet smaller (larger). For ease of presentation I assume throughout that preferences are single-peaked with respect to house sizes.

A mechanism maps each profile of all agents' preferences to a matching. A matching, in turn, is a one-to-one function between agents and houses. The crawler, a new matching mechanism for the single-peaked domain, determines matchings by screening all houses in order of their size, starting with the smallest. Once a house whose current owner $i$ wants to either stay put or move to a smaller house is found, the crawler matches this agent $i$ with his most preferred house. If $i$'s most preferred house differs from the house he owned at the beginning of this step, then each current occupant of a house sized between these two "crawls" to the next biggest house. This process is repeated until all agents are matched.

Theorem 1 shows that the crawler is efficient, strategy proof, and individually rational. A mechanism is strategyproof if no agent can ever benefit from misrepresenting his preferences. It is efficient if it maps each profile of preferences to a matching for which there does not exist an alternative matching weakly preferred by all and strictly by some. It is individually rational if no agent is ever matched with a house he deems worse than the one he was endowed with.

Without the assumption of single peakedness, exactly one mechanism satisfies these three criteria: when all linear orders are permitted as preferences, then Gale's top trading cycles is the unique efficient, strategy proof, and individually rational matching mechanism. In Gale's top trading cycles each agent points to the owner of his most preferred house. Any agent in a pointing cycle is matched with the house he points to. The procedure is repeated with all unmatched agents and the restriction of their preferences to the unmatched houses and the algorithm terminates once a matching is reached.

Shapley and Scarf's [15] and Roth's [12] proofs that Gale's top trading cycles is efficient, strategy proof, and individually rational apply directly to the domain of single-peaked preferences. However, Ma's [10] proof, that Gale's top trading cycles is the only such mechanism, does not transfer. Theorem 2 shows that Ma's [10] uniqueness result holds if each agent can be picky about any house, in the sense that this house is the only house he prefers to his endowment. Since many picky preferences are not single-peaked, Ma's [10] uniqueness result does not apply to the single-peaked domain. Indeed, when there are at least 3 agents, the crawler differs from Gale's top trading cycles (Proposition 1).

On the domain of single-peaked preferences the crawler has two advantages over Gale's top trading cycles. It has an extensive form implementation that is - in a well-defined sense - easier to understand than any extensive form implementation of Gale's top trading cycles. The crawler can, moreover, also be applied to ongoing matching problems where some agents have to be matched before the full extent of all agents' preferences becomes known.

To define mechanisms that are more or less easy to understand, consider a strategy for some agent $i$ in an extensive form mechanism. Arbitrarily fix a history where agent $i$ moves and that can be reached if the agent plays the given strategy. This strategy is obviously dominant following Li [9] if $i$ (weakly) prefers the worst outcome associated with the continuation of his strategy to the best outcome following a deviation at the current history (and all later histories). To calculate the relevant worst (best) payoff the agent considers the most harmful (favorable) choices by all other agents in all histories following the current one. Li [9] argues that even cognitively impaired agents or agents who suspect the designer of fraud never see a reason to deviate from an obviously dominant strategy. Theorem 3 shows that the crawler can be implemented in obviously dominant strategies. Conversely I show that even on the restricted domain of single-peaked preferences Gale's top trading cycles cannot be implemented in obviously dominant strategies.[1]

[1]Li [9] already showed that Gale's top trading cycles cannot be implemented in obviously dominant strategies on the domain of all linear preferences.

In Section 6 I define a variant of the crawler that can be used on a larger domain of single-peaked preferences where agents may be indifferent between some houses. Theorem 4 shows that this variant inherits the three crucial properties of the crawler: it is efficient, individually rational and implementable obviously dominant strategies.

Section 7 concerns matching problems where some agents need to be matched before the full extent of the problem becomes known. Bade [5] motivates such problems as shift exchanges, where the reassignment of some shifts typically has to occur before the preferences of all future workers become known. In line with this motivation I assume here that agents' preferences are single-peaked with respect to time. The agents, as well as the shifts they are endowed with, are represented as the set of natural numbers $\mathbb{N}$. There is some fixed number $T \in \mathbb{N}$ such that agents never find any shift further than $T$ periods from their initial endowment acceptable. Theorem 5 defines an efficient, strategy-proof and individually rational variant of the crawler for shift exchange problems. I show that there exists a number $K$ such that it suffices to know the first $i+K$ preferences to match any agent $i$. Bade [5] in contrast shows that no efficient and individually rational matching mechanism satisfies this criterion of decision-making in finite time, if any agent $i$ may rank all shifts during his "lifespan" $\{i-T,\dots,i+T\}$ in any order.

# 2 Definitions

There is a set of agents $N\colon = \{1,\dots,n\}$. Each agent $i \in N$ is initially endowed with house $i$, so the set of houses is also $N$. Each agent $i$ has a transitive and complete preference $\succsim_i$ over all houses. A profile of all agents' preferences is denoted $\succsim$. A house $j$ is $\succsim_i$-**acceptable** if $j \succsim_i i$. The preference $\succsim_i^{[j]}$ is **picky about** $j$ if no house other than $i$ and $j$ is $\succsim_i^{[j]}$-acceptable. If $j \succsim_i j'$ holds for some $j \in N$ and all $j'$ in some set $N' \subset N$ I write $j \succsim_i N'$

The names of all houses reflect an objective linear order on all houses. Houses are ordered by their sizes and $i < j$ means that $i$ is smaller than $j$. The preference $\succsim_i$ is **single-peaked** (with respect to the order $<$ on all houses) if there exists a $i^* \in N$ such that $j \succsim_i k$ holds if either $k < j \leq i^*$ or $k > j \geq i^*$. A preference is a **linear** order if it is antisymmetric. Arbitrary domains of agent $i$'s preferences and of preference profiles are denoted $\Omega_i$ and $\Omega :\, = \Omega_1 \times \ldots, \times \Omega_n$. The domains of all linear preferences, of all single-peaked preferences and of all linear, single-peaked preferences respectively are $\Omega^l$, $\hat{\Omega}$, and $\hat{\Omega}^l = \Omega^l \cap \hat{\Omega}$.[2]

A **submatching for** $N'$ is a bijection $\nu : N' \to \nu(N')$ with $\nu(N') \subset N$. Under $\nu$ agent $i \in N'$ is matched with house $\nu(i)$. The submatching that matches no one is called $\emptyset$. A submatching $\rho : N' \to N'$ is a cycle if for each $i, j \in N'$ there exists some $m \in \mathbb{N}$ such that $j = \rho^m(i)$. Any submatching $\nu : N' \to \nu(N')$ entails an **indexation with respect to a submatching** $\nu$ for the set $N'$, so that agents with lower index own smaller houses: $\nu(i_t) < \nu(i_{t+1})$ for all $1 \leq t <\mid N' \mid$. If are indexed with respect to $\nu$ and if $t < t'$, then agent $i_t$ occupies a smaller house than $i_{t'}$ given $\nu$.

A submatching $\mu$ that matches all agents is a **matching**. If $\nu(i) = \mu(i)$ for all $i$ matched by $\nu$ then I write $\nu \subset \mu$. The sets of all matchings and submatchings respectively are $\mathcal{M}$ and $\overline{\mathcal{M}}$. The initial endowment is represented by the matching $id : N \to N$ where $id(i) = i$ for all $i \in N$. Agents only care about their own houses, so agent $i$ prefers matching $\mu$ to matching $\mu'$ if and only if $\mu(i) \succsim_i \mu'(i)$. A matching $\mu$ is **efficient** at $\succsim$ if any matching $\mu'$ that is strictly better than $\mu$ for some agent is strictly worse than $\mu$ for a different agent. The same $\mu$ is **individually** rational at $\succsim$ if $\mu(i)$ is $\succsim_i$-acceptable for each $i$.

A social choice function $scf : \Omega \to \mathcal{M}$ maps each profile $\succsim$ in the arbitrary domain $\Omega$ to a matching in $\mathcal{M}$. Any social choice function can be viewed as a direct revelation mechanism with the understanding that each agent $i$ declares a preference $\succsim_i$ to the designer, who in turn chooses the matching $scf(\succsim)$ given that $\succsim$ is the profile of stated preferences. The mechanism $scf$ is efficient (individually rational) if $scf(\succsim)$ is efficient (individually rational) at $\succsim$ for each $\succsim \in \Omega$. It is **strategy proof** if no agent has an incentive to misrepresent his preferences, so $scf(\succsim)(i) \succsim_i scf(\succsim'_i, \succsim_{-i})(i)$ holds for all

[2]The preference represented by $u_i(j) = - \mid j - i \mid$ is in $\hat{\Omega}$ but not in $\hat{\Omega}^l$ since $i$ indifferent between $i+1$ and $i-1$. Since any $\succsim \in \hat{\Omega}$ may have multiple most preferred houses, such preferences are sometimes called single-plateaued.

$i, \succsim'_i$ and $\succsim$.

# 3 The Crawler

The crawler $C : \hat{\Omega}^l \to \mathcal{M}$ is defined via a trading process that screens all houses in order of their size. The smallest house whose current occupant, say agent $i$, wants to either stay or move to a yet smaller house, leaves the mechanism with his most preferred house as his match. All agents who currently occupy houses that are at least as big as agent $i$'s choice and smaller than the house vacated by agent $i$ "crawl" to the next largest house. This process is repeated until all agents are matched. To calculate $C(\succsim)$ for any $\succsim \in \hat{\Omega}^l$ go to Step 1, initialized with $N^1 := N$, $\nu^1 = id$ and let $t = i_t$ for all $t \in N$.

Step $k$:

Screening: If $\nu^k(i_t) \succ_{i_t} \nu^k(i_{t+1})$ holds for some $t$, let $t^*$ be the minimal such $t$. If not, let $t^* := \mid N^k \mid$. Let $i_{t^*} := i^k$.

Matching: Let $C(\succsim)(i^k) = \nu^k(i_r)$ be the $\succsim_{i^k}$-best house among all remaining houses.

Crawling: For each agent $i_t$ with $r \leq t < t^*$ let $\nu^{k+1}(i_t) := \nu^k(i_{t+1})$.

Updating: Let $N^{k+1}$ be the set of all unmatched agents. If $N^{k+1} = \emptyset$ terminate. If not let $\nu^{k+1}(i) = \nu^k(i)$ for each $i \in N^{k+1}$ for whom $\nu^{k+1}(i)$ is not yet defined. Index all agents in $N^{k+1}$ with respect to $\nu^{k+1}$, and go to Step $k+1$.

Exactly one agent, the agent $i_{t^*} = i^k$ identified in Screening, is matched at each step. The superscript $k$ keeps track of the step $k$ at which agent $i^k$ is matched. While subscripts change with the current occupancy of the unmatched houses at every step, any superscript remains fixed after it has been assigned.

If agent $i^k$ gets matched with the house he currently occupies no agent crawls from $\nu^k$ to $\nu^{k+1}$. If not then all occupants of houses at least as large as the match of agent $i^k$ and smaller than the house occupied by $i^k$ at the beginning of Step $k$ crawl to the next largest unmatched house. Any crawling agent $i$ prefers his new house $\nu^{k+1}(i)$ to $\nu^k(i)$ the house he occupied at the beginning of this step. Any temporary submatching $\nu^k$ respects the ordering of houses in the sense that $\nu^k(i) < \nu^k(j)$ holds for two unmatched agents $i$ and $j$ if $i < j$. A combination of some (or even all) sub-steps would render the definition of the crawler more concise. However, the detailed definition can more easily be amended to cover infinitely many agents, indifferences and obvious dominance. To see the crawler at work consider the following example.

**Example 1** Define a profile of preferences $\succsim$ with $6 \succsim_4 N$, $3 \succsim_6 N$, $5 \succsim_7 N$, $\succsim_2 = \succsim_2^{[3]}$ so that agent agent 2 only prefers house 3 to his own, and where all other agents want to move to the largest possible house, so $7 \succsim_i N$ for $i = 1, 3, 5$. Figure 1 illustrates the crawling process. Each line represents the 7 houses. The top labels denote the houses, the bottom labels denote the current occupants. The boxes represent finalized matches.

In the first line each house is occupied by its initial owner. Screening all houses starting with the smallest, Step 1 finds that agent 6 is the first owner who wants to move to a smaller house. The solid arrow in the first line shows that agent 6 would like to move to house 3. Agent 6 is boxed with house 3 in the next line, as they no longer participate in the crawling process. The dashed arrows show how agents 3, 4, and 5 crawl to the next largest house. Since agent 2 is picky about house 3 he becomes matched in Step 2, as shown in line three. In Step 3 agent 7 is the first agent who would like to move to a smaller house and the solid arrow represents the fact that agent 7 most likes house 5. In line 5 agent 7 is boxed with house 5. The dashed arrows represent agents 4 and 5's, respective crawls to the next largest house. Since none of the agents remaining unmatched after Step 3 would like to move to a house smaller than the one they currently occupy the temporary matching achieved in Step 3 is also the final matching.

**Theorem 1** *The crawler $C : \hat{\Omega}^l \to \mathcal{M}$ is a well-defined, efficient, strategyproof, and individually rational mechanism.*

**Proof** Fix a profile $\succsim \in \hat{\Omega}^l$.

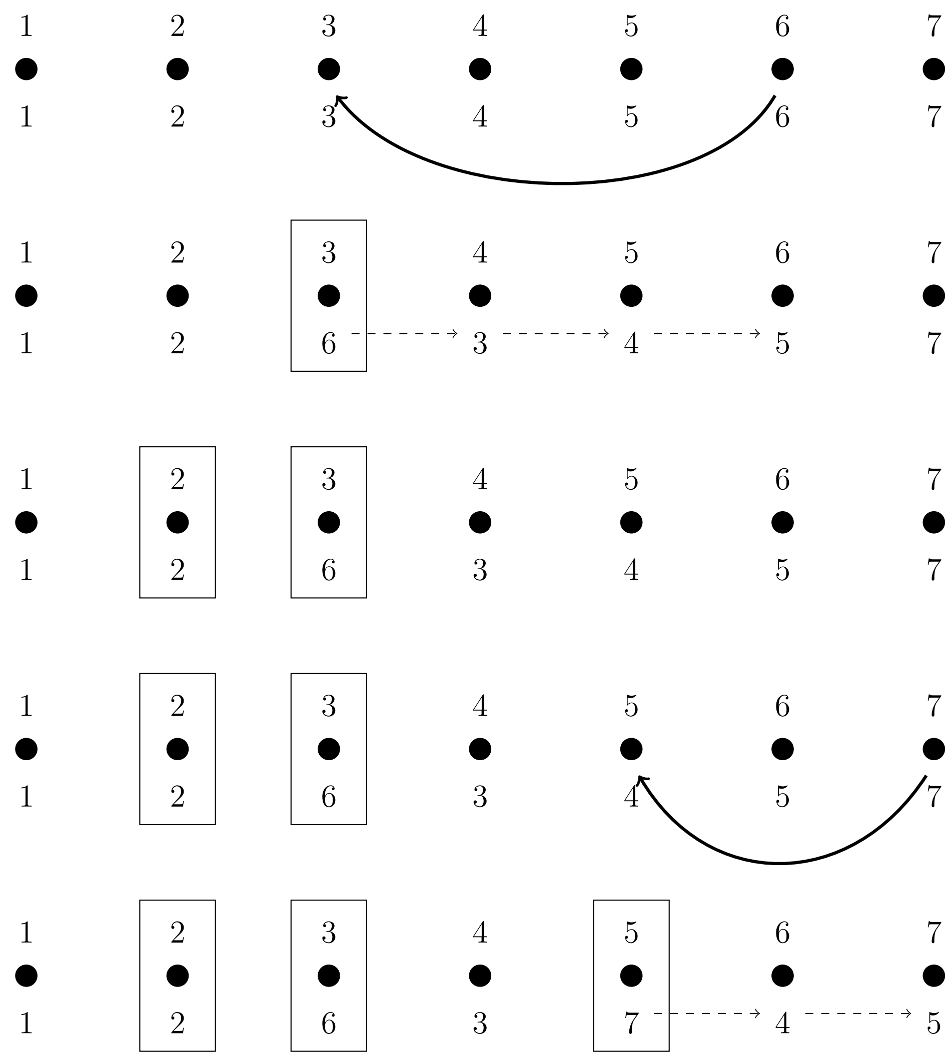


Figure 1: The crawling process

At any Step $k$, Screening finds an agent $i_{t^*} := i^k$. Since this agent $i^k$ is the only agent matched at Step $k$ and since $i^k$ is matched with an as of yet unmatched house $C(\succsim)$ is a matching and the crawler is well-defined.

To see that $C$ is efficient note that $i^1$, the agent matched in Step 1, prefers $C(\succsim)(i^1)$ to all houses. Conditioning on the match between $i^1$ and $C(\succsim)(i^1)$ the second matched agent $i^2$ obtains his most preferred house out of the remainder. Proceeding inductively we see that $i^k$ prefers his match $C(\succsim)(i^k)$ to all houses that remain after all agents $\{i^1, \ldots, i^{k-1}\}$ have been matched, and $C$ is efficient.

To see that $C$ is individually rational consider an arbitrary agent $i$. If $i$ stays unmatched at Step $k$ we have $\nu^{k+1}(i) \succsim_i \nu^k(i)$, if $i$ does get matched at Step $k$ we have $C(\succsim)(i) \succsim_i \nu^k(i)$. Using transitivity and $\nu^1(i) = i$ we obtain $C(\succsim)(i) \succsim_i i$ and $C$ is individually rational.

Finally $C$ is strategy proof since it can be implemented in obviously dominant strategies as will be shown in Theorem 3. $\square$

# 4 Gale's top trading cycles

When there are at least 3 agents and 3 houses then the Crawler differs from Gale's top trading cycles. Gale's top trading cycles $G : \Omega \to \mathcal{M}$ is defined for any arbitrary domain $\Omega$ of linear preferences. The following algorithm finds $G(\succsim)$ for any profile $\succsim \in \Omega$. Initialize with $N^1 := N$ and $\nu^0 = \emptyset$.

Step $k$:

Let each agent in $N^k$ point to his most preferred house in $N^k$ and let each house point to its owner. At least one cycle forms. Match each agent in a cycle with the house he is pointing to. Let $\nu^{k+1}$ summarize all matches made so far. If $\nu^{k+1}$ is a matching let $k = K$ and terminate with $G(\succsim) = \nu^K$. If not let $N^{k+1}$ be the set of unmatched agents and go to Step $k+1$.

Shapley and Scarf [15] and Roth [12] show that $G$ is efficient, strategyproof, and individually rational on any domain $\Omega$ of linear preferences. Ma [10] shows that $G$ is the unique such mechanism on $\Omega^l$ the domain of all linear preferences. To put the case of single-peaked preferences into relief I slightly strengthen Ma's [10] result to Theorem 2, which shows that the uniqueness result applies to any domain $\Omega$ that allows for all picky preferences.

**Theorem 2** *Fix an efficient, strategy proof, and individually rational mechanism $M : \Omega \to \mathcal{M}$ for a domain of linear preferences $\Omega$. If $j \succsim_i i$ for some $\succsim_i \in \Omega_i$ and $i, j$ implies $\succsim_i^{[j]} \in \Omega_i$, then $M$ is Gale's top trading cycles.*

Ma's [10] proof uses exactly one feature of the domain $\Omega^l$: if an agent finds a house acceptable according to some preference, then he may be picky about this house. Theorem 2 is consequently covered by Ma's [10] proof. For convenience I also state a proof here. Over the years Svensson [16], Anno [1] and Sethuraman [14] gave substantially more concise uniqueness proofs. My proof combines some of their simplifying ideas: Following Svensson [16] I use induction over the set of cycles that form in trading process. Following Sethuraman [14] I directly work with efficient, strategyproof and individually rational mechanisms which allows me to avoid the detour to relate such mechanisms to the core.

**Proof** Fix a profile $\succsim^* \in \Omega$. Since Gale's trading process starts with $\nu^0 = \emptyset$ and ends with $G(\succsim^*) = \nu^K$, the claim holds if $\nu^{k-1} \subset M(\succsim^*)$ implies $\nu^k \subset M(\succsim^*)$ for all $0 < k \leq K$. So assume $\nu^{k-1} \subset M(\succsim^*)$ for some $k \leq K$. Say $\rho : N' \to N'$ is a cycle at Step $k$ of $G(\succsim^*)$. For each $i \in N'$ let $\succsim_i' :\ = \succsim_i^{[\rho(i)]}$, so $\succsim_i'$ is picky about $\rho(i)$. Considering all $D \subset N'$, use induction over $\mid D \setminus N' \mid$ to see $\rho \subset M(\succsim_D', \succsim_{-D}^*)$ for all $D \subset N'$, in particular $\rho \subset M(\succsim^*)$ for $D = \emptyset$.

At $\mid D \setminus N' \mid = 0$ (so $D = N'$) the efficiency and individual rationality of $M$ implies $\rho \subset M(\succsim_D', \succsim_{-D}^*)$. Now assume $\rho \subset M(\succsim_D', \succsim_{-D}^*)$ for any $\mid N' \setminus D \mid \leq m$ with $D \subset N'$. Fix $D \subset N'$ with $\mid N' \setminus D \mid = m + 1$. For any $i \in N' \setminus D$, the inductive hypothesis and the definition of Gale's top trading cycles respectively imply $\rho \subset M(\succsim_{D\cup\{i\}}', \succsim_{-(D\cup\{i\})}^*)$ and $\rho(i) \succsim_i N^{k-1}$. Since $M$ is strategyproof $M(\succsim_D', \succsim_{-D}^*)(i)$ must for any such $i$ equal $\rho(i)$. Since $\rho$ is a cycle and since $M$ is individually rational, the definition of $\succsim'$, then implies $M(\succsim_D', \succsim_{-D}^*)(i) = \rho(i)$ for each $i \in D$ and we have $\rho \subset M(\succsim_D', \succsim_{-D}^*)$. $\square$

As most picky preferences are excluded from the linear domain of single-peaked preferences $\hat{\Omega}^l$, there may be multiple efficient, strategy proof and individually rational mechanisms $M : \hat{\Omega}^l \to \mathcal{M}$. The next proposition shows that this is indeed the case: the crawler differs from Gale's top trading cycles.

**Proposition 1** *If $N \geq 3$ then the crawler $C : \hat{\Omega}^l \to \mathcal{M}$ differs from Gale's top trading cycles $G : \hat{\Omega}^l \to \mathcal{M}$.*

**Proof** Define $\succsim \in \hat{\Omega}^l$ such that the ideal houses of agents 1, 2, and 3 respectively are 3, 1, and 1. If there are any agents $i > 3$ these agents are endowed with their ideal house. To calculate $C(\succsim)$ note that Step 1 matches agent 2 with house 1, whereas $G(\succsim)(1) = 3$. $\square$

The alignment of interests implied by restriction to single-peaked preferences allows for a larger set of efficient, strategy proof and individually rational mechanisms. Clearly the crawler and Gale's top trading cycles are not the only such mechanisms: The crawler has a dual mechanism that starts screening with the largest house and then moves to the smaller ones. Mutatis mutandis all of the arguments in the present paper also apply to this dual mechanism.

# 5 Obvious Dominance

Some implementations of strategy proof mechanisms are easier to understand than others. To make this intuitive idea precise, Li [9] defines "obvious dominance" to distinguish strategies that are merely dominant from strategies that can easily be recognized as such. Fix a strategy for an agent in some extensive form game. This strategy is obviously dominant if, whenever the agent gets to choose at a history that can be reached if the agent follows the given strategy, the agent's minimal payoff if he continues the strategy given any possible follow-up choices of all other agents is at least as high as the agent's maximal payoff given any collective deviation starting at the present history. Obvious dominance distinguishes between different extensive forms that implement the same strategyproof social choice function.[3] Li [9] provides experimental evidence that agents indeed find obviously dominant implementations of strategy proof mechanisms easier to understand than alternative implementations that do not satisfy this criterion.

[3] Li's [9] lead example to motivate obvious dominance shows that ascending clock auctions implement second price auction in obviously dominant strategies whereas the corresponding direct revelation mechanism does not.

Li [9] shows that Gale's top trading cycles with at least 3 agents cannot be implemented in obviously dominant strategies. The crawler is simpler to understand: there is an extensive form mechanism that implements the crawler in obviously dominant strategies.

An **extensive form mechanism** $M$ is an extensive game form where a rooted tree represents the a set of **histories** $H$. A history is **terminal** if it is not a subhistory of any other history. The set of all terminal histories is $Z$. The set of possible actions after the nonterminal history $h$ is $A(h) := \{a \mid (h, a) \in H\}$. The set of players is $N$. The player function $P$ maps any nonterminal history $h \in H \setminus Z$ to a player $P(h) \in N$ who gets to choose from all actions $A(h)$ at $h$.[4] Each terminal history $h \in Z$ is associated with a matching $\mathcal{M}$. A **behavior** $B_i$ for player $i$ is a vector of actions that specifies a choice $a \in A(h)$ for each history $h$ with $P(h) = i$. The path $Path(B)$ or a behavior profile $B := B_1 \times \cdots \times B_n$ is the set of all histories that are reached if all agents follow $B$. So $\emptyset \in Path(B)$ and $(h, a) \in Path(B)$ if $h \in Path(B)$ and $B_{P(h)}(h) = a$. The outcome $M(B)$ is associated with the unique terminal history $h \in Path(B)$. A **strategy** $S_i$ for player $i$ maps each $\succsim_i \in \Omega_i$ to a behavior $S_i(\succsim_i)$. A strategy profile $S = S_1 \times \cdots \times S_n$ consists in strategies for all agents.

A strategy $S_i$ is **obviously dominant** (Li [9]) for agent $i$ if for every $\succsim_i \in \Omega_i$, behavior profiles $B$ and $B'$, and history $h$, with $h \in Path(S_i(\succsim_i), B_{-i})$, $h \in Path(B')$, $P(h) = i$, $S_i(\succsim_i)(h) \neq B'_i(h)$ we have

$$M(S_i(\succsim_i), B_{-i})(i) \succsim_i M(B')(i).$$

A social choice function (or direct revelation mechanism) $scf : \Omega \to \mathcal{M}$ is implementable in obviously dominant strategies if there exists an extensive form mechanism $M$ and a profile of obviously dominant strategies $S$ such that $M(S(\succsim)) = scf(\succsim)$ for all $\succsim \in \Omega$.

**Theorem 3** *The crawler $C : \hat{\Omega}^l \to \mathcal{M}$ can be implemented in obviously dominant strategies.*

[4] Bade and Gonczarowski [6] show that simultaneous moves can be ignored without loss of generality when considering extensive form mechanisms that implement an outcome in obviously dominant strategies.

**Proof** Define an extensive form mechanism where each history is labeled $(k,t)$ for some $k$, $t$. All histories $(k,t)$ for a fixed $k$ subdivide Step $k$ of $C$ into separate choices for all agents $i_t$ whose choices matter at this Step. At the history $(k,t)$ agent $i_t$ chooses from the choice set $A(k,t) \colon = \{c, 1, \cdots, t\}$ if $t <\mid N^k \mid$ and $A(k, \mid N^k \mid) \colon = \{1, \ldots, \mid N^k \mid\}$. If agent $i_t$ chooses $c$ go to $(k, t+1)$, if $i_t$ chooses a number $r \in \{1, \ldots, t\}$, let $t = t^*$ and follow Matching (letting $C(\succsim)(i_{t^*}) = \nu^k(i_r)$), Crawling, and Updating as in the definition of $C$. At the end of Updating go to the history labeled $(k+1, 1)$ (as opposed to so Step $k+1$).

Consider the strategy profile $S$ where agent $i_t$ chooses $c$ if $\nu^k(i_{t+1}) \succ_{i_t} \nu^k(i_t)$ and chooses $r \in \{1, \ldots, t\}$ if $\nu^k(i_r)$ is his most preferred unmatched house at Step $k$. As long as $\nu^k(i_{t+1}) \succ_{i_t} \nu^k(i_t)$ holds agents choose $c$ (for continue) and we move from $(k,t)$ to $(k,t+1)$ and stay within Step $k$. Once $\nu^k(i_{t+1}) \succ_{i_t} \nu^k(i_t)$ is not satisfied the other sub-steps of Step $k$ are triggered and agent $i_{t^*}$ is matched to his most preferred remaining house. Since agents are asked in order of their index with respect to $\nu^k$, each Step $k$ finds the smallest house whose owner does not want to move to a larger house and the behavior $S(\succsim)$ in the histories $(k,t)$ for a fixed $k$ induces the choices prescribed by Step $k$. In sum we obtain $M(S(\succsim)) = C(\succsim)$ for each $\succsim \in \hat{\Omega}^l$.

To see that $S$ is obviously dominant fix an arbitrary $(k,t)$. Say that $j$ is the $\succsim_{i_t}$-best house that remains unmatched at Step $k$. If $j \leq \nu^k(i_t)$ then $i_t$ is matched with $j$ if he follows $S_{i_t}(\succsim_i)$. Since $j$ is the $\succsim_{i_t}$-best unmatched house agent $i_t$ cannot be made strictly better off by any collective deviation at $(k,t)$ and the following histories. If $j > \nu^k(i_t)$ and if agent $i_t$ follows $S_{i_t}(\succsim_i)$ then he obtains a house he weakly prefers to $\nu^k(i_t)$ - no matter the behavior of all agents in the subsequent steps. However, if he deviates, he gets immediately matched with a house $j' \leq \nu^k(i_t)$ which is by the assumption on $i_t$'s preference no better than $\nu^k(i_t)$. Since $(k,t)$ was chosen arbitrarily, $S$ is obviously dominant. $\square$

The test whether a social choice function can be implemented in obviously dominant strategies is tougher than the test whether it is strategy proof. Theorem 3 therefore implies that the crawler $C : \hat{\Omega}^l \to \mathcal{M}$ is strategy proof, completing the proof of Theorem 3.

Li [9] has shown that Gale's top trading cycle with at least 3 agents is not implementable in obviously dominant strategies. Given Ma's [10] uniqueness result (Theorem 2) no efficient and individually rational mechanism for house matching problems with more than 2 agents can be implemented in obviously dominant strategies. Bade and Gonczarowski [6] study obvious dominance in a variety of settings and come to the conclusion that only very few Pareto optimal social choice functions can be implemented in obviously dominant strategies. They show in particular that median voting with single-peaked preferences is not implementable in obviously dominant strategies. Arribillaga, Masso and Neme [3] come to a similar conclusion on the dearth of obviously strategyproof voting mechanisms.

Theorem 3 then simultaneously applies the domain restrictions of the impossibility results by Li [9] (housing market) and Bade and Gonczarowski [6] as well as Arribillaga, Masso and Neme [3] (single peakedness) to obtain a possibility result for the single-peaked housing markets. The crawler is efficient and individually rational and can be implemented in obviously dominant strategies. This possibility result differs in two dimensions from the preceding impossibility results. It concerns a novel mechanism (the crawler) as well as a novel domain of preferences (the single-peaked matching domain). So one may now wonder whether the restriction on the domain is sufficient for Gale's top trading cycles to be implementable obviously dominant strategies. In the appendix I give a negative answer. I show that $G : \hat{\Omega}^l \to \mathcal{M}$ cannot be implemented in obviously dominant strategies if there are at least 4 agents.

# 6 Indifferences

While Gale's top trading cycles mechanism is the unique efficient, strategy proof, and individually rational mechanism for the linear domain $\Omega^l$, there exist a plethora of different such mechanisms on the grand domain $\Omega$ that permits indifferences. Jaramillo and Manjunath [8], Alcalde-Unzu and Molis [2], Saban and Sethuraman, Aziz and De Keijzer [4], as well as Plaxton [11] all describe and study such mechanisms.

With single-peaked preferences there are - of course - even more such mechanisms, given that Ma's [10] uniqueness result does not apply. The circle crawler $C^\sim : \hat{\Omega} \to \mathcal{M}$ adapts the crawler to the case of indifferences and can be implemented in obviously dominant strategies.[5]

To define the circle crawler $C^\sim(\succsim)$ for each $\succsim \in \hat{\Omega}$ modify the crawling process in the definition of $C$ by starting each step with a new Matching$^\sim$ sub-step and by a slight modification of the original Matching sub-step to Circling$^\sim$ keeping all else equal. The new Circling$^\sim$ sub-step shares with the original Matching sub-step that agent $i_{t^*}$ moves to a weakly smaller house. In Matching agent $i_{t^*}$ moves to his most preferred house. The Circling$^\sim$ sub-step addresses the problem that an agent may have multiple most preferred houses by letting $i_{t^*}$ move to the smallest among all his most preferred houses. The second difference between Matching and Circling$^\sim$ is that agent $i_{t^*}$ permanently moves to $\nu^k(i_r)$ in Matching while this move is temporary in Circling$^\sim$. Since the downward move of $i_{t^*}$ does not determine agent $i_{t^*}$'s match a different rule is needed to govern matches in $C^\sim$. Agents are periodically matched using the first sub-step Matching$^\sim$. For convenience the following definition repeats the sub-steps that are identical to the sub-steps of step $k$ in the crawler $C$.

Step $k$:

Matching$^\sim$: If $\nu^k = \nu^{k'}$ for some $k < k'$ then let $C^\sim(\succsim)(i^{k''}) :\, = \nu^k(i^{k''})$ for each each agent $i^{k''}$ with $k' \leq k'' < k$ and go to Updating. If not go to Screening.

Screening: If $\nu^k(i_t) \succ_{i_t} \nu^k(i_{t+1})$ holds for some $t$, let $t^*$ be the minimal such $t$. If not, let $t^* :\, =\mid N^k \mid$. Let $i_{t^*} :\, = i^k$.

Circling$^\sim$: Let $\nu^{k+1}(i^k) :\, = \nu^k(i_r)$ be the smallest $\succsim_{i^k}$-best house among all remaining houses.

Crawling: For each agent $i_t$ with $r \leq t < t^*$ let $\nu^{k+1}(i_t) :\, = \nu^k(i_{t+1})$.

[5]Ehlers' [7] proof that there exists no efficient, group-strategyproof and individually rational matching mechanism for the grand domain of all preferences directly transfers to the case of the single-peaked domain $\hat{\Omega}$. However the question whether there exists a non-bossy efficient, strategy proof and individually rational mechanism for $\hat{\Omega}$ remains open.

Updating: Let $N^{k+1}$ be the set of all unmatched agents. If $N^{k+1} = \emptyset$ terminate. If not let $\nu^{k+1}(i) = \nu^k(i)$ for each $i \in N^{k+1}$ for whom $\nu^{k+1}(i)$ is not yet defined. Index all agents in $N^{k+1}$ with respect to $\nu^{k+1}$, and go to Step $k+1$.

Matching$^\sim$ only calls for any matches to be made once agents come full circle. If the new temporary endowment $\nu^k$ at some Step $k$ coincides with a preceding temporary endowment $\nu^{k'}$ at some Step $k' < k$, then Matching$^\sim$ matches each agent who assumed the role of $i_{t^*}$ at one of the intermediate steps, so each $i^{k''}$ with $k' \leq k'' < k$, with the house $\nu'(i^{k''})$ he currently occupies. Example 2 illustrates how delaying the matching of some agents with multiple most preferred houses facilitates Pareto improvements.

**Example 2** Let $N = \{1,2,3\}$ and let $\succsim_1$, $\succsim_2$, and $\succsim_3$ respectively top rank the sets $\{1,2,3\}$, $\{1,2\}$, and $\{1\}$. In the unique Pareto optimum at $\succsim$, agents 1 and 3 swap houses, while 2 keeps his endowment. In Step 1 of $C^\sim(\succsim)$ agent $2 := i^1$ is the first agent who strictly prefers the house he currently occupies (house 2) to the next largest remaining house (house 3). So Circling$^\sim$ and Crawling yield $\nu^2(1) = 2$ and $\nu^2(2) = 1$. If we were to immediately match agent $2 = i^1$ with house 1, we would obtain an inefficient matching. However following the definition of $C^\sim$ no agent is matched at Step 1. At Step 2, agent 3 is $i^2$ and the temporary matching $\nu^3$ has $\nu^3(3) = 1$, $\nu^3(2) = 2$, and $\nu^3(1) = 3$. At Step 3, agent 3 is once again the first agent who does not strictly prefer the next largest house and we obtain $\nu^4(3) = 1$, $\nu^4(2) = 2$, and $\nu^4(1) = 1$. Since $\nu^4 = \nu^3$ agent $3 = i^3$ is matched with $\nu^4(3) = 1$ at Step 3. Agents 2 and 1 are then respectively matched with houses 2 and 3 at Steps 5 and 6.

To see that the circle crawler $C^\sim$ reduces to the standard crawler $C$ on the domain $\hat{\Omega}^l$ fix any $\succsim \in \hat{\Omega}^l$. At the first step the first clause of Matching$^\sim$ cannot be satisfied, so the algorithm of $C^\sim$ must continue with Screening. Since the Screening sub-step of $C$ and $C^\sim$ is identical, the same agent $i^1 := i_{t^*}$ is identified by both algorithms. While this agent under the algorithm for $C$ is immediately matched with his most preferred house $j := \nu^1(i_r)$, the same agent temporarily moves into this house, so we have $j = \nu^2(i^1)$.

If $r = t^*$, then the preceding condition means that agent $i_{t^*} = i^1$ stays put and we have $\nu^1 = \nu^2$ and agent $i_{t^*}$ gets matched at Step 2. If not then Step 2 again identifies the same agent $i :\ = i^2$ in Screening. Since $i$, due to the preceding move, now occupies his most preferred house agent $i$ stays put in this Step 2 and we have $\nu^2 = \nu 3$ so that $i$ is matched with the same house $j$ in Step 3. (This is indeed how agent $3 = i^3$ is matched at Step 4 in Example 2.) Proceeding inductively we see that $C^{\sim}(\succsim) = C(\succsim)$ holds for any $\succsim \in \hat{\Omega}$.[6]

**Theorem 4** *The circle crawler $C^{\sim} : \hat{\Omega} \to \mathcal{M}$ is an efficient, strategyproof and individually rational matching mechanism that can be implemented in obviously dominant strategies.*

**Proof** Fix a profile $\succsim \in \hat{\Omega}$.

Since any Step $k$ either matches no one or bijectively matches a set of agents with a set of houses, the modified crawling process at $\succsim$ yields a submatching $\nu : N' \to \nu(N')$ for some subset $N' \subset N$. Suppose $\nu$ was not a matching so $N \setminus N' \neq \emptyset$. Since $N \setminus N'$ is finite, there are only finitely many submatchings mapping $N \setminus N'$ to $N \setminus \nu(N')$. Therefore there exists some Steps $k < k'$ such that $\nu^k = \nu^{k'}$. By the definition of the crawling process any $i^{k''} \in N \setminus N'$ with $k \leq k'' < k'$ must be matched at Step $k'$ - a contradiction.

The proof that $C$ is individually rational (see Theorem 1) applies unchanged to the case of $C^{\sim}$.

To see that $C^{\sim}(\succsim)$ is efficient it suffices to show that each agent $i$ matched at Step $k$ weakly prefers his match to any unmatched house at Step $k$ and strictly prefers his match to any house that remains unmatched after that step. For $i$ to be matched at Step $k$, $\nu^k = \nu^{k'}$ must hold for some $k' \leq k$ and $i$ must equal $i^{k''}$ for some $k' \leq k'' < k$.

At Step $k''$ agent $i^{k''}$ is temporarily matched with one of his most preferred houses out of $\nu^k(N^k) = \nu^{k''}(N^{k''})$. Since $\nu^{s+1}(j) \succsim_j \nu^s(j)$ holds for all Steps $s$ and any unmatched agent $j$, transitivity implies $C^{\sim}(\succsim)(i^{k''}) = \nu^k(i^{k''}) \succsim_{i^{k''}}$

[6] It should be noted though that the superscript $k$ of agent $i^k$ does not denote the step at which an agent is matched by $C^{\sim}(\succsim)$.

$\nu^k(j)$ for any $j \in N^k$. So $i$ weakly prefers his match $C^\sim(\succsim)(i)$ to all houses $\nu(N^k)$ remaining at Step $k$.

Now fix any house $j \in \nu^k(N^k)$ that agent $i^{k''}$ ranks at the top of all houses remaining at Step $k$, so $j \succsim_{i^{k''}} \nu^k(N^k)$. Since preferences are single-peaked, since any agent who moves to a larger house at some step moves to the next largest house, and since $\nu^{k'}(i^{k''}) = \nu^k(i^{k''})$, there must exist some Step $s$ with $k' < s \leq k$ with $\nu^s(i^{k''}) = j$. Since $j$ is not continuously owned by the same agent on the path from $k'$ to $k$, the agent $i'$ with $\nu^k(i') :\ = j$ moves houses on the path from $k'$ to $k$. So $i' = i^{\tilde{k}}$ for some $k' \leq \tilde{k} < k$ and $i'$ is matched with $\nu^k(i') = j$ at Step $k$. Since $j$ was chosen arbitrarily, agent $i$ strictly prefers his match to all houses that remain in the mechanism after he gets matched.

The implementation of $C^\sim$ in obviously dominant strategies closely follows the path laid out in the proof that the crawler $C$ can be implemented in obviously dominant strategies (Theorem 3): modify the crawling process defining $C^\sim$ such that each Step $k$ is further subdivided into nodes $(k,t)$ as defined in the proof of Theorem 3 (along with the modified conditions to start and continue the crawling process). The strategy $S$ prescribes the choice $c$ if the agent choosing at $(k,t)$ (weakly) prefers the next largest house to his current house and prescribes to circle to the smallest most preferred house if not, is - by the same arguments as the ones given in the proof of Theorem 3 - also obviously dominant in this modified extensive form game. □

# 7 Shift Exchange Problems

Shift exchange problems as defined by Bade [5] differ only in one respect from housing markets: the designer has to start matching agents and objects before the full extent of the problem becomes known. There is an endless stream of agents and objects. However these agents have finite life spans and the match of any given agent may only depend on the preferences known during his life span. To allow for such perpetual problems the sets of agents and objects are both modelled as the set of natural numbers $\mathbb{N}$. As a reminder of this new assumption the objects are called shifts and time is the ordering with respect to which all agents' preferences are single-peaked. Each agent only finds shifts within a some finite time window acceptable. Each preference $\succsim_i$ in the domain $\hat{\Omega}_i^T$ is a linear and single-peaked order, where any shift before $i - T$ and after $i + T$ is $\succsim_i$-unacceptable. Just as in the case of housing markets the identity function $id$ is the initial endowment, and each agent $i$ is initially endowed with shift $i = id(i)$. A matching $\mu : \mathbb{N} \to \mathbb{N}$ is a one-to-one and onto function: so each agent works a shift and each shift is taken by someone. The definitions of mechanisms, efficiency, individual rationality, strategyproofness, and obvious dominance transfer verbatim from Sections 2 and 5.

A mechanism $M : \hat{\Omega}^T \to \mathcal{M}$ is **practicable** if there exists some $K \in \mathbb{N}$ such that for each $i$ it suffices to know the first $i + K$ preferences to match agent $i$. In generic shift exchange problems, efficiency and individual rationality conflict with practicability: Assuming $T \geq 2$, and fixing an arbitrarily large number $K$, Bade [5] shows that no efficient and individual rational mechanism $M : \Omega^T \to \mathcal{M}$ - strategy proof or not - only uses the first $K$ preferences to match at least one agent. For any such mechanism we must at some profile $\succsim$ elicit more than $K$ preferences to match even a single agent.

The single-peaked domain differs sharply: a practicable variant of the crawler can be applied to shift exchange problems. This variant is moreover not just efficient and individually rational but also strategyproof. It can even be implemented in obviously dominant strategies. The infinite crawler $C^\infty : \Omega^T \to \mathcal{M}$ extends the definition of the crawler $C$ to shift exchange problems. The crawling process used to define $C^\infty$ is identical to the crawling process used to define the crawler $C$, except that Screening in each step $k$ is replaced by Screening$^\infty$ as follows.

Screening$^\infty$: If $\nu^k(i_t) \succ_{i_t} \nu^k(i_{t+1})$ holds for some $i_t \leq v^k(i_1) + T$, let $t^*$ be the minimal such $t$ and go to Matching. If not, let $t^* :\ = 1$, $C^\infty(\succsim)(i_1) :\ = \nu^k(i_1)$ and go to Updating. In either case let $i_{t^*} :\ = i^k$.

The trading process that defines the standard crawler $C$ does not map all shift exchange problems to matchings. If all agents want to work later, then no agent $t^*$ is found via the original Screening sub-step. Screening$^\infty$, in contrast, only screens finitely many agents. Once it is clear that all unmatched agents in the set $\{i_1, \ldots, \nu^k(i_1) + T\}$ want to work later, the second clause of Screening$^\infty$ calls for the current owner of the earliest unmatched shift to be matched with that shift. In this case the owner of $\nu^k(i_1)$ as well as each other agent $i$ who may possibly find $\nu^k(i_1)$ acceptable (so each $i \leq \nu^k(i_1)+T$) wants to work later and there is no individually rational matching where $i_1$ works later than $\nu^k(i_1)$. Recognizing this inevitability, the second clause of Screening$^\infty$ then matches $i_1$ with $\nu^k(i_1)$.[7]

The infinite crawler $C^\infty$, just like the original crawler, matches exactly one agent per step: $i^k$ is the agent matched at Step $k$. Say that an agent $j$ **enters** the trading process with the first step where some agent $j' \geq j$ gets matched, so $j$ enters at Step $k^*$ where $k^*$ is the minimal $k$ with $i^k \geq j$. If agent $j$ enters at Step $k$, then $\nu^{k'}(i) = i$ holds for all $k' \leq k$, $i \geq j$: no agent $i \geq j$ moves shift at any Step $k'$ before $k$. Since the trading process matches infinitely many agents, there is for each agent a unique step at which he enters. Reconsider Example 1, assuming that the example only shows the preferences of the first 7 agents over the first 7 shifts. Then agents 1, 2, ..., 6 all enter at Step 1 when agent 6 is the first agent who wants to work an earlier shift. The next step matches agent 2 with shift 2. No new agent enters at this step. All choices in the trading process up to Step $k$ are based on preferences of agents that entered so far. Conversely if no agent $i > j$ entered the trading process of $C^\infty(\succsim)$ until Step $k$, then $C^\infty(\succsim_{\{1,\ldots,j\}}, \succsim'_{\{j+1,\ldots\}})$ follows the exact same trading process up to Step $k$.

The infinite crawler $C^\infty$ embeds the standard crawler $C$ for $n$ agents as follows: fix any $\succsim \in \hat{\Omega}^l$ the set of all preference profiles of $n$ agents over equally many objects. Define $\succsim' \in \hat{\Omega}^n$ such that $\succsim$ is the restriction of $\succsim'$ to the first $n$ agents and objects and such that any $i > n$ only finds his own endowment acceptable. Then we have $C(\succsim)(i) = C^\infty(\succsim')(i)$ for each $i \in \{1, \ldots, n\}$. Conversely if there exists some $j \in \mathbb{N}$ such that each agent $i > j$ only finds his own object acceptable according to $\succsim \in \hat{\Omega}^T$, then $C^\infty(\succsim)(i) = C(\overline{\succsim})(i)$ holds for each $i \leq j$ where $\overline{\succsim}$ is the restriction of $\succsim$ to the agents and shifts

[7] To obtain a more concise definition, one could disregard the condition that $N^{k+1} = \emptyset$ in Updating, since it is not possible to match all agents in finitely many steps. One could replace Updating with Updating$^\infty$: Let $\nu^{k+1}(i) = \nu^k(i)$ for each $i \in N^{k+1} \colon = N^k \setminus N^k$ for whom $\nu^{k+1}(i)$ is not yet defined. Index all agents in $N^{k+1}$ with respect to $\nu^{k+1}$, and go to Step $k+1$.

$\{1, \ldots, j\}$

**Theorem 5** *The infinite crawler $C^\infty : \hat{\Omega}^T \to \mathcal{M}$ is a well defined, efficient, individually rational, and practicable mechanism that can be implemented in obviously dominant strategies.*

**Proof** Fix an arbitrary profile $\succsim \in \hat{\Omega}^T$. To see that each agent must get matched by the trading process suppose not and say that $i^*$ is the minimal agent who does not get matched. Consider a Step $k^*$ at which all agents $i < i^*$ have been matched. By the definition of the crawling process the smallest unmatched agent at some step must own the earliest unmatched shift at that step, so $i^* = i_1$ holds at any Step $k$ later than $k^*$. For $i^*$ to remain unmatched, any Step $k > k^*$ must (by Screening$^\infty$) match some agent $i \leq \nu^k(i_1) + T$. Since $\succsim \in \hat{\Omega}^T$ and since $i_1$ only crawls at some Step $k$ if he prefers $\nu^k(i_1)$ to his preceding temporary match, we have $\nu^k(i_1) \leq i_1 + T$. In sum we obtain that any Step $k > k^*$ matches an agent $i \leq i_1 + 2T = i^* + 2T$. Since there are only finitely many agents $i \leq i^* + 2T$ there must come a step where no agent is matched - a contradiction.

The proof that $C$ is individually rational (see Theorem 1) applies unchanged to the case of $C^\infty$.

To see that $C^\infty$ is practicable, I fix an arbitrary $i, \succsim, \succsim'$ and show that $C^\infty(\succsim)(i) = C^\infty(\succsim_{\{1,\ldots,i+2T\}}, \succsim'_{\{i+2T+1,\ldots\}})(i)$ holds for this $i, \succsim, \succsim'$. Say that agent $i$ is matched at Step $k$ of the trading process $C^\infty(\succsim)$. As long as $i$ remains unmatched in any trading process, an agent $j$ can only enter the trading process if $j \leq \nu^k(i_1)+T \leq \nu^k(i)+T \leq i+2T$. So no agent $j > i+2T$ enters the trading process of $C^\infty(\succsim)$ before $i$ gets matched. So according to $\succsim$ is identical to $(\succsim_{\{1,\ldots,i+2T\}}, \succsim'_{\{i+2T+1,\ldots\}})$ the preferences of all agents who entered by Step $k$ are identical. Since all matches made up to Step $k$ may only depend on the preferences of the agents who entered so far and since $i$ is matched at Step $k$, we obtain $C^\infty(\succsim)(i) = C^\infty(\succsim_{\{1,\ldots,i+2T\}}, \succsim'_{\{i+2T+1,\ldots\}})(i)$.

To see that $C^\infty(\succsim)$ is efficient note that the agent $i^k$ matched at Step $k$ obtains his most preferred shift conditioning on all preceding matches if $i^k$ is selected via the first clause of Screening$^\infty$. If not then any individually rational matching $\mu$ that is consistent with the matches achieved in all preceding steps has $\mu(i^k) = \nu^k(i^k)$. In either case agent $i^k$ is matched with his most preferred shift among all remaining shifts that he could be matched with in some individually rational matching. So $C^\infty(\succsim)$ is efficient.

To see that $C^\infty$ can be implemented in obviously dominant strategies consider the extensive form mechanism $C'$ that that subdivides each Step $k$ of $C^\infty$ into separate choices $(k,t)$ for all agents who move during this step. Each history of this extensive form mechanism is labelled with a tuple $(k,t)$ where $k$ corresponds to the Step $k$ of $C^\infty$ and $t$ is the index of the agent $i_t$ who chooses at the current the history $(k,t)$. The choice set at $(k,t)$ is $A(k,t) \colon = \{c, 1, \cdots, t\}$. If agent $i_t$ chooses $c$ and if $i_{t+1} \leq \nu^k(1_1) + T$ go to $(k, t+1)$. If agent $i_t$ chooses $c$ and if $i_{t+1} > \nu^k(1_1) + T$ follow the second clause of Screening$^\infty$ and go to the history labelled $(k+1, 1)$. If $i_t$ chooses a number $r \in \{1, \ldots, t\}$, let $t = t^*$ and follow Matching, Crawling, and Updating as in the definition of $C^\infty$. At the end of Updating go to the history labelled $(k+1, 1)$ (as opposed to so Step $k+1$). The proof that this extensive form mechanism indeed implements $C^\infty$ tightly follows the arguments proof of Theorem 3 and is therefore omitted. $\square$

Theorem 5 sharply contrasts with the non-existence result for the generic shift exchange problems. Efficient and individually rational mechanisms for the generic domain may not even be able to match a single agent using finitely many preferences. The infinite crawler is, however, efficient, individually rational and also practicable: for every single agent $i$ it suffices to elicit the first $i + 2T$ preferences to match $i$. The infinite crawler is moreover implementable in obviously dominant strategies.

Combing all preceding results one can prove a yet stronger claim that applies to shift exchange problems with single-peaked preferences that allow for indifferences. Replacing Screening with Screening$^\infty$ in the circle crawler $C^\sim$ one obtains the infinite circle crawler which has all the desirable properties of the other crawlers defined in this paper. The infinite circle crawler is a practicable, efficient, and individually rational matching mechanism for shift exchange problems with single-peaked preferences (with indifferences) that can be implemented in obviously dominant strategies. The proof of this result is omitted since it tightly follows the proofs for Theorems 1, 3, 4, and 7.

# 8 Conclusion

The crawler is a new efficient, strategyproof and individually rational matching mechanism on the domain of single-peaked preferences. On this domain it has two main advantages over Gale's top trading cycles. It can be implemented in obviously dominant strategies following Li [9]. It also can be used for shift exchange problems following Bade [5] where some agents have to be matched before all preferences can be elicited.

While I have shown that the crawler is implementable in obviously dominant strategies, I have not provided a characterization of all efficient and individually rational mechanisms for the domain of single-peaked preferences $\hat{\Omega}^l$ that are implementable in obviously dominant strategies. Clearly the dual crawler that starts the screening process with the largest house is also implementable in obviously dominant strategies. When there are only 3 agents then even Gale's top trading cylces is implementable in obviously dominant strategies. I conjecture that any social choice function on the matching domain with single-peaked preferences that is implementable in obviously dominant strategies is a combination of these three mechanisms.

Many of the studies on adapting Gale's top trading cycles to the case of indifferences focus on computational complexity. In this vein one might consider the set of matching problems $\hat{\Omega}^l$ as $n$ grows large. How hard is it to calculate the outcome $C^{\sim}(\succsim)$ as $n$ grows large? Even abstracting away from indifferences one could compare the calculation of $G$ and $C$ in terms of computational complexity.

Alternatively one could investigate how the crawler fares under the assumption that agents need to endogenously acquire information about the objects to figure out which one would be best for them. Comparing the crawler and Gale's top trading cycles it appears at a first glance as if the crawler requires less information about the agents preferences than Gale's top trading cycles. In Gale's top trading cycles all agents have to immediately declare which is their most preferred house. Conversely the agents occupying larger houses initially only observe how the set of available houses dwindles. Once it is their turn to enter they only need to consider a restricted set of options. The crawler may therefore yield better results than Gale's top trading cycles if agents can only devote limited resources to learning their preferences.

Finally in the context of shift exchange problems it would be interesting to see whether any efficient, individually rational, and strategyproof mechanism eventually coincides with the infinite crawler $C^\infty$. It is clearly also efficient, strategy proof and individually rational to use Gale's top trading cycles for a first batch of agents to then proceed with the rules for $C^\infty$. However it is not clear how to continue an ongoing matching problem with single-peaked preferences in a efficient, individually rational and strategy proof way that differs from the the rules of $C^\infty$.

# 9 Appendix

In Section 5 I claimed that Gale's top trading cycles is also on $\hat{\Omega}^l$, the domain of linear single-peaked preferences not implementable in obviously dominant strategies.[8] The upcoming proof of this claim crucially relies on Bade and Gonczarowski's [6] gradual revelation principle. In their Theorem 1 Bade and Gonczarowsi [6] show that a social choice function is implementable in obviously dominant strategies if and only if it is implementable by an obviously incentive compatible gradual revelation mechanism. A gradual revelation mechanism is an extensive form mechanism where each action corresponds to a set of preferences. There are no simultaneous moves, singleton choice sets, or directly consecutive moves for the same agent. An agent's strategy is truthful if he, wherever possible, chooses an action that corresponds to the set of preferences containing his true preference.

Formally, the player function maps each non-terminal history $h$ to a single player $P(h)$ who chooses from $A(h)$ with $\mid A(h) \mid> 1$. Moreover $P(h) \neq P(h, a)$ for any $a \in A(h)$. Each history $h$ is associated with a set of

[8]Li [9] already showed that Gale's top trading cycles on the domain $\Omega^l$ is not implementable in obviously dominant strategies.

preferences $\Omega_i(h) \subset \Omega_i$, with $\Omega(h) := \Omega_1(h) \times \cdots \times \Omega_n(h)$. The mechanism starts with $\Omega(\emptyset) = \Omega$. If $P(h) = i$, then $\{\Omega_i(h,a)\}_{a \in A(h)}$ partitions $\Omega_i(h)$, if not then $\Omega_i(h) = \Omega_i(h,a)$ holds for each $a \in A(h)$. A strategy $S_i$ in a gradual revelation mechanism is truthful if at each history $h$ agent $i := P(h)$ with preference $\succsim_i \in \Omega_i(h)$ chooses the action $a \in A(h)$ with $\succsim_i \in \Omega_i(h,a)$.[9] If all agents follow the truthful strategy then $\Omega(h)$ describes everything that the agents revealed about their preferences up to history $h$. Finally the gradual revelation mechanism $M$ is obviously incentive compatible if truthtelling is obviously dominant for each agent.

**Theorem 6** *Let there be at least 4 agents. Then Gale's top trading cycles $G : \hat{\Omega}^l \to \mathcal{M}$ cannot be implemented in obviously dominant strategies.*

**Proof** Suppose $M$ was an obviously incentive compatible mechanism that implements $G$ for $N = \{1, \ldots, 4\}$. Let $\Omega^*$ be the set of all preference profiles for which agents 1 and 2 want to move to larger houses while agents 3 and 4 want to move to smaller houses. So $\Omega^*_i := \{\succsim_i \in \hat{\Omega}^l \mid j \succsim_i H \text{ implies } j > i\}$ for $i = 1,2$ and $\Omega^*_i := \{\succsim_i \in \hat{\Omega}^l \mid j \succsim_i H \text{ implies } j < i\}$ for $i = 3,4$.

**Claim (*)** For any history $h$ with $\Omega^* \subset \Omega(h)$, there exists an action $a \in A(h)$ such that $\Omega^* \subset \Omega(h,a)$: fixing any history on the path of the truthtelling strategy for any $\succsim \in \Omega^*$, where no agent has revealed more than the direction in which he wants to move, the agent moving at the present history will not reveal any more than the direction in which he wants to move.

To see claim (*) fix a history $h$ with $\Omega^* \subset \Omega(h)$, say that $i = P(h) \in \{1,2\}$ and let $j$ be the other agent in $\{1,2\}$. Suppose there existed two actions $a, a' \in A(h)$ with $\succsim_i \in \Omega_i(h,a)$ and $\succsim'_i \in \Omega_i(h,a')$ for some for some $\succsim_i, \succsim'_i \in \Omega^*_i$ with $i+1 \succsim_i \{1,2,3,4\}$. For $\succsim \in \Omega^*$ with $4 \succsim_j \{1,2,3,4\}$, $j \succsim_3 \{1,2,3,4\}$, and $3 \succsim_4 \{1,2,3,4\}$ we have $G(\succsim)(i) = i$. On the other hand, $G(\succsim')(i) = i+1$ holds for $\succsim' \in \Omega^*$ with $j+1 \succsim'_j \{1,2,3,4\}$ and $1 \succsim'_3 \{1,2,3,4\}$. Since $\succsim_i \in \Omega^*_i$ and $i \in \{1,2\}$, $G(\succsim') = i+1$ is strictly $\succsim_i$-preferred to $G(\succsim)(i) = i$, and

[9] This definition does not determine the agents choice at a history $h$ with $\succsim_i \notin \Omega_i(h)$ and an agent may have multiple different truthful strategies. Since such "off the path"-choices are irrelevant for the present analysis and I speak of "the" truthful strategy.

the action $a$ is not an obviously dominant choice for agent $i$ with preference $\succsim_i$ at $h$. There must be an action $a \in A(h)$ with $\Omega_i^* \subset \Omega_i(h,a)$. Since $\Omega_{i'}(h,a) = \Omega_{i'}(h)$ holds for all $i' \neq P(h)$ we obtain $\Omega^* \subset \Omega(h,a)$. Claim (*) then holds since the same arguments apply mutatis mutandis to the case of $P(h) \in \{3,4\}$.

Now fix any $\succsim^* \in \Omega^*$. Since no agent has revealed anything before the game starts we trivially have $\Omega^* \subset \hat{\Omega}^l = \Omega(\emptyset)$. Since $\succsim^* \in \Omega^*$ the inductive application of claim (*) to all agents choices yields that $\Omega^* \subset \Omega(h)$ holds for each history $h$ with $\succsim \in \Omega(h)$. We in particular have for $h^*$, the unique terminal history of on the path of the truhtelling behavior associated with $\succsim^*$ that $\Omega^* \subset \Omega(h^*)$, and no agent reveals more than the direction he would like to move in. But $G$ is not constant on $\Omega^*$: we for example have that $G(\succsim) \neq G(\succsim')$ holds for the two profiles $\succsim, \succsim' \in \Omega^*$ constructed in the preceding paragraph. So the terminal history $h^*$ cannot be mapped to a unique outcome - a contradiction. □

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