Shift Exchange problems

Sophie Bade*

June 8, 2017

Abstract

Consider a matching problem where agents and objects arrive over time. Assume matchmaking has to start before all agents' preferences become known: the decision on who works which shift in the current month cannot be based on the preferences of agents who are set to work next year. To capture this ongoing nature I model the sets of agents and shifts as countably infinite. Each each agent must work within a finite time window around the shift he was endowed with. Shift exchange problems are ongoing versions of housing markets as defined by Shapley and Scarf [9] and much of the theory for housing markets transfers. However any Pareto optimal, strategy proof and individually rational mechanism must elicit infinitely many preferences to match any finite subset of agents. To overcome this flaw I suggest two alternative individually rational mechanisms with reasonably weakened welfare and incentive properties.

1 Introduction

A shift exchange problem starts with a schedule that sets each agent to work exactly one shift. These initial endowments may be inconvenient for

*sophie.bade@rhul.ac.uk, 44 (0) 1784 437520, Royal Holloway, University of London and Max Planck Institut for research on Collective Goods, Bonn. Funding through the ARCHES prize is gratefully acknowledged. I would like to thank the seminar audience at Haifa University, Guillaume Haeringer, Annika Johnson, Bettina Klaus, Michael Mandler, Herve Moulin, Marcus Pivato, and Karol Swzagrzak.

the workers and there may be room for Pareto improving rearrangements of work schedules. To reassign shifts in the immediate future we cannot wait to learn the preferences of agents who are set to work in the distant future. The decision which shift an agent works may only depend on the preferences that are known at the time of this decision.

For a typical shift exchange problem, think of the Park Slope Food Coop in Brooklyn. The Park Slope Food Coop is a large supermarket run by its more than 16,500 members each of whom works once every four weeks in exchange for a $20-40\%$ savings on groceries. Often a member is assigned a shift that is inconvenient - or even impossible - to work for that member. Such members may swap shifts.[1] Exchange problems with unlimited sets of agents and objects abound: Flight attendants and bus drivers are given schedules that may not always suit them. MRI appointments and time slots to use large scale telescopes may require rearrangements. The kidney matching problem is another case in point: The pool of patients with willing and incompatible donors is constantly changing.

Shift exchange problems are similar to housing markets as defined by Shapley and Scarf [9]: In both types of problems each agent is endowed with exactly one object. Any agent's preference over matchings only depends on the object he is matched with. The crucial difference is that it is not possible to simultaneously elicit all agents preferences in a shift exchange problem. To capture the ongoing nature of shift exchange problems I model the set of agents and their shifts as the set of natural numbers $\mathbb{N}$. Initially each agent $i \in \mathbb{N}$ is endowed with shift i, which must be worked on date i. There is a fixed number T, so that any agent i has to work during the time window $\{i-T, \ldots, i, \ldots i+T\}$.

How should we organize shift exchanges? A good mechanism should be Pareto optimal so that it never maps any profile of preferences to an outcome for which there exists an alternative matching μ strictly preferred by some agents and weakly by all. It should be individually rational in the sense

[1] In practice the Parks Slope Food Coop until recently used a black-board for members to organize shifts swaps on an ad hoc basis. I thank Guillaume Haeringer - a matching theorist and Park Slope Food Coop member - for introducing me to the shift swapping problem of the Park Slope Food Coop.

that no agent strictly prefers his endowment to his match. Finally no agent should have an incentive to misrepresent his preference.

Exactly one mechanism satisfies these three criteria on the domain of housing problems: Shapley and Scarf [9], Roth [8], and Ma [7] showed that a mechanism is Pareto optimal, strategy proof, and individually rational if and only if it is Gale's top trading cycles mechanism. In Gale's top trading cycles each agent points to the owner of his most preferred house. All agents in cycles are matched with the house they point to. The pointing is then repeated with all remaining agents and houses. The process terminates once a matching is found.

Since shift exchange problems embed housing markets, any result on Pareto optimal, strategyproof and individually rational mechanisms for shift exchange problems must extend the above result for housing markets. This extension is, however, not straightforward as the application of Gale's trading cycles process to shift exchange problems does not define a mechanism. To see this consider a profile where each each agent prefers to work later than the shift he was originally assigned to. There is not a single trading cycle and no agent is matched via Gale's trading process.

Theorem 1 shows that Gale's top trading cycles process is also on the domain of shift exchange problems strongly linked with Pareto optimality, strategy proofness, and individual rationality. Any mechanism M that uses Gale's trading process whenever possible satisfies some intermediate welfare and incentive properties: if for some profile of preferences the outcome of such a mechanism M is Pareto dominated, then the dominating matching and the outcome of M must coincide for all agents matched via Gale's trading process. If an agent in a shift exchange problem is matched via Gale's trading process, then he cannot improve his shift by misstating his preferences. Conversely, any good mechanism for a shift exchange problem must sequentially match top trading cycles whenever possible. Theorem 2 uses Theorem 1 to characterize all good mechanisms, for the case that agents never find it acceptable to anticipate or postpone their shift by more than a day.

There is no bound on the set of preferences that must be elicited for a good mechanism to start matching agents. Theorem 3 shows that for any number K and any good mechanism there is a profile of preferences, such that more than K preferences have to be elicited to match at least one of the first K agents. When it is acceptable anticipate or postpone shifts by more than a day, this problem becomes more severe. In that case there is not even a Pareto optimal and individually rational mechanism that always determines at least one match in finite time. To show this I construct a profile with a unique Pareto optimal and individually rational matching μ^*, where each odd agent works two days later, agent 2 works shift 1 and all other even agents work two days earlier. To obtain this matching μ^* all agents have to engage in an infinite chain swap: if each agent points to the agent whose shift he obtains according to μ^*, the resulting pointing chain links any two agents. Since this chain swap fails if a single agent finds only his own shift acceptable, the mechanism has to elicit all preferences to match even a single agent i with $\mu^*(i)$. These results do not depend on there being countably many shifts and agents, they also apply to house matching markets where agents only find houses within (short) windows around their endowments acceptable.

Given that good mechanisms cannot guarantee to make any matching decisions in finite time, we need to find mechanisms that satisfy reasonably weakened incentive and welfare properties. Taking Gale's top trading process as my starting point Section 7 proposes the closed and open intervals mechanism as two such compromises. Either mechanism partitions all agents into intervals of equal length. The *closed intervals mechanism* then applies Gale's top trading cycles to each of these "closed" intervals. This mechanism is - just like Gale's top trading cycles - strategyproof and individually rational. It may, however, fail to reach the Pareto frontier: If we only allow shift exchanges within the same calendar month of the Park Slope Food Coop's problem, we rule out some potentially Pareto improving swaps. Two members who are set of work on January 30th and February 2nd and most like each other's shifts cannot trade.

The *open intervals mechanism* does not rule out any potentially individually rational swaps. The open intervals mechanism starts out like the closed intervals mechanism: In a first round it applies Gale's trading process to the first interval of agents (and their shifts) to obtain preliminary matches. While the closed intervals mechanism immediately finalizes all these matches, the open one only finalizes matches of the agents or shifts whose trading window does not reach into the next interval. The agents whose matches are not yet finalized move on to the next round with their preliminary matches as their new endowments. The next round (and any following round) applies Gale's top trading cycles to the agents in the next interval and to the agents whose matches where not finalized in the preceding round.

Proposition 1 shows that the open intervals mechanism outperforms the the closed one in terms of efficiency. Theorem 4 shows that the open intervals mechanism finds Pareto optima in a restricted class of possible matchings and that truhtelling is a dominant strategy for a large portion of all agents. Corollary 1 shows that the open intervals mechanism is Pareto optimal in a very strong sense when each agent's trading window is very short. Assuming that an agent only misrepresents his preferences if he strictly benefits from such a misrepresentation, the open intervals mechanism maps any profile of announced preferences to a matching that is Pareto optimal at the true underlying preferences. An Appendix contains all proofs that do not appear in the main text.

Shift exchange problems fit into a growing literature on dynamic matching problems with infinite streams of agents that cannot wait infinitely long to get matched. Similarly to the matching markets studied in Unver [11], Akbarpour, Li, and Oveis Gharan [1], Anderson, Ashlagi, Gamarnik, and Kanoria [2] shift exchange problems are unilateral matching problems where each agent is initially endowed with exactly one object. The cited papers make two assumptions on preferences that sharply differ from the present assumptions: Unver [11], Akbarpour, Li, and Oveis Gharan [1], Anderson, Ashlagi, Gamarnik, and Kanoria [2] all assume dichotomous preferences that are objectively certifiable. The models of Unver [11], Akbarpour, Li, and Oveis Gharan [1], Anderson, Ashlagi, Gamarnik, and Kanoria [2] are all motivated by the kidney exchange problem where we can reasonably assume that agents are indifferent between all compatible kidneys and that compatibility is objective. Conversely it would be a stretch to assume that agents are indifferent between all shifts they prefer to their endowment. Preferences over shifts are moreover subjective.

The assumption of objective dichotomous preferences has two main benefits: Unver [11], Akbarpour, Li, and Oveis Gharan [1], Anderson, Ashlagi, Gamarnik, and Kanoria [2] can, on the one hand, safely ignore the incentives for the truthful revelation of preferences. The size of the set of all agents not matched with their endowment is, on the other hand, a straightforward measure of welfare. Unver [11], Akbarpour, Li, and Oveis Gharan [1], Anderson, Ashlagi, Gamarnik, and Kanoria [2] all focus the trade off between two measures of welfare: the size of the set of matched agents and the time agents spend waiting to get matched. To evaluate this trade-off Unver [11], Akbarpour, Li, and Oveis Gharan [1], Anderson, Ashlagi, Gamarnik, and Kanoria [2] assume explicit random processes that govern the arrival and/or departure of agents and compatibilities.

Leshno [6] and Bloch and Cantala [5] both study unilateral matching markets without initial endowment, such as market for social housing. Motivated by adoption Baccara, Lee, and Yariv [3] study a bilateral matching market in which prospective parents and children of two possible types stochastically enter the adoption pool. Leshno [6], Bloch and Cantala [5], and Baccara, Lee, and Yariv [3] share the assumption of dichotomous preferences with the papers cited above. Without endowments, though, there is the danger of mismatches. There is a tradeoff between immediately getting an object of the less preferred type and waiting for the more preferred type. The present paper shares the concern with incentives with Leshno [6], Bloch and Cantala [5], and Baccara, Lee, and Yariv [3]. However, whether or not agents have initial endowments is crucial: the results in Leshno [6], Bloch and Cantala [5], and Baccara, Lee, and Yariv [3] all focus on different ways to organize waiting lists for objects.

I am aware of two papers on dynamic matching that do not model preferences as dichotomous: Bloch and Cantala [5] and Schummer [10]. Both consider unilateral matching problems without initial endowments. Since allowing for all linear orders on objects greatly complicates any analysis of a dynamic matching problem, both Bloch and Cantala [5] as well as Schummer [10] starkly simplify other aspects of the matching problem. Bloch and Cantala's [5] section on general linear orders only considers the shortest possible waiting lists. Conversely Schummer [10] assumes that all agents have the same preferences over objects.

2 Definitions

There are countably many agents and shifts both modelled as $\mathbb{N}$. Initially agent i **owns** shift i which must be worked on day i. Agent i's preference $\succ_i$ over shifts is a linear order[2] on $\mathbb{N}$. Agent i finds shift j **acceptable** if $j \succsim_i i$ (implying that each agent finds his own shift acceptable). There exists some fixed number T such that agent i never finds working more than T days before or after his own shift acceptable, $\mid j - i \mid > T$ implies $i \succ_i j$ for all $i, j \in \mathbb{N}$.

The notation $\succ_i$: j, k, i means that j and k respectively are the $\succ_i$-best and second best shifts and that no shift $j' \notin \{i, j, k\}$ is $\succ_i$-acceptable. The preference $\succ_i$: i where agent i only finds his own shift acceptable is denoted $\succ_i^e$. The preference $\overline{\succ}_i$ is a restriction of $\succ_i$ if $\overline{\succ}_i$ is defined on some subset $S \subset \mathbb{N}$ such that $j \overline{\succsim}_i k$ implies $j \succsim_i k$ for all $j, k \in S$. The set S to which $\succ_i$ is restricted to will always be clear from the context, the notation $\overline{\succ}_i$ does therefore not keep track of this set. A profile of preferences $\succ = (\succ_i)_{i \in \mathbb{N}}$ sums up all agents preferences. The preferences of a subset $S \subset \mathbb{N}$ of agents are denoted $\succ_S$. The domain of agent i's preferences is Ω_i^T. The domain of all preferences is $\Omega^T := \times_{i \in \mathbb{N}} \Omega_i^T$. An economy is defined by the profile $\succ \in \Omega^T$ of all agents preferences over all shifts.

To illustrate shift exchange problems I throughout represent agents and shifts as dots on a horizontal line. If a solid arrow points from i to j then agent i would most like to work the shift j. Dotted arrows stand in for second most preferred shifts. Figure 1 represents the preferences of the first five agents in the profile $\succ$ with $\succ_1$: $2, 1$, $\succ_2$: 2, $\succ_3$: $6, 2$, $\succ_4$: $5, 4$, and $\succ_5$: $2, 4$.

A **submatching** for a set of agents $S \subset \mathbb{N}$ is a permutation $\nu : S \to S$, and $Dom(\nu)$ is the domain of the submatching. Agent $i \in Dom(\nu)$ is matched with shift $\nu(i)$ by ν. The submatching that matches no one is denoted $\emptyset$. A submatching ν is a **cycle** if for each $i, j \in Dom(\nu)$ there exists an $n \in \mathbb{N}$ such that $i = \nu^n(j)$. If ν is a cycle then $Dom(\nu)$ is finite. Cycles are alternatively denoted as vectors $(i_1, \ldots, i_n)$ with the understanding that $\nu(i_t) = i_{t+1}$ for all $t < n$ and $\nu(i_n) = i_1$. So agents 5 and 8 swap shifts in the cycle $(5, 8)$, conversely in the cycle $(1, 3, 5)$ agent $i \in \{1, 3\}$ works shift $i + 2$

[2]So $\succ_i$ is complete transitive and antisymmetric.

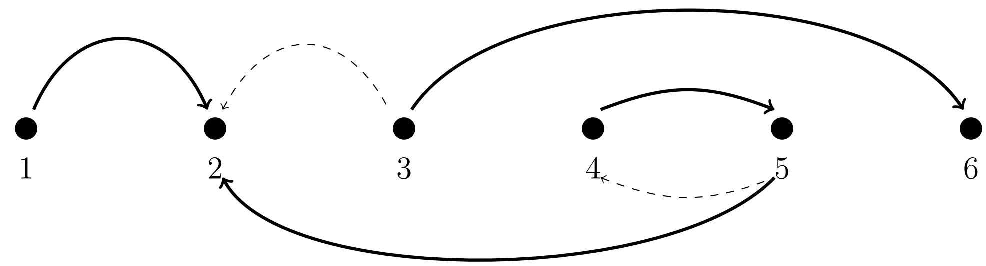

Figure 1: First and second best shifts

while agent 5 works shift 1.

A **matching** $\mu : \mathbb{N} \to \mathbb{N}$ is a submatching with $Dom(\mu) = \mathbb{N}$. So for μ to qualify as a matching, each shift has to be taken, and each agent has to work a shift: schemes according to which some agent does not work any shift (such as $\mu(i) = i - 1$ for all $i > 1$) are ruled out. The identity function $id : \mathbb{N} \to \mathbb{N}$ is the matching where each agent works his own shift ($id(i) = i$ for all $i \in \mathbb{N}$). A matching μ is **consistent with a** submatching $\nu : S \to S$ if $\mu(i) = \nu(i)$ holds for all $i \in S$. In this case I write $\nu \subset \mu$. The set of all matchings and submatchings respectively are $\mathcal{M}$ and $\overline{\mathcal{M}}$. Each agent only cares about the shift he works, so i prefers matching μ to μ' if and only if $\mu(i) \succsim_i \mu'(i)$.

A matching μ is **Pareto optimal** at a profile $\succ$ if there is no alternative matching $\mu' \neq \mu$ with $\mu'(i) \succsim_i \mu(i)$ for all $i \in \mathbb{N}$. A submatching $\nu : S \to S$ is **globally Pareto optimal** if there exists no matching μ such that $\mu(i) \succsim_i \nu(i)$ for all $i \in S$ and $\mu(i) \succ_i \nu(i)$ for some $i \in S$. Conversely the submatching is only **locally Pareto optimal** if there exists no submatching $\nu' \neq \nu$ with $Dom(\nu) = Dom(\nu')$ and $\nu'(i) \succsim_i \nu(i)$ for all $i \in S$. For matchings the three

Pareto properties coincide.[3] A submatching $\nu : S \to S$ is **individually rational** at $\succ$ if each agent $i \in S$ finds $\nu(i)$ $\succ_i$-acceptable.

A (direct revelation) **rule** $R : \Omega^T \to \overline{\mathcal{M}}$ maps each profile $\succ \in \Omega^T$ to a submatching $R(\succ) \in \overline{\mathcal{M}}$; agent $i \in Dom(R(\succ))$ is matched with $R(\succ)(i)$. If R maps each $\succ \in \Omega^T$ to a matching then $R : \Omega^T \to \mathcal{M}$ is a (direct revelation) **mechanism**. The rule $R : \Omega^T \to \overline{\mathcal{M}}$ is consistent with the mechanism $M : \Omega^T \to \mathcal{M}$ if $R(\succ)$ is consistent with $M(\succ)$ for each $\succ \in \Omega^T$. Agent i with preference $\succ_i$ behaves truthfully in a mechanism (or rule) if he reports his true preference $\succ_i$.

Mechanisms and rules are respectively Pareto optimal, globally Pareto optimal, locally Pareto optimal and individually rational if they map each profile of preferences to Pareto optimal, globally Pareto optimal, locally Pareto optimal or individually rational (sub-)matchings. A rule $R : \Omega^T \to \overline{\mathcal{M}}$ is **strategyproof** if there does not exist any mechanism $M : \Omega^T \to \mathcal{M}$, profile $\succ$, agent i, and deviation such that $i \in Dom(R(\succ))$, $M(\succ_i', \succ_{-i})(i) \succ_i R(\succ)(i)$, and R consistent with M. So R is strategyproof if R is not consistent with any mechanism M according to which some agent finds it for some profile of preferences beneficial to declare a preference other than his own. When R is a mechanism then the present definition reduces to the standard definition of strategyproofness. A mechanism is strategyproof if truthtelling is a dominant strategy for each agent. A Pareto optimal, strategyproof, and individually rational mechanism is **good**.

3 Housing Markets

In a housing market, as defined by Shapley and Scarf [9] there are finitely many agents and the preferences of these agents can be elicited simultane-

[3]Global Pareto optimality implies local Pareto optimality. Conversely, even if for some submatching μ and each $i \in \mathbb{N}$ there exists a locally Pareto optimal submatching ν with $\nu(i) = \mu(i)$, μ need not be globally Pareto optimal. Consider $\succ$ with $\succ_1$: $3, 2, 1$, $\succ_2$: $4, 1, 2$, $\succ_3$: $1, 3$, $\succ_4$: $2, 4$, and $\succ_i$: i for all $i > 4$. Then the cycles $(1, 2)$ and (i) for all $i \in \mathbb{N} \setminus \{1, 2\}$ are all locally Pareto optimal at $\succ$. However for μ to be (globally) Pareto optimal at $\succ$ it has to be consistent with the cycles $(1, 3)$ and $(2, 4)$. If for some submatching μ and each $i \in \mathbb{N}$ there exists a globally Pareto optimal submatching ν with $\nu(i) = \mu(i)$, then μ is globally Pareto optimal.

ously. Otherwise shift exchange problems are identical to housing markets: Each agent is endowed with exactly one object (a house or a shift). Matchings are one-to-one and onto mappings between agents and objects. The agents' preferences over matchings are derived from their preferences over objects which are in either case linear orders. The restriction that each agent i finds all shifts outside $\{i-T, \ldots i+T\}$ unacceptable has no bite in housing markets if we let T be large enough.

A housing market is defined by the (finite) sets of agents N, and houses H, the initial endowment $\tau : N \to H$ (a bijection), and by the profile $\succ$ of all agents' (linear) preferences over all houses H. Calling the domains of preferences and matchings $\hat{\Omega}$ and $\hat{\mathcal{M}}$, **Gale's top trading cycles** $G : \hat{\Omega} \to \hat{\mathcal{M}}$ is defined via the following algorithm that finds a matching $G(\succ)$ for each $\succ \in \hat{\Omega}$: Initialize the algorithm so that all agents and houses remain unmatched. Go to Step 1.

Step k: Each unmatched agent points to the owner of his most preferred unmatched house. Match all agents in cycles to the houses they point to. Terminate if all agents are matched. Otherwise, go on to Step $k+1$.

Shapley and Scarf [9], Roth [8], and Ma [7] showed that a mechanism $M : \hat{\Omega} \to \hat{\mathcal{M}}$ is good if and only if it is Gale's top trading cycles mechanism.

4 Embedding Gale's top trading cycles

To see that Gale's trading process applied to a shift exchange problem need not yield a matching, reconsider the profile where each agent i would like to work tomorrow (shift $i+1$). In this case Gale's trading process does not make a single match. Gale's trading process therefore only defines a rule for shift exchange problems. Taking this restriction into account, Theorem 1 shows how to extend Shapely and Scarf's [9], Roth's [8], and Ma's [7] results on the existence and uniqueness of good house matching mechanisms to shift exchange problems. Any good mechanism for shift exchange problems must use Gale's trading process wherever possible. Conversely a mechanism that combines Gale's trading process with a locally Pareto optimal and individually rational rule that does not interfere with strategy proofness is good.

The **permacycles** rule $PC : \Omega^T \to \overline{\mathcal{M}}$ adapts Gale's trading process to shift exchange problems. To calculate a submatching $PC(\succ)$ for any $\succ \in \Omega^T$ initialize the the process so that all agents and shifts remain unmatched. Go to Step 1.

Step k: Each unmatched agent points to the owner of his most preferred unmatched shift. Match all agents in cycles to the shifts they point to. Terminate if no cycles formed or if all agents are matched. Otherwise, go on to Step $k+1$.

To see that PC is a well-defined rule fix a Step k. Since any preference $\succ_i \in \Omega_i^T$ is a linear order with no more than $2T+1$ $\succ_i$-acceptable shifts, any unmatched agent at Step k has a most preferred unmatched shift. Moreover, if some cycles form at Step k, then some unmatched agents are bijectively matched to some unmatched shifts. So $PC(\succ)$ is indeed a submatching for each $\succ \in \Omega^T$.

The set of all agents not matched under permacycles at some profile $\succ$ is $Q(\succ)$, so $Q(\succ)$ is empty if and only if $PC(\succ)$ is a matching. To see that PC is not a mechanism consider the profile $\succ$ where each agent i would most like to work shift $i+1$. At Step 1 there is not a single pointing cycle and $PC(\succ) = \emptyset$. The descriptions of PC and Gale's top trading cycles differ only in one respect. While PC requires the trading process to terminate at a round without a single pointing cycle, Gale's top trading cycle does not foresee this case. The number of agents is crucial for this difference: with finitely many agents any step has a cycle. Permacycles therefore reduces to Gale's top trading cycles if we replace the set of agents $\mathbb{N}$ with any finite set N. Example 1 shows that both finitely and infinitely long trading process may but need not result in matchings.

Example 1 1. With $\succ_1$: $2, 1$ and $\succ_i$: $i-1, i+1, i$ for all $i > 1$, illustrated in Figure 2 the trading process does not terminate and finds a matching. The only cycle that forms at Step 1 is $(1, 2)$. At each following Step k the cycle $(2k-1, 2k)$ forms. So $(i, i+1) \subset PC(\succ)$ holds for all odd $i \in \mathbb{N}$.

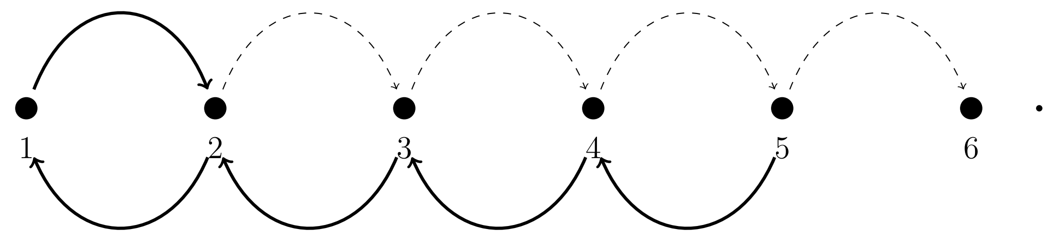

Figure 2: PC goes on forever, all agents are matched

2. The matching $PC(\succ^e) = id$ is found with the first step of the trading process where each agent points to his own shift.

3. With $\succ_2$: $1, 2$, $\succ_i$: $i+3, i$ for each $i \in 3\mathbb{N}$, $\succ_i$: $i+1, i$ for each $i \in 3\mathbb{N}+1$ and $\succ_i$: $i-3, i-1, i$ for each $i \in 3\mathbb{N}+2$, illustrated in Figure 3 the trading process neither terminates nor finds a matching.

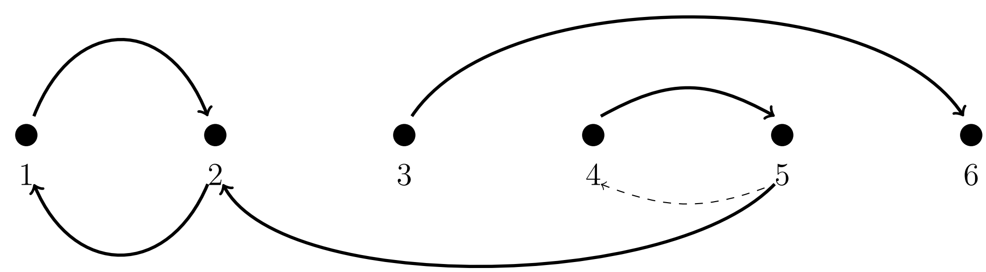

Figure 3: PC neither finds a matching nor terminates

In Step k the cycle $(3(k-1)+1, 3(k-1)+2)$ forms. All agents in $3\mathbb{N} = Q(\succ)$ remain unmatched.

4. If $\succ_i$: $i+1, i$ holds for all i the trading process immediately terminates without matching a single agent.

Just as in Gale's top trading cycles the order in which trading cycles are eliminated does not matter in permacycles. To see this consider a version of permacycles $\widetilde{PC} : \Omega^T \to \overline{\mathcal{M}}$ that sequentially matches arbitrarily chosen trading cycles to achieve a submatching where no cycles form among the unmatched. So the algorithms used to calculate $\widetilde{PC}(\succ)$ and $PC(\succ)$ are identical except that each Step of the former only matches the agents in a selection of cycles (as opposed to in all cycles). There are, moreover, no pointing cycles among $\mathbb{N} \setminus Dom(\widetilde{PC}(\succ)))$ the agents that are not matched by $\widetilde{PC}(\succ)$.[4]

Lemma 1 $\widetilde{PC}(\succ) = PC(\succ)$ *for all* $\succ \in \Omega^T$.

The strong relation between Gale's trading process and the properties of Pareto optimality, strategy proofness and individual rationality transfers from housing markets to shift exchange problems. I first show in Lemma 2 that permacycles is an individually rational, globally Pareto optimal, and strategyproof rule. Lemma 2 is then used to show in Theorem 1 that the combination of permacycles with an appropriate rule for the unmatched yields a good mechanism. Conversely any good mechanism for shift exchange problems must be consistent with permacycles.

Lemma 2 *The permacycles rule* $PC : \Omega^T \to \overline{\mathcal{M}}$ *is individually rational, globally Pareto optimal, and strategyproof.*

Proof Fix any $\succ \in \Omega^T$

Since any agent $i \in Dom(PC(\succ))$ is matched via a trading cycle, any such i weakly prefers his match $PC(\succ)(i)$ to his endowment i and PC is individually rational.

To see that $PC(\succ)$ is globally Pareto optimal, suppose not. So say there existed some matching μ such that $\mu(i) \succsim_i PC(\succ)(i)$ for all $i \in Dom(PC(\succ))$ and $\mu(j) \succsim_j PC(\succ)(j)$ for some $j \in Dom(PC(\succ))$. Say j is matched at Step k of $\widetilde{PC}(\succ)$ and assume without loss of generality that $\mu(i) = PC(\succ)(i)$

[4]If $\widetilde{PC}$ did allow pointing cycles among the unmatched, $\widetilde{PC}$ may match fewer agents than PC. To see this, consider a rule $\widetilde{PC}$ defined as follows. Fix any step. If an even agent i points to his own shift, then match the smallest such agent. If not match all agents in all cycles. Each agent points to himself at the first steps of $PC(\succ^e)$ and respectively $\widetilde{PC}(\succ^e)$. So Step 1 already finds $PC(\succ^e) = id$. However, each Step $k \in \mathbb{N}$ of $\widetilde{PC}(\succ^e)$ only matches agent $2k$ with his own shift. No odd agent gets matched by $\widetilde{PC}(\succ)$. The problem here is that $\widetilde{PC}$ not only rearranges the order in which cycles are matched but goes so far as to reassign some cycles to an infinite future that never occurs.

holds for all agents matched by $PC(\succ)$ at Steps 1 through $k-1$. Since j is matched at Step k of $PC(\succ)$, $PC(\succ)(j)$ is $\succ_j$-preferred to all shifts that remain unmatched at Step k. For $\mu(j) \succ_j PC(\succ)(j)$ to hold $\mu(j)$ must equal $PC(\succ)(i)$ for some agent i matched at one of the preceding rounds - a contradiction.

To see that PC is strategyproof, suppose there existed a mechanism $M : \Omega^T \to \mathcal{M}$, profile $\succ$, agent $i \in Dom(PC(\succ))$, and deviation $\succ_i'$ such that M is consistent with PC, and $M(\succ_i', \succ_{-i})(i) \succ_i PC(\succ)(i)$. Say that i is matched at Step k of the trading process of $PC(\succ)$, which reaches the submatching ν in Steps 1 through $k-1$. The definition of PC and $M(\succ_i', \succ_{-i})(i) \succ_i PC(\succ)(i)$ together imply $M(\succ_i', \succ_{-i})(i) \in Dom(\nu)$. Now consider the modified permacycles process $\widetilde{PC}(\succ_i', \succ_{-i})$ where any cycle ν' with $Dom(\nu') \subset Dom(\nu)$ that also forms under $PC(\succ)$ is matched before all other cycles. Once ν is reached $\widetilde{PC}(\succ_i', \succ_{-i})$ goes on to match all agents in all cycles that form at any step.[5] Noting that $\widetilde{PC}(\succ_i', \succ_{-i})$ reaches the submatching ν in Step $k-1$, we see that $\nu \subset M(\succ_i', \succ_{-i})$ since $PC = \widetilde{PC}$ (Lemma 1) is by assumption consistent with M. So $M(\succ_i', \succ_{-i})(i)$ cannot be contained in $Dom(\nu)$ - a contradiction. $\square$

Lemma 2 entails a recipe for the construction of good mechanisms for shift exchange problems: use permacycles wherever possible and then make sure that the properties also hold for all agents not matched by permacycles. Theorem 1 makes this explicit and shows that the converse also holds: any good mechanism must use PC wherever possible.

Theorem 1 *A mechanism $M : \Omega^T \to \mathcal{M}$ is good if and only if it is consistent with $PC : \Omega^T \to \overline{\mathcal{M}}$ and an individually rational and locally Pareto optimal rule $R : \Omega^T \to \overline{\mathcal{M}}$ with $Dom(R(\succ)) = Q(\succ)$ and $R(\succ)(i) \succsim_i' M(\succ_i', \succ)(i)$ for all $\succ$, $i \in Q(\succ)$ and $\succ_i'$.*

Proof Fix a mechanism $M : \Omega^T \to \mathcal{M}$.

[5] Since $i \notin Dom(\nu)$ and since $\nu(j) \succ_j i$ holds for all $j \in Dom(\nu)$, the submatching ν is indeed reached by $\widetilde{PC}(\succ_i', \succ_{-i})$. The reordering, moreover, matters only for finitely many steps, implying that $\widetilde{PC}$ does not leave any cycles unmatched. $\widetilde{PC}(\succ_i', \succ_{-i})$ is therefore well-defined.

Say that M is consistent with PC and an individually rational and locally Pareto optimal rule $R : \Omega^T \to \overline{\mathcal{M}}$ with $Dom(R(\succ)) = Q(\succ)$ and $R(\succ)(i) \succsim_i' M(\succ_i', \succ_{-i})(i)$ for all $\succ$, $i \in Q(\succ)$ and $\succ_i'$. Fix a profile $\succ$, an agent i, and a deviation $\succ_i'$.

$M(\succ)(i) \succsim_i i$ holds by Lemma 2 if $M(\succ)(i) = PC(\succ)(i)$ and by assumption if $M(\succ)(i) = R(\succ)(i)$. M is in sum individually rational.

Suppose some matching μ did Pareto dominate $M(\succ)$. By Lemma 2 $\mu(j) = PC(\succ)(j)$ holds for all $j \notin Q(\succ)$. Since R is assumed to be locally Pareto optimal, $\mu(j) = PC(\succ)(j)$ must then also hold for each $j \in Q(\succ)$ a contradiction, and M is Pareto optimal.

If $i \in Q(\succ)$, then $M(\succ)(i) = R(\succ)(i) \succsim_i M(\succ_i', \succ_{-i})(i)$ holds by assumption. If not then $M(\succ)(i) = PC(\succ)(i) \succsim_i M(\succ_i', \succ_{-i})(i)$ holds since PC is strategyproof as was shown in Lemma 2. M is in sum strategy proof.

Now assume that M is good. Define $PC^k : \Omega^T \to \mathcal{M}$ as the rule that matches all trading cycles that form in any of the first k steps of Gale's trading process. Note that PC is consistent with M if and only if PC^k is consistent with M for each $k \in \mathbb{N}$. It therefore suffices to show that PC^1 is consistent with M and that PC^k is consistent with M if PC^{k-1} is. To see this fix a profile $\succ \in \Omega^T$ and any cycle $\nu : S \to S$ at some Step k of $PC(\succ)$ assuming that either $k = 1$ or PC^{k-1} consistent with M. It then suffices to show that $\nu \subset M(\succ)$. Since $\nu : S \to S$ forms at Step k of $PC(\succ)$, Lemma 1 implies that $PC^{k-1}(\succ)(i) = PC^{k-1}(\succ_S', \succ_{-S})(i)$ for each $\succ'$ and each $i \in Dom(PC^{k-1}(\succ))$. By the inductive hypothesis we then have $PC^{k-1}(\succ) \subset M(\succ_S', \succ_{-S})$ for any $\succ'$.

To see that $\nu \subset M(\succ)$, let $\succ_i''$: $\nu(i), i$ for each $i \in S$. The following induction over $\mid S \setminus D \mid$ then shows that $\nu \subset M(\succ_D'', \succ_D)$ holds for all $D \subset S$. Since $(\succ_D'', \succ_{-D})$ equals $\succ$ if $D = \emptyset$, we in particular obtain $\nu \subset M(\succ)$.

Start of the induction. Let $\mid S \setminus D \mid = 0$ and $D \subset S$, so $S = D$. Since M is Pareto optimal and individually rational we have that $\nu \subset M(\succ_D'', \succ_{-D}) = M(\succ_S'', \succ_{-S})$.

Step of the induction. Suppose that the claim holds for all $D \subset S$ with $\mid S \setminus D \mid \leq m$. Consider some $D \subset S$ with $\mid S \setminus D \mid = m + 1$. The inductive hypothesis implies $\nu \subset M(\succ''_{D\cup\{i\}}, \succ_{-(D\cup\{i\})})$. By the definition of PC, $\nu(i)$ is the $\succ_i$-best shift out of all shifts that are not matched by $PC^{k-1}(\succ)$. Since $PC^{k-1}(\succ) \subset M(\succ'_S, \succ_{-S})$ for any $\succ'$, agent i cannot be matched with a $\succ_i$-better shift than $\nu(i)$ under $M(\succ''_D, \succ_{-D})$. Since M is strategyproof, $M(\succ''_D, \succ_{-D})(i) = \nu(i)$ holds for each $i \in S \setminus D$. These equalities together with the assumption that $\succ''_i$: $\nu(i), i$ holds for all $i \in D$, the individual rationality of M, and the fact that ν is a cycle imply that $M(\succ''_D, \succ_{-D})(i) = \nu(i)$ must also hold for all $i \in D$.

We can conclude that PC must be consistent with M. Define a rule $R : \Omega^T \to \overline{\mathcal{M}}$ such that $Dom(R(\succ)) = Q(\succ)$ and $R(\succ)(i) := M(\succ)(i)$ for all $\succ \in \Omega^T$ and $i \in Q(\succ)$. Since PC is globally Pareto optimal and since M is Pareto optimal R must be locally Pareto optimal. Since M is individually rational, the rule R must be too. Since M is strategy proof each agent $i \in Q(\succ)$ must weakly $\succ_i$-prefer $M(\succ)(i) = R(\succ)(i)$ to $M(\succ'_i, \succ_{-i})(i)$ for all $\succ'_i \in \Omega^T_i$. $\square$

The first part of Theorem 1 gives a recipe how to construct a good mechanism using permacycles. Since permacycles inherits some crucial welfare and incentive properties of Gale's top trading cycles, this recipe asks for permacycles to be used wherever possible. All agents not matched by permacycles must then be matched via an individually rational and locally Pareto optimal rule that is consistent with strategy proofness.[6]

The only aspect of Ω^T that matters for the proof of the first part of Theorem 1 is that PC is well-defined on Ω^T. The definition of Ω^T was only indirectly used in the proofs of Lemmas 1 and 2: The mechanism $PC : \Omega^T \to \overline{\mathcal{M}}$ is well-defined since there exists a unique optimal shift j for each agent

[6] To see that combining PC with any individually rational rule is not enough define the mechanism $PC^* : \Omega^T \to \mathcal{M}$ by $PC^*(\succ)(i) = i$ if $i \in Q(\succ)$ and $PC^*(\succ)(i) = PC(\succ)(i)$ otherwise. While PC^* is individually rational, it is neither Pareto optimal nor strategy proof. To see this consider a profile $\succ$ where each agent i most likes to work shift $i+1$ so that $PC(\succ) = \emptyset$ and $PC^*(\succ)(i) = i$ for all $i \in \mathbb{N}$. Now assume that agent 2 ranks agent 1's shift above his own. Agent 2 has an incentive to deviate to $\succ'_2$: $1, 2$ as $PC^*(\succ'_2, \succ_{-2})(2) = 1$ which is strictly $\succ_2$-preferred to shift 2. The matching $PC^*(\succ'_2, \succ_{-2})$ moreover strictly Pareto dominates $PC^*(\succ)$ at $\succ$.

$i \in \mathbb{N}$ set $S \in \mathbb{N}$ with $i \in S$ and preference $\succ_i \in \Omega_i^T$. As long as each set S contains for each $\succ_i \in \Omega_i'$ and $i \in N$ a unique best element, the rule PC is well-defined on Ω'.[7] Lemmas 1 and 2 as well as the first part of Theorem 1 immediately extend to any domain Ω' on which PC is welldefined: on any such domain the recipe for the construction of good mechanisms works.

The proof that good mechanisms must be constructed following this recipe (second part of Theorem 1) relies on a different aspect of Ω^T. If a shift j is $\succ_i$-acceptable for some $\succ_i \in \Omega_i$, then there exists a preference $\succ_i' \in \Omega^T$ for which shift j is the only $\succ_i'$-acceptable shift (other than shift i). The second part of Theorem 1 then applies to any domain Ω' that satisfies this richness condition. This same richness condition is used in Ma's [7] proof that Gale's top trading cycle is the unique good mechanism for housing markets. On less rich domains there may indeed be more good mechanisms. Restricting attention to the domain of single peaked preferences Bade [4] defines alternative good mechanisms for housing markets as well as for shift exchange problems.[8]

5 Short Trading Spans

Theorem 2 characterizes the set of all good mechanisms for the domain Ω^1 where agents never find working two or more days before or after their own shift acceptable. Given Theorem 1 this characterization boils down to defin-

[7]To see that PC need not be well-defined if this condition is violated say that each agent i's preference $\succ_i$ is such that $j \succ_i j'$ holds for all $j > j'$. Then no agent at Step 1 has a most preferred unmatchted shift.

[8]For a different example consider the set of housing markets for three agents and houses $\{1, 2, 3\}$ and a domain of preferences Ω' which does not contain any preferences where an agent finds only one house other than the house he was endowed with acceptable. Call the two matchings where each agent obtains a house that differs from his endowment μ° and μ'. Define a good mechanism M^* that uses Gale's top trading cycles whenever at least one agent finds only his own house acceptable. In the alternative case where all agents find all houses acceptable the mechanism chooses agent 1's preferred matching among μ° and μ'. The mechanism M^* differs from Gale's top trading cycles: If $\succ$ is such that agents 1 and 3 top rank house 2, while agent 2 top ranks house 3, then agent 1 is matched with house 1 according to Gale's top trading cycles ($G(\succ)(1) = 1$). However $M^*(\succ)$ is agent 1's preferred matching for which everyone moves, and we have $M^*(\succ)(1) = 3$.

ing an appropriate rule $R : \Omega^1 \to \overline{\mathcal{M}}$ to match all agents not matched by permacycles. The following observations on the domain of very short trading spans simplify this task: If a cycle ν involving more than one agent is individually rational at some $\succ \in \Omega^1$, then it is a swap between two adjacent agents, so we have $\nu = (i, i+1)$ for some $i \in \mathbb{N}$. There, moreover, exists some cutoff $i^* \in \mathbb{N}$ such that agent i is not matched by $PC(\succ)$ if and only if $i \geq i^*$ and each agent $i \geq i^*$ would like to delay his shift by a day.[9] Finally if each agent i in some infinite tail of the problem has the preference $\succ_i: i+1, i-1, i$ then it is locally Pareto optimal to match each odd agent (or each even agent) with his direct successor. These preliminary observations are summarized in Lemma 3 for which I define the submatchings $\nu^E[i^\circ]$ and $\nu^O[i^\circ]$ for $\{i^\circ, \dots\}$ by $(i, i+1) \subset \nu^X[i^\circ]$ for each even (odd) $i \geq i^\circ$ if $X = E$ (if $X = O$) as well as $\nu^X[i^\circ](i^\circ) = i^\circ$ if the match of i° is not determined by the preceding condition.[10]

Lemma 3 *Fix a profile $\succ \in \Omega^1$.*

1. *Fix $\mu \in \mathcal{M}$ and $i \in \mathbb{N}$ such that $i \neq \mu(i)$. If μ is individually rational then it is either consistent with $(i, i+1)$ or with $(i-1, i)$.*
2. *If $Q(\succ) \neq \emptyset$ then there exists an agent $i^* \in \mathbb{N}$ such that $Q(\succ) = \{i^*, \dots\}$ and each $i \in Q(\succ)$ $\succ_i$-prefers $i+1$ to all shifts $Q(\succ)$.*
3. *If there exists an $i^\circ \in \mathbb{N}$ such that $\succ_{i^\circ}: i^\circ+1, i^\circ$ and $\succ_i: i+1, i-1, i$ for all $i > i^\circ \in \mathbb{N}$ then $\nu^E[i^\circ]$ and $\nu^O[i^\circ]$ are globally Pareto optimal at $\succ$.*

To define **permacycles with even privilege** $PC^E : \Omega^1 \to \mathcal{M}$ fix an arbitrary $\succ \in \Omega^1$ and define S, $\succ_S^\circ$, and i° as follows. Let $S := \{i \in Q(\succ) \mid i+1 \succ_{i+1} i\}$. For each $i \in S$ let $\succ_i^\circ := \succ_i^e$ if $\succ_i: i+1, i$ and $\succ_i^\circ: i-1, i$ if

[9]Even at $T = 2$ this need not hold. At the profile $\succ$ with $\succ_2: 2$, $\succ_i: i+2, i$ for each odd i and $\succ_i: i-1, i$ for each even $i > 2$ we have $Q(\succ) = \mathbb{N} \setminus \{2\}$ and some agents $i \in Q(\succ)$ most prefer to work earlier shifts.

[10]Considering $\nu^E[7]$ note that the first condition implies $(i, i+1) \subset \nu^E[7]$ for all even $i < 7$. Since this condition does not determine $\nu^E[7](7)$, agent 7 is by the second condition matched with his own shift.

$\succ_i$: $i+1, i-1, i$.[11] Agent i° is such that $\succ_{i^\circ}$: $i^\circ+1, i^\circ$ and $\succ_i$: $i+1, i-1, i$ for all $i > i^\circ$. $PC^E(\succ)$ is then defined such that $PC(\succ_S^\circ, \succ_{-S}) \subset PC^E(\succ)$. If $PC(\succ_S^\circ, \succ_{-S})$ is not a matching then $\nu^E[i^\circ] \subset PC^E(\succ)$. The present notation suppresses the dependence of S, $\succ_S^\circ$, and i° on $\succ$. This is possible, since all upcoming arguments refer to an arbitrary but fixed profile $\succ$.

To understand PC^E fix a profile $\succ$ and note that $PC^E(\succ)$ is consistent with $PC(\succ)$. If $PC(\succ)$ is not a matching, part 2 of Lemma 3 implies the existence of an agent i^* such that each $i \in Q(\succ) :=\{i^*, \dots\}$ ranks $i+1$ at the top. If the shift of agent $i \in Q(\succ)$ is considered unacceptable by $i+1$ then $i \in S$. By part 1 of Lemma 3 no agent $i \in S$ is matched with $i+1$ for any individually rational matching. Since $\succ_i^\circ$ is derived from $\succ_i$ by deleting $i+1$ from the list of acceptable shifts, a matching is individually rational at $\succ$ if and only if it is individually rational at $(\succ_S^\circ, \succ_{-S})$. Depending on whether $i \in S$ finds $i-1$ acceptable or not $PC^E(\succ)(i) = PC(\succ_S^\circ, \succ_{-S})(i)$ either equals $i-1$ or i. If S has a maximal agent then this agent is $i^\circ - 1$ and $\succ_i$: $i+1, i-1, i$ holds for any agent $i > i^\circ$. $PC^E(\succ)$ then matches the agents $\{i^\circ, \dots\}$ in accord with the submatching $\nu^E[i^\circ]$, which is by part 3 of Lemma 3 globally Pareto optimal and individually rational.

Permacycles with even privilege is a mechanism: If $PC(\succ_S^\circ, \succ_{-S})$ is a matching then $PC^E(\succ)$ equals this matching. If not then $PC^E(\succ)$ is consistent with $PC(\succ_S^\circ, \succ_{-S})$ and $\nu^E[i^\circ]$. Since $\nu^E[i^\circ]$ matches all agents not matched by $PC(\succ_S^\circ, \succ_{-S})$, $PC^E(\succ)$ is also in this case a matching and PC^E is well-defined.

The set of good matching mechanisms not only contains PC^E, but any permacycles mechanisms PC^f with incentive compatible termination rule $f : \Omega^T \to \{E, O\}$. To calculate $PC^f(\succ)$ fix an arbitrary $\succ$ and define S, $\succ^\circ$, and i° as above. Let $PC(\succ_S^\circ, \succ_{-S}) \subset PC^f(\succ)$. If $PC(\succ_S^\circ, \succ_{-S})$ is not a matching then $\nu^{f(\succ)}[i^\circ]$ matches all agents in $Q(\succ_S^\circ, \succ_{-S}) = \{i^\circ, \dots\}$. The termination rule f is incentive compatible if $PC^f(\succ)(i) \succsim_i PC^f(\succ_i', \succ_{-i})$ holds for all $i \in Q(\succ)$. The outcomes PC^f and PC^E differ at $\succ$ if and only if $Q(\succ_S^\circ, \succ_{-S}) \neq \emptyset$ and $f(\succ) = O$. The characterization of all incentive compatible termination rules $f : \Omega^1 \to \{E, O\}$ intricate. Since it is also irrelevant for all practical purposes it is omitted. To show the existence of

[11] Since $S \subset Q(\succ)$, any such $i \in S$ most prefers $i+1$ and $\succ_i^\circ$ is well-defined.

some termination rule f for which PC^f is indeed strategyproof, I show that PC^E, the permacycles mechanism PC^f with the termination rule f that maps each $\succ$ to E, is strategy proof.

Theorem 2 *A mechanism $M : \Omega^1 \to \mathcal{M}$ is good if and only if it can be represented as permacycles with termination rule PC^f. Permacycles with even privilege PC^E is a good mechanism.*

Theorem 2 shows that Gale's trading process goes a long way in shift exchange problems with short trading spans. To determine a Pareto optimal and individually rational matching in a strategyproof way, one first has to use Gale's trading process wherever possible. If only finitely many agents can be matched in this way, then each remaining agent i wants to work "tomorrow". If $i + 1$ finds shift i unacceptable then i will, under no individually rational matching, get to work tomorrow. The preference of each such agent i is edited to delete $i+1$ from the set of acceptable shifts. Gale's trading process is then applied to the remaining agents given the edited profile of preferences. If even this step matches only finitely many agents, then the remaining agents are matched such that either all odd or all even ones among them get their most preferred shift, while the others get their second most preferred shift.

In the next section I show that this mechanism - as well as any good mechanism for a domain Ω' that embeds Ω^1 is inherently infinite.

6 Endless Mechanisms

In a shift exchange problem there are infinitely many agents whose preferences cannot be elicited simultaneously. The preceding two sections considered the infinite size of shift exchange problems while ignoring the restriction that the match of any particular agent may only depend on a finite set of preferences. The present section the focusses on the latter requirement and finds that no good mechanism for shift exchange problems satisfies it. To demonstrate that the results in the present section are driven by the impossibility to simultaneously elicit all preferences rather than the infinite amount of agents, I also consider house matching problems with $N = H = \{1, \ldots, n\}$ where each agent i never finds houses with j with $j >\mid i - T \mid$ acceptable. To distinguish such problems from standard house matching problems as well as from infinite shift exchange problems I denote the set of all preference profiles in such problems by $\hat{\Omega}^T$.

A mechanism is endless if there is no (strict) subset $\{1,\dots,K\}$ of all agents, such that eliciting the preferences of these first K agents is sufficient to match at least one agent. So $M:\Omega^T\to\mathcal{M}$ ($M:\hat{\Omega}^T\to\hat{\mathcal{M}}$) is **endless** if for each $K\in\mathbb{N}$ ($K<n$) there exist profiles $\succ$, $\succ'$ such that $M(\succ)(i)\neq M(\succ_{\{1,\dots,K\}},\succ'_{\{K+1,\dots\}})(i)$ holds for all $i\leq K$.

Theorem 3 first shows that any good mechanism for the domains Ω^1 and $\hat{\Omega}^1$ of short trading spans in endless. With longer trading spans ($T\geq 2$) the nonexistence problem becomes more severe: in that case any Pareto optimal and individually rational mechanism is endless.

Theorem 3 *Fix a Pareto optimal and individually rational mechanism* $M:\Omega^T\to\mathcal{M}$ *or* $M:\hat{\Omega}^T\to\hat{\mathcal{M}}$. *If*

1. *M is strategyproof and $T=1$, or*
2. *$T>1$*

then M is endless.

Proof Fix an arbitrary $K\in\mathbb{N}$ ($K<n$). The construction of two profiles $\succ^T\in\Omega^T$ for $T=1,2$ such that $M(\succ^T_{\{1,\dots,K\}},\succ^e_{\{K+1,\dots\}})(i)\neq M(\succ^T)(i)$ for all $i\in\{1,\dots,K\}$, shows that M is endless in all four cases.[12]

1. If $T=1$ and M is strategyproof, define $\succ^1$ such that $\succ^1_1$: $2,1$, $\succ_i$: $i+1,i-1,i$ for all $1<i\leq K+1$, $\succ_i=\succ^e_i$ for all $i>K+1$. Figure 4 illustrates $\succ^1_i$ for the agents up to agent K.

   If $\succ^1\in\Omega^1$ then $PC(\succ^1)\subset M(\succ^1)$ and $PC(\succ^1_{\{1,\dots,K\}},\succ^e_{\{K+1,\dots\}})\subset M(\succ^1_{\{1,\dots,K\}},\succ^e_{\{K+1,\dots\}})$ hold by Theorem 1. If $\succ^1\in\hat{\Omega}^1$ the same conclusion follows from the fact that Gale's top trading cycles is the unique strategyproof, efficient and individually rational matching mechanism for housing problems. While the the cycles $(K,K+1)$, $(K-2,K-1)$, $(K-4,K-3)$, ... form under $PC(\succ^1)$, the cycles $(K-1,K)$, $(K-$

[12] This completes the proof since $\Omega^2\subset\Omega^T$ and $\hat{\Omega}^2\subset\hat{\Omega}^T$ holds for all $T>1$.

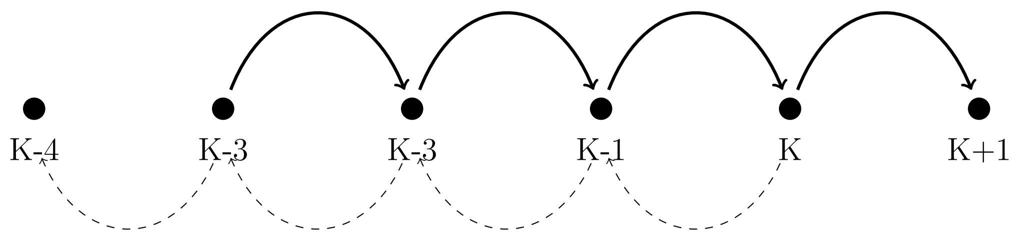

Figure 4: The choice of $K+1$ matters

3, $K-2$), ... form under $PC(\succ^1_{\{1,\dots,K\}}, \succ^e_{\{K+1,\dots\}})$. So for each $i \leq K$, $M(\succ^1)(i)$ differs from $M(\succ^1_{\{1,\dots,K\}}, \succ^e_{\{K+1,\dots\}})(i)$.

2. If $T > 1$, define the profile $\succ^2 \in \Omega^T$ with $\succ^2_i$: $i+2, i$ for each odd i, $\succ^2_i$: $i-2, i$ for each even $i > 2$, and $\succ^2_2$: $1, 2$. Figure 5 illustrates $\succ$. There is exactly one individually rational Pareto optimum at $\succ^2$: the

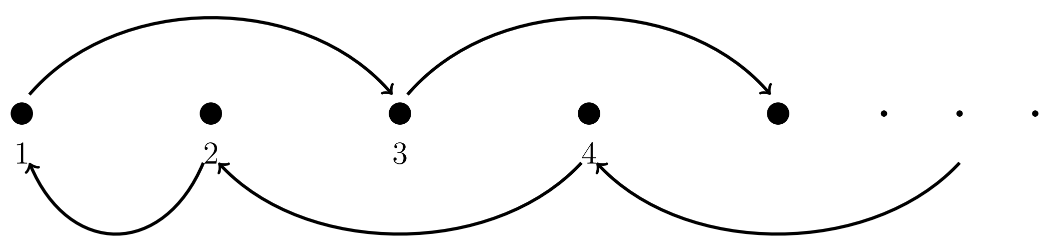

Figure 5: An infinite "cycle" as the unique Pareto optimum

matching μ with $\mu(i) = i+2$ for each odd i, $\mu(i) = i-2$ for each even $i > 2$, and $\mu(2) = 1$. Since M is individually rational and strategyproof we have $M(\succ^2) = \mu$. However the endowment is the only individually rational matching at $(\succ^2_{\{1,\dots,K\}}, \succ^e_{\{K+1,\dots\}})$.

In the alternative case of a house problem let $\succ^2$ be defined as above for all agents $i < K$. If K is odd, let $\succ^2_K$: $K+1, K$ and $\succ^2_{K+1}$: $K-1, K+1$, if K is even let $\succ^2_K$: $K-2, K$ and $\succ^2_{K+1}$: $K, K+1$. Furthermore let $\succ^2_i = \succ^e_i$ for all $i > K+1$. Note that all agents from 1 to $K+1$ form a single cycle in the unique individually rational Pareto optimum at $\succ^2$.

Conversely, the endowment is the only individually rational matching at $(\succ^2_{\{1,\ldots,K\}}, \succ^e_{\{K+1,\ldots,n\}})$.

□

Theorem 3 implies that there are no good mechanisms for shift exchange problems where some decisions must be made in finite time. When $T > 1$ this non-existence result holds even if we disregard incentive constraints. But any imaginable real life infinite horizon matching problem will require some decisions to be made in finite time: The shifts of September 2019 will have to be worked before we can elicit preferences over shifts in 2040. We cannot wait with kidney transplants that are feasible today until get a full picture of renal disease in 14 years. Considering only mechanisms where participation is voluntary - and therefore keeping the requirement of individual rationality in place, we need content ourselves with matching mechanisms that violate strategy proofness and/or Pareto optimality. We need to find compromise mechanisms that can be implemented in finite time and do reasonably well. Ideally truthtelling is a dominant strategy for a large share of agents and the outcomes of such compromise mechanisms can only be Pareto dominated by a small set of of matchings. The next section proposes two such compromises.

7 The Intervals Mechanisms

This section compares two different mechanisms for shift exchange problems that apply Gale's top trading cycles to all shifts and agents in trading intervals of length $I > T$. While one of these two mechanisms prohibits all trade across interval boundaries the other allows shifts swaps between agents from different intervals. Both these mechanisms start by running Gale's top trading cycles with all agents in the first trading interval. While the closed intervals mechanism C^I finalizes all resulting matches, the open intervals mechanism O^I only finalizes the matches involving agents and or shifts that cannot wait. The agents whose matches have not been finalized join the next trading interval with their temporary matches from the first round as their endowments. In either case the inductive application of Gale's top trading cycles to all upcoming sets of agents yields a matching.

To formally define $C^I : \Omega^T \to \mathcal{M}$ and $O^I : \Omega^T \to \mathcal{M}$ fix an arbitrary $\succ \in \Omega^T$. Let $\succ^1$ be the restriction of $\succ$ to the sets of agents and shifts $I^1 := \{1, \dots, I\}$ and let $\tau^1 = id : I^1 \to I^1$ be the initial endowment. Go to Round 1.

Round r

Gale's top trading cycles Calculate $G(\succ^r)$ given the initial endowment $\tau^r : I^r \to \tau^r(I^r)$.

Matching Let $C^I(\succ)(i) := G(\succ^r)(i)$ for all $i \in I^r$, and let $O^I(\succ)(i) := G(\succ^r)(i)$ for all $i \in I^r$ with $i \leq rI - T$ or $G(\succ^r)(i) \leq rI - T$. Let J be the set of all agents in I^r not matched at the current step.

Updating Let $I^{r+1} := \{rI + 1, (r+1)I\} \cup J$, $\tau^{r+1}(i) = i$ for each $i \in \{rI+1, (r+1)I\}$, $\tau^{r+1}(i) = G(\succ^r)(i)$ for each $i \in J$ and $\succ^r$ the restriction of $\succ$ to all agents I^{r+1} and all shifts $\tau^{r+1}(I^{r+1})$. Go to Round $r + 1$.

To see that C^I and O^I are well-defined fix an arbitrary $\succ \in \Omega^T$ and $i \in \mathbb{N}$. There exists a Round $r \in \mathbb{N}$ with $rI < i \leq (r+1)I$ when agent and shift i enter the trading process. If agent or shift i is not matched in this round, then he or it must be matched in Round $r + 1$ since $i \leq r(I + 1)$ and $I > T$ together imply $i < (r + 2)I - T$. So each agent and each shift get matched and $C^I(\succ) : \mathbb{N} \to \mathbb{N}$ as well as $O^I(\succ) : \mathbb{N} \to \mathbb{N}$ are onto. They are also one-to-one, since all agents are matched via trading cycles, so that no two agents are matched with the same shift.

Both intervals mechanisms satisfy the requirement that we cannot wait for all preferences to be elicited to match a particular agent or shift. According to C^I it suffices to elicit the preferences of all agents $\{rI + 1, \dots, (r + 1)I\}$ to match any agent in this interval. The mechanism O^I allows the (later) agents and shifts whose span intersects with the next trading interval to also participate in this next interval where they must get matched. So O^I is not endless. Indeed it satisfies a much stronger criterion of decision making in finite time. To match agent and shift i it - always - suffices to know the first $i + I + T$ preferences.

Since C^I matches each i via a predetermined round of Gale's top trading cycles (namely the Round r for which $(r-1)I < i \leq (r+1)I$) and since Gale's top trading cycles is strategyproof and individually rational, C^I is too. But C^I rules out many potentially individually rational matches: if i, j, and r are such that $i \leq rI < j$, then C^I never matches agent i with shift j. However if $j - i \leq T$, then any Pareto optimal and individually rational matching μ at some $\succ' \in \Omega^T$ with $\succ'_i$: j, i and $\succ'_j$: i, j must be consistent with (i, j) the cycle where i and j swap shifts. Conversely the open intervals mechanism $O^I : \Omega^T \to \mathcal{M}$ allows for such matches. Considering the same profile $\succ'$ note that agent i enters the trading process in Round r where he points to his own shift. Since $j - i \leq T$ and $j > rI$ the match between i and his shift is not finalized in Round r and agent i also participates in Round $r + 1$, where agent j enters. The cycle (i, j) now forms and is finalized since $i < (r+1)I - T$. So we obtain $O^I(\succ)(i) = j$.

Example 2 Fix a profile $\succ'$ with with $\succ'_i$: $i+2, i+1, i$ for each odd i, $\succ'_i$: $i-2, i-1, i$ for each even $i > 2$, and $\succ'_2$: $1, 2$ illustrated in Figure 6. So the only difference between $\succ'$ and $\succ^2$, defined in proof of Theorem 3 and illustrated in Figure 5, is that each even (odd) agent $i > 2$ also accepts to work shift $i-1$ $(i+1)$ under $\succ'$.

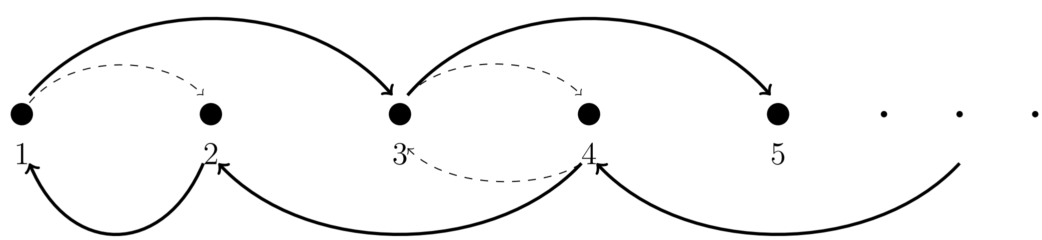

Figure 6: The profile $\succ'$

The new profile $\succ'$ shares with the profile $\succ^2$, defined in the proof of Theorem 3, that μ with $\mu(i) = i + 2$ for each odd i, $\mu(i) = i - 2$ for each even $i > 2$, and $\mu(2) = 1$ is the unique individually rational Pareto optimum. Let I: $= 5$ and T: $= 3$. While C^5 as well as O^5 map the $\succ^2$ to id, the acceptance of "intermediate" shifts under $\succ'$ yields better results.[13]

The cycles (5) and $(1, 3, 4, 2)$ form in Round 1 of $C^5(\succ')$ and $O^5(\succ')$ (to see this consider the arrows among agents 1 through 5 Figure 6). Under C^5 these matches are immediately finalized and we obtain $(1, 3, 4, 2) \subset C^5(\succ')$ as well as $C^5(\succ')(5) = 5$. The cycles (6) and $(7, 9, 10, 8)$ form in Round 2 of $C^5(\succ')$ so that $(7, 9, 10, 8) \subset C^5(\succ')$ and $C^5(\succ')(6) = 6$. Any odd (even) trading round of $C^5(\succ')$ looks just like any other odd (even) trading round and we obtain $G^5(\succ')(10n + i) = G^5(\succ')(i) + 10n$ for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, 10\}$.

Under $O^5(\succ)$ only $O^5(\succ')(1) = \mu(1) = 3$, $O^5(\succ')(2) = \mu(2) = 1$, and $O^5(\succ')(4) = \mu(4) = 2$ are finalized with the first round. Agents 3 and 5 enter the next trading round with shifts 4 and 5 as their respective endowments. Round 2 of $O^5(\succ')$ applies Gale's top trading cycles to the profile of agents' 3, 5, 6, 7, ..., 10 preferences over shifts 4, 5, 6, ..., 10 where agent 3 is endowed with shift 4 and each agent $i \in \{5, 6, 7, 8, 9, 10\}$ is endowed with his own shift i. Round 2 finalizes the match $\mu(i)$ for each $i \in \{3, 5, 6, 7, 8\}$; agents 9, and 10 go on to Round 3 with shifts 10 and 8 as their respective new endowments. Proceeding inductively we obtain $O^5(\succ') = \mu$

In sum we obtain $C^5(\succ') \neq O^5(\succ') = \mu$. At $\succ'$ $C^5(\succ')$ is strictly Pareto ranked between the initial endowment id and μ.[14] While both C^I and O^I let the match of any single agent depend on only finitely many preferences, only O^I results in the infinite chain μ. The decisive difference between the

[13] In Step 1 of Round 1 of $C^5(\succ)$ and $O^5(\succ)$ there is a chain involving all agents. The agent at the helm of this chain, agent 5, only finds his own shift $\succ_5$-acceptable in the set $\{1, \ldots, 5\}$. Once he is matched with his own shift at Step 1, agent 3 is matched with his own shift at step 2, ... So in the first round of $C^5(\succ)$ and $O^5(\succ)$ each agent keeps his own shift. Under the C^5 trading process these matches are finalized, under the O^5 trading process only agents 1 and 2 are definitively matched with their own shifts, the other three agents move on to the next trading round. While differently many agents take part in the second trading rounds of C^5 and O^5 at $\succ$, this second trading round is in either case very similar to the first. Each agent in the second round (and in each subsequent round) using either trading process is matched with his own shift and $C^5(\succ) = O^5(\succ) = id$.

[14] While each agent obtains his most preferred shift under μ, only agents 2, $10n + 1$, $10n + 4$, $10n + 7$, and $10n + 10$ for all $n \in \mathbb{N}_0$ do so under $C^5(\succ')$. Agents $10n + 5$ and $10n + 6$ for all $n \in \mathbb{N}_0$ stay with their endowment which is their third most preferred shift. All other agents obtain their second most preferred shift.

two mechanisms is that trading cycles which form in one round may (only) under O^I be reopened in the next round.

While $O^I(\succ)$ may Pareto dominate $C^I(\succ)$ (as shown in Example 2), the converse never holds. To see this, fix $\succ$ such that $C^I(\succ) \neq O^I(\succ)$. Fix $r, j \in \mathbb{N}$ such that $C^I(\succ)(i) = O^I(\succ)(i)$ holds for all i matched in Rounds $1, \ldots, r-1$ of $C^I(\succ)$ whereas $C^I(\succ)(j) \neq O^I(\succ)(j)$ holds for agent j who is matched in Round r of $C^I(\succ)$. Any agent i matched in some Round $r' < r$ under $C^I(\succ)$ who enters Round r of $O^I(\succ)$ must be matched with his temporary endowment (which equals $C^I(\succ)(i)$) at Round r of $O^I(\succ)$ and can therefore be ignored in Round r of $O^I(\succ)$. Both mechanisms (temporarily) match agent j to $C^I(\succ)(j)$ in Round r. Under C^I this match is finalized. For $C^I(\succ)(j) \neq O^I(\succ)(j)$ to hold agent j must enter Round $r+1$ under $O^I(\succ)$. Since $C^I(\succ)(j)$ is agent j's temporary endowment in that round we have $O^I(\succ)(j) \succ_j C^I(\succ)(j)$ and $C^I(\succ)$ cannot Pareto dominate $O^I(\succ)$. The following example shows that the outcomes of the two intervals mechanisms need not be Pareto ranked.

Example 3 Let $T = 2$ and $I = 4$. Define $\succ$ such $\succ_4$: $6, 4$, $\succ_5$: $6, 5$, $\succ_6$: $4, 5, 6$ and $\succ_i^e = \succ_i$ otherwise (see Figure 7 for an illustration). Under $C^4(\succ)$ and $O^4(\succ)$ exactly 2 agents swap shifts while all other agents work their own shift: $(5, 6) \subset C^4(\succ)$ and $(4, 6) \subset O^4(\succ)$.

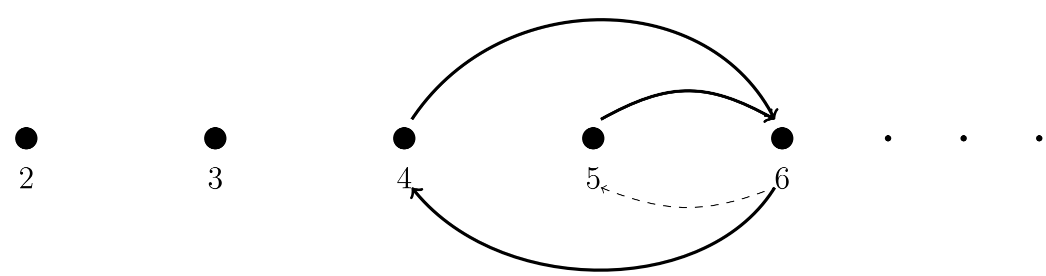

Figure 7: Agent 5 prefers C^4 to O^4

Barriers to trade generally incur benefits and losses to different parties. Here it is agent 5 who benefits from the trading barrier between agents 4 and 6.

In the face of this barrier agent 6 makes do with his second most preferred shift, namely agent 5's shift. Conversely agents 4 and 6 are harmed by the trading barrier between them. The following proposition summarizes the welfare comparisons between the two intervals mechanisms.

Proposition 1 *Fix some $I > T$. There exists some $\succ^* \in \Omega^T$ such that $O^I(\succ^*)$ Pareto dominates $C^I(\succ^*)$. For some other $\succ^\circ \in \Omega^T$, $O^I(\succ^\circ)$ and $C^I(\succ^\circ)$ are not Pareto ranked. There exists no $\succ \in \Omega^T$ such that $C^I(\succ)$ Pareto dominates $O^I(\succ)$.*

While the open intervals mechanism outperforms the closed one in terms of welfare, it fares less well in terms of incentives. The following Theorem 4 on the open intervals mechanism shows that truthtelling is only for the earlier agents in each interval a dominant strategy. The later agents may be able to misrepresent their preferences to obtain a better shift. The open intervals mechanism is individually rational. Finally, to Pareto dominate a matching $O^I(\succ)$ distant enough agents have to engage in swapping chains.

Theorem 4 *Fix an open intervals mechanism $O^I : \Omega^T \to \mathcal{M}$ and a profile $\succ \in \Omega^T$. If some matching $\mu \in \mathcal{M}$ Pareto dominates $O^I(\succ)$ then there exist $i, j, r, n \in \mathbb{N}$ such that $i \leq rI - T$, $j > rI$ and $\mu^n(i) = j$. If $O^I(\succ_i', \succ_{-i})(i) \succ_i O^I(\succ)(i)$ holds for some $\succ_i'$ and i, then there exists a number $r \in \mathbb{N}$ such that $rI - T < i \leq rI$. $O^I(\succ)(i) \succsim_i i$ holds for all i.*

The incentives for truthtelling in O^I are illustrated in Example 4. Example 5 then shows that the outcome of the open intervals mechanism may be dominated by a matching that involves reasonably distant agents who belong to different trading intervals.

Example 4 Let $T = 1$ and $I = 5$ and consider $\succ^*$ with $\succ_4^*$: $5, 4$, $\succ_5^*$: $6, 4, 5$ and $\succ_6^*$: $5, 6$ with $\succ_i^* = \succ_i^e$ for all $i \in \mathbb{N} \setminus \{4, 5, 6\}$ illustrated in Figure 8.

Under $O^5(\succ^*)$ agent 4 and 5 swap shifts in the first round. Since $4 < I + 1 - T = 5$ these matches are finalized in the first round and we obtain $(4, 5) \subset O^5(\succ^*)$. Each cycle forming in the first round of $O^5(\succ_5', \succ_5^*)$ with $\succ_5'$: $6, 5$ consists of one agent and his shift. Given $T = 1$ these matches are finalized for agents 1, 2, 3, and 4. Agent 5 (endowed with shift 5) goes on

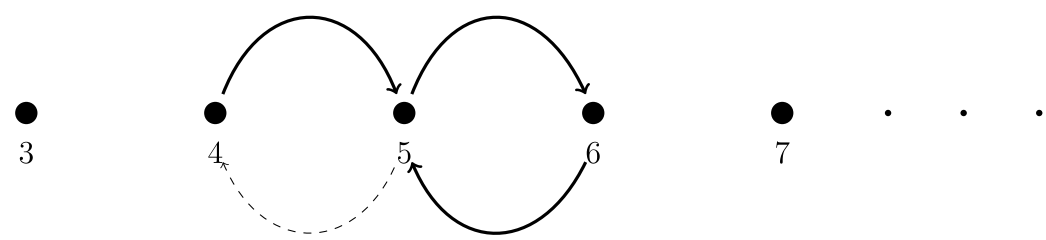

Figure 8: Tuthtelling need not be strategyproof under O^I

to the next round where he swaps shifts with agent 6. Since $2I+1-T>6$ both these matches are finalized in round two. Since $O^5(\succ'_5, \succ^*_{-5})(5) = 6 \succ_5 4 = O^5(\succ^*)$, O^5 is not strategy proof.

To understand the incentives for the earlier agents in any given interval consider agent 7 who enters the trading process with Round 2. Since $7 < 2I+1-T = 10$ agent 7 is for each $\succ$ matched by the application of Gale's top trading cycles to $\succ^2$. Since G is strategy proof and since agent 7 has no impact on the size of the set I^2, the truthful revelation of his preference is weakly dominant for agent 7.

Example 5 Let $T=2$ and $I=5$ and consider $\succ^*$ with $\succ^*_3$: $4,3$, $\succ^*_4$: $6,4$, $\succ^*_5$: $3,5$, $\succ^*_6$: $5,6$ and $\succ^*_i = \succ^e_i$ for all $i \in \mathbb{N} \setminus \{3,4,5,6\}$ illustrated in Figure 9.

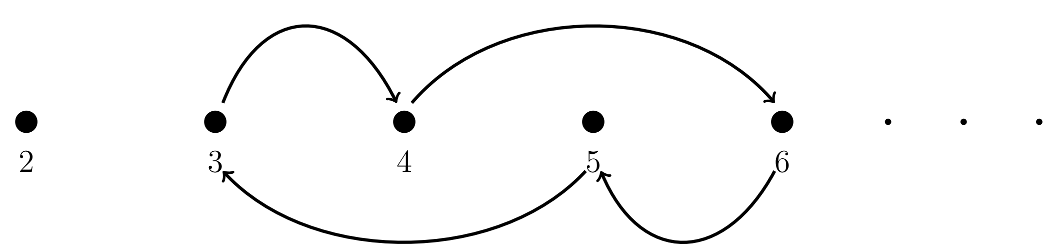

Figure 9: A matching that dominates O^I

The first round of $O^5(\succ^*)$ matches each agent with his own shift. These matches are finalized for agents 1, 2, and 3. Agents 4 and 5, endowed with their original shifts, move on. The second round (and any round thereafter) presents the same picture: each participating agent is matched with his own shift, and we obtain $O^5(\succ^*) = id$. But in Figure 9 there is a pointing cycle $(3, 4, 6, 5)$ and the matching μ with $(3, 4, 6, 5) \subset \mu$ and $\mu(i) = i$ for $i \notin \{3, 4, 5, 6\}$ Pareto dominates $O^5(\succ^*) = id$. Letting $i = 3$, $j = 6$, $r = 1$ and $n = 2$ the conditions $i \leq rI - T$, $j > rI$ and $\mu^n(i) = j$ are satisfied by μ.

To see that a dominating matching μ must reach far enough across interval boundaries vary the interval length I. If $I = 6$, all agents in the pointing cycle Figure 9 take part in Round 1 of $O^6(\succ^*)$ and we obtain $O^6(\succ^*) = \mu$. If $I = 4$, then each agent is matched with his own shift in the first round of $O^4(\succ^*)$. Since $T = 2$ agents 3 and 4 then move on to the next round with their original shifts as their endowments. The agents 3, 4, 5, and 6 all take part in Round 2 and $O^4(\succ^*)$ equals μ.

Theorem 4 implies that the open intervals mechanism does reasonably well if T is small in comparison to I. If $T = 1$, then the statement in Theorem 4 can be strengthened: whether we allow for strategic behavior or not, the outcomes of open intervals mechanism are Pareto optimal. Consider an actual profile of preferences $\succ$ and a profile of reported preferences $\succ'$ where only agents who strictly benefit from a misrepresentation of their preferences do so ($\succ_i \neq \succ'_i$ implies $O^I(\succ')(i) \succ_i O^I(\succ_i, \succ'_{-i})(i)$). The open intervals mechanism maps the profile of reported preferences $\succ'$ to a matching that is Pareto optimal at the actual preferences $\succ$.

Corollary 1 *Fix two profiles of preferences $\succ, \succ' \in \Omega^1$ with either $\succ'_i = \succ_i$ or $O^I(\succ') \succ_i O^I(\succ_i, \succ'_{-i})$ for each i. Then $O^I(\succ')$ is Pareto optimal at $\succ$.*

Proof Suppose some matching μ Pareto dominated $O^I(\succ')$ at $\succ$. First suppose that μ Pareto dominated $O^I(\succ')$ at the reported profile $\succ'$. By Theorem 4 there would have to exist $i, j, r, n \in \mathbb{N}$ such that $\mu^n(i) = j$, $i \leq rI - T$, and $j > rI$, implying $i - j > 1$. So μ is by Lemma 3 part 1 not individually rational at $\succ'$ and can therefore not Pareto dominate $O^I(\succ')$ at $\succ'$, which is by Theorem 4 individually rational at $\succ'$.

For μ to Pareto dominate $O^I(\succ')$ at $\succ$, $O^I(\succ)(j) \neq \mu(j)$ must therefore hold for some j with $\succ'_j \neq \succ_j$. Since truthtelling is by Theorem 4 a dominant strategy for any agent $i \notin \{I, 2I, 3I, \dots\}$, agent j must equal nI for some $n \in \mathbb{N}$.

For $j = nI$ to (strictly) benefit from reporting $\succ'_j$ instead of $\succ_j$ we must have $O^I(\succ_j, \succ'_j)(j) = j - 1$ and $O^I(\succ')(j) = j + 1$, $\succ_j$: $j + 1, j - 1, j$ and $\succ'_j$: $j + 1, j$.[15] Consequently $O^I(\succ')(j) \neq \mu(j)$ implies that j strictly prefers $O^I(\succ')(j)$ to $\mu(j)$ and μ cannot Pareto dominate $O^I(\succ')$ at $\succ$. □

8 Conclusion

In a housing market there are finitely many agents and equally many objects. The designer simultaneously decides over all matches between agents and objects. This is unproblematic since all agents in a housing market know their preferences. Conversely only a few agents know their preferences at the start of a shift exchange problem. As time goes on more and more agents learn their preferences. It is impossible to simultaneously elicit all preferences as some agents have to be matched before all preferences become known. The ongoing nature of shift exchange problems is captured via the assumption of countably infinite sets of agents and shifts.

The result that Pareto optimality, strategy proofness and individual rationality can only be achieved by using Gale's top trading cycles - wherever possible - transfers to the case with infinitely many objects (Theorem 1). However, with infinitely many objects Gale's trading process need not match all agents (and shifts). To obtain a good mechanism Gale's trading process needs to be combined with an additional rule to match the agents that are not part of any trading cycle.

In Theorem 2 I provide an example of such a good mechanism for the case that each agent may only swap shifts within a very short time span. Theorem 3 then shows that any good mechanism for a domain that contains this - very

[15]Since O^I is individually rational we have $O^I(\succ_j, \succ'_{-j})(j) \succsim_j j$; since $O^I(\succ')(j) \succ_j O^I(\succ_j, \succ'_{-j})(j)$, $O^I(\succ')(j) \neq j$. Since Gale's top trading cycles, which is used in each trading round, is strategyproof, the report $\succ'_j$ must be such that agent j's match is determined by a different trading round under $\succ'$ and $(\succ_j, \succ'_{-j})$. But this is only possible if agent j is under $(\succ_j, \succ'_{-j})$ definitively matched in the trading round with which he enters, while he stays for a next trading round under $\succ'$. For j to be definitively matched in the trading round in which he enters we must have $O^I(\succ_j, \succ'_{-j})(j) = j - 1$. Since $O^I(\succ')(j) \neq j$, we must then have $O^I(\succ')(j) = j + 1$.

small - domain may have to elicit infinitely many preferences to decide how to match the first agent. With longer trading spans this problem turns out to be yet more severe: in that case any Pareto optimal and individually rational mechanism (strategyproof or not) may require the elicitation of all agents preferences to decide on the shift that the first agent should be matched to.

The design of reasonable mechanisms consequently requires us to either weaken our efficiency and incentive compatibility standards or to content ourselves with good mechanisms for smaller domains. Keeping the domain fixed, Section 7 compares two different approaches to apply Gale's top trading cycles to the same fixed intervals of agents. According to the first C^I no trade may occur across intervals. Conversely the intervals mechanism O^I allows for some "smoothing" across intervals. Any agent for whom there remains scope to improve their match after they participated in a trading interval of O^I goes on to participate in the following interval. While O^I outperforms C^I in terms of its welfare properties as shown in proposition 1, O^I does less well in terms of incentives. While C^I is strategyproof, truthtelling is only dominant for the earlier agents in each trading interval of O^I. On the domain with very short trading spans O^I is Pareto optimal - no matter the extent to which agents act strategically.

Besides the restriction to very short trading spans I did not explore domain restrictions in this paper. In Bade [4] I show that good mechanisms for shift problems exist when all agents preferences are single peaked. The second part of Theorem 1 which states that Gale's trading process must be used wherever possible in a good mechanism for a shift exchange problem does not apply to the domain of single peaked preferences.

One dimension that was entirely ignored in the present paper stands out: uncertainty. But, when agents submit their own preferences they may know only a few other agents preferences. One might specify a distribution over possible preferences and assume that agents have preferences over lotteries. Suppose that all agents know all preferences in the current trading interval but don't know the preferences of agents in the upcoming interval. Now suppose furthermore that agents are expected utility maximizers who consider the utility differential between their own shift and any other acceptable shift large. Finally suppose that agents consider the event that a shift is acceptable to be rare. Under these assumptions truthtelling is a (Bayes-Nash) equilibrium strategy in O^I. To see this consider an agent i who under truthtelling is definitively matched with some shift $j \neq i$ in the trading round with which he entered the trading process. Under a misreport this same agent may obtain a better shift in the next trading round. However, given that such compatibility is rare the expected utility of a misreport falls below the utility of obtaining j for sure.

In addition to such exogenous uncertainty one could investigate random matching mechanisms which introduced probabilistic matching rules to incentivize truthtelling. Considering agents who don't know whether the designer uses C^I or O^I to match them with shifts, truthtelling is a best reply if the probability of C^I being chosen is high enough. In a similar vein one could investigate the effect of the interval length I being drawn from some distribution.

Appendix

Proof of Lemma 1: Suppose there exists a profile $\succ \in \Omega^T$ where either $PC(\succ)(i) \neq \widetilde{PC}(\succ)(i)$ or $i \notin Dom(\widetilde{PC}(\succ))$ holds for some $i \in Dom(PC(\succ))$. Say that $PC(\succ)$ matches i via the cycle $\nu : S \to S$ at Step k, while $PC(\succ)(j) = \widetilde{PC}(\succ)(j)$ holds for all $j \in S'$, the set of all agents matched by $PC(\succ)$ in Steps 1 through $k-1$. As long as some agents in S' remain unmatched in the trading process of $\widetilde{PC}(\succ)$ the agents in S either all take part in pointing chains terminating with some shifts in S' or they form the cycle ν. If all agents in S form the cycle ν at some step, they cannot become part of any other cycle at any later step. Since $PC(\succ)(j) = \widetilde{PC}(\succ)(j)$ holds for each $j \in S'$, the cycle ν then forms among the agents who remain unmatched by $\widetilde{PC}(\succ)$ a contradiction. So we must have $PC(\succ) \subset \widetilde{PC}(\succ)$ for each $\succ$. Since any cycle ν that forms at Step k of the of $\widetilde{PC}(\succ)$ forms at some Step $k' \leq k$ under $PC(\succ)$, the inverse subset relation also holds. $\square$

Proof of Lemma 3:

1. Say agent i is the smallest agent such that μ is not consistent with (i), $(i, i+1)$, or $(i-1, i)$. Since μ is individually rational, $\succ_i \in \Omega_i^1$, and

$\mu(i) > i$, we have $\mu(i) = i+1$. Since $(i, i+1)$ is not consistent with μ, $\mu(i+1)$ must by the same reason equal $i+2$. For μ' to be a matching some agent $i' \geq i+2$ must then be matched with shift i, a contradiction to individual rationality given that $\succ_{i'} \in \Omega^1_{i'}$.

2. Let $i^* := \min Q(\succ)$. Define a function $f : Q(\succ) \to Q(\succ)$ such that $f(i)$ is agent i's most preferred shift in $Q(\succ)$. If $f(i^*) = i^*$ then $PC(\succ)(i^*) = i^*$ and i^* cannot be in $Q(\succ)$. Since $\succ_{i^*} \in \Omega^T_{i^*}$, i^*+1 must be agent i^*'s most preferred shift in $Q(\succ)$ we have $f(i^*) = i^*+1 \in Q(\succ)$. Now suppose that $f(i) = i+1 \in Q(\succ)$ holds for all $i^* \leq i < n$. If $n-1$ or n is agent n's most preferred shift in $Q(\succ)$ then $PC(\succ)(n) \in \{n, n-1\}$ and n cannot be in $Q(\succ)$. So $f(n) = n+1 \in Q(\succ)$. In sum we obtain that $\{i^*, \dots,\} \subset Q(\succ)$ and consequently $Q(\succ) = \{i^*, \dots\} = \mathbb{N} \setminus (Dom(PC(\succ)))$.

3. Say there exists some i° such that $\succ_{i^\circ}: i^\circ+1, i^\circ$ and $\succ_i: i+1, i-1, i$ for all $i > i^\circ$. Under $\nu^E[i^\circ]$ each even $i \geq i^\circ$ works his most preferred shift. Conditioning on each even agent working their most preferred shift each odd agent gets to work his most preferred shift in $\{i^\circ, \dots\}$ and $\nu^E[i^\circ]$ is locally Pareto optimal. Only one agent finds a shift in $\{1, \dots, i^\circ - 1\}$ acceptable for some profile in Ω^1: this is agent i°. But according to the particular profile $\succ$, i° only finds shifts in $\{i^\circ, \dots\}$ acceptable. So $\nu^E[i^\circ]$ is also globally Pareto optimal. Mutatis mutandis the same applies to $\nu^O[i^\circ]$.

□

Proof of Theorem 2: Fix a profile $\succ \in \Omega^1$, an agent i, a deviation $\succ'_i$. For $\succ$, define S, $\succ^\circ$ and i° as in the main body of the text.

Fix a mechanism PC^f.

Individual rationality: If $i < i^\circ$ then agent i considers $PC^f(\succ)(i) = PC(\succ^\circ_S, \succ_{-S})(i)$ by Lemma 2 acceptable at $(\succ^\circ_S, \succ_{-S})$. Since the set of $\succ^\circ_i$-acceptable shifts is a subset of the set of $\succ_i$-acceptable shifts for each $i \in S$, $PC^f(\succ)(i)$ is $\succ_i$-acceptable for any $i \in Dom(PC(\succ^\circ_S, \succ_{-S})(i))$. If $i \geq i^\circ$ then

$PC^f(\succ)(i)$ either equals $\nu^E[i^\circ](i)$ or $\nu^O[i^\circ](i)$ and is therefore $\succ_i$-acceptable. PC^f is in sum individually rational.

Pareto optimality: Suppose there exists a matching μ such that $\mu(i) \succsim_i PC^f(\succ)(i)$ for all $i \in \mathbb{N}$ and $\mu(j) \succ_j PC^f(\succ)(j)$ for some $j \in \mathbb{N}$. If $j \in S$ then j only prefers $j+1$ to $PC^f(\succ)(j)$. So we must have $\mu(j) = j+1$ and - by part 1 of Lemma 3 - $\mu(j+1) = j$. The Pareto dominance of μ (over $PC^f(\succ)$), the individual rationality of PC^f and $j \in S$ then imply the contradiction $j = \mu(j+1) \succsim_{j+1} PC^f(\succ)(j+1) \succsim_{j+1} j+1 \succ_{j+1} j$. So we must have $PC^f(\succ)(i) = \mu(i)$ for all $i \in S$.

Combining $PC^f(\succ)(i) = \mu(i)$ for all $i \in S$ with $\mu(i) \succsim_i PC(\succ_S^\circ, \succ_{-S})(i)$ for all $i \in Dom(PC(\succ_S^\circ, \succ_{-S}))\backslash S$ the global Pareto optimality $PC(\succ_S^\circ, \succ_{-S})$ at $(\succ_S^\circ, \succ_{-S})$ (established in Lemma 2) implies that $\mu(i)$ must also equal $PC^f(\succ)(i) = PC(\succ_S^\circ, \succ_{-S})(i)$ for all $i \in Dom(PC(\succ_S^\circ, \succ_{-S})) \setminus S$. Since $PC^f(\succ)$ matches the remaining agents $\{i^\circ, \dots\} = \mathbb{N} \setminus Dom(PC(\succ_S^\circ, \succ_{-S}))$ via $\nu^{f(\succ)}[i^\circ]$, which is by part 3 of Lemma 3 globally Pareto optimal, μ cannot Pareto domiante $PC^f(\succ)$ at $\succ$ and PC^f is Pareto optimal.

Strategyproofness If $i \in S$ then $i+1 \succ_{i+1} i$ and $PC^f(\succ_i', \succ_{-i})(i) \neq i+1$ holds by the individual rationality of PC^f. Since $PC^f(\succ)(i)$ is agent i's preferred shift in $\mathbb{N} \setminus \{i+1\}$ we have that $PC^f(\succ)(i) \succsim_i PC^f(\succ_i', \succ_{-i})(i)$. If $i < i^\circ$ and $i \notin S$ then $PC^f(\succ)(i) \succsim_i PC^f(\succ_i', \succ_{-i})(i)$ holds by Lemma 2. If $i \geq i^\circ$ then $PC^f(\succ)(i) \succsim_i PC^f(\succ_i', \succ_{-i})(i)$ holds by the assumption on f made in the definition of PC^f. In sum PC^f is strategy proof.

Say that $M : \Omega^1 \to \mathcal{M}$ is good. Theorem 1 implies that M must be consistent with permacycles. So if $PC(\succ)$ is a matching we are done.

So assume that $S \neq \emptyset$ and consider some $i \in S$. Since M is individually rational and since $i \in S$, $(i,j) \not\subset M(\succ)$. By part 1 of Lemma 3 $M(\succ)(i) \neq i+1$. As a good mechanism M is by Theorem 1 consistent with PC. Since M is strategy proof we therefore have $M(\succ)(i) \succsim PC(\succ_i^\circ, \succ_{-i})(i) = M(\succ_i^\circ, \succ_{-i})(i)$. Since $M(\succ)(i) \neq i+1$ and since $PC(\succ_i^\circ, \succ_{-i})(i)$ is the $\succ_i$-best shift in $\mathbb{N} \setminus \{i+1\}$, we have $M(\succ)(i) = PC(\succ_i^\circ, \succ_{-i})(i) = PC^f(\succ)(i)$. Given the matches of any $i \in S$, $PC(\succ_I^\circ, \succ_{-I})(\succ) \subset M(\succ)$ must hold by the arguments in the proof of Theorem 1.

If S has a maximum, suppose that $M(\succ)$ restricted to $\{i^\circ, \dots\}$ was neither $\nu^E[i^\circ]$ nor $\nu^O[i^\circ]$. So suppose there exists some agent $j > i^\circ$ with $M(\succ)(j) = j$. Since M is consistent with permacycles $M(\succ'_j, \succ_{-j})(j) = PC(\succ'_j, \succ_{-j})(j) = j-1$ holds for $\succ'_j$: $j-1, j$. But since $j > i^\circ$ we have $\succ_j$: $j+1, j-1, j$ and therefore $M(\succ'_j \succ_{-j})(j) \succ_j M(\succ)(j)$ a contradiction to the strategyproofness of M.

The first part of the proof implies that PC^E is Pareto optimal and individually rational. It moreover implies that no agent $i < i^\circ$ can benefit from misstating his preferences. To see that $PC^E(\succ)(i) \succsim_i PC^E(\succ'_i, \succ_{-i})(i)$ also holds for $i \geq i^\circ$ note that each even $i \geq i^\circ$ is matched with $PC^E(\succ)(i) = i+1$ his most preferred shift and therefore has not incentive to deviate. If $i \geq i^\circ$ is odd then $PC^E(\succ)(i)$ is the second best shift according to $\succ_i$. Given that each $i > i^\circ$ ranks $i+1$ at the top, $PC^E(\succ'_i, \succ_i)(i) = i+1$ would have to hold for i to improve upon $PC^E(\succ)(i)$. However $PC^E(\succ'_i, \succ_i)(i) \neq i+1$ holds for all $\succ'_i$ and PC^E is strategyproof. □

Proof of Theorem 4: Suppose that $O^I(\succ)$ was dominated by a matching μ where for any agent i there exists a cycle ν and a number r such that $\nu \subset \mu$ $i \in Dom(\nu) \subset \{(r-1)I+1-T, \ldots, rI\}$. For each such cycle ν let $r(\nu)$ be the smallest number such that $Dom(\nu) \subset \{(r-1)I+1-T, \ldots, rI\}$.[16]

Now choose ν^* such that $\nu^* \subset \mu$, $\nu^* \not\subset O^I(\succ)$, and $\nu \subset O^I(\succ)$ for all $\nu \subset \mu$ with $r(\nu) < r(\nu^*)$. Note that each agent $i \in \{1, \ldots, (r(\nu^*)-1)I - T)\}$ belongs to to some cycle ν with $r(\nu) < r(\nu^*)$. The open intervals mechanism finds and finalizes all these matches by Round $r(\nu^*)$ at the latest. Since $Dom(\nu^*) \subset \{(r(\nu^*)-1)I+1-T, \ldots, r(\nu^*)I\}$ and since $O^I(\succ)$ does not match any agent or shift $i \in Dom(\nu^*)$ with an agent or shift $j \leq (r(\nu^*)-1) - T$, all agents and shifts in $Dom(\nu^*)$ take part in Round $r(\nu^*)$ of $O^I(\succ)$.

First suppose that $\nu^* = \rho$ for some trading cycle ρ that occurs in the application of Gale's top trading cycles in Round $r(\nu^*)$. Since $O^I(\succ)(i) \neq \nu^*(i)$ for some i, this agent i must go on to Round $r(\nu^*)+1$ to swap $\rho(i) = \nu^*(i)$ against the different and strictly $\succ_i$-preferred shift $O^I(\succ)(i)$. But then we have $O^I(\succ)(i) \succ_i \rho(i) = \nu^*(i) = \mu(i)$ and μ cannot Pareto dominate $O^I(\succ$

[16] To see that a cycle ν may belong to two different such sets note that the cycle $\nu : \{2I\} \to \{2I\}$ belongs to $\{I+1-T, \ldots, 2I\}$ as well as to $\{2I+1-T, \ldots, 3I\}$.

). So ν^* cannot equal to a trading cycle that forms at Round $r(\nu^*)$. Therefore there exist two agents $j \in Dom(\nu^*)$ and $j' \neq j$ respectively matched via trading cycles ρ and ρ' at Round $r(\nu^*)$ of $O^I(\succ)$ with $\rho(j) \neq \nu^*(j) = \rho'(j')$ and ρ' forms no earlier than ρ at Round $r(\nu^*)$.[17] Since ρ forms no later than ρ' we have $\rho(j) \succ_j \rho'(j') = \nu^*(j)$. Since agent j can only improve his shift if he goes on to Round $r(\nu^*)$ we have $O^I(\succ)(j) \succsim_j \rho(j)$. By transitivity we obtain $O^I(\succ)(j) \succ_j \nu^*(j) = \mu(j)$ and μ cannot Pareto dominate $O^I(\succ)$.

To see that truthtelling is a dominant strategy for $i \in \{(r-1)I+1, \dots, rI\}$ if and only if $i \leq rI-T$, first say that $i \leq rI-T$ so that $O^I(\succ)(i) = G(\succ^r)(i)$. For any deviation $\succ_i'$ we have $O^I(\succ_i', \succ_{-i}) = G(\succ_i'', \succ_{-i}^r)(i)$ where $\succ_i''$ is the restriction of $\succ_i'$ to $\tau^r(I^r)$ the set of shifts available in trading Round r. Since G is strategy proof we have $G(\succ^r)(i) \succ_i^r G(\succ_i'', \succ_{-i}^r)(i)$. Since $\succ_i^r$ is a restriction of $\succ_i$ the preceding preference statement implies $O^I(\succ)(i) = G(\succ^r)(i) \succsim_i G(\succ_i'', \succ_{-i}^r)(i) = O^I(\succ_i', \succ_{-i})$ and truthelling is a dominant strategy for i.

If $i > rI-T$, let $j := rI-T$ and $j' := rI+1$, $\succ_j: i, j$, $\succ_i: j', j, i$, $\succ_i': j', i$, $\succ_{j'}: i, j'$, and $\succ_i = \succ_i^e$ for all other agents i. Agent i enters with Round r. If he truthfully reveals his preference he is matched with j in round r. This match is finalized in round r since $j \leq rI - T$, yielding $O^I(\succ)(i) = j$. If i instead claims his preference is $\succ_i'$ then agent i is temporarily matched with his own shift in round r. In round $r+1$ i swaps shifts with j' and we obtain $O^I(\succ_i', \succ_{-i})(i) = j'$. Since $j' \succ_i j$ agent i has an incentive to misrepresent his preference at the profile $\succ$. □

References

[1] Akbarpour, M., S. Li, and S. Oveis Gharan "Thickness and Information in Dynamic Matching Markets" *mimeo* (2016).

[2] Anderson, R. I. Ashlagi, Kanoria, Y., and Gamarik D.: "A dynamic model of barter exchange" (2015) *mimeo*

[17] Note that this includes the case where $\rho = \rho'$ and $j' \neq j$.

[3] Baccara, M., S. Lee, and L. Yariv. "Optimal Dynamic Matching" *mimeo* 2016.

[4] Bade, S.: "Pareto opitmal, strategy proof and individually rational matching with single peaked preferences" *mimeo* 2016.

[5] Bloch, F. and D. Cantala: "Dynamic assignment of objects to queuing agents", forthcoming *American Economic Journal: Microeconomics* 2017.

[6] Leshno, J. "Dynamic Matching in Overloaded Waiting Lists," (2017) *mimeo* Columbia Univeristy.

[7] Ma, J.: "Strategyproofness and the Strict Core in a Market with Indivisibilites", *International Journal of Game Theory*, 23, (1995), pp. 75-83.

[8] Roth, A.: "Incentive Compatibility in a Market with Indivisibilities" *Economics Letters*, 9, (1982), pp. 127-132.

[9] Shapley, L. and H. Scarf: "On Cores and Indivisibility", *Journal of Mathematical Economics*, 1, (1974), pp. 23 - 37.

[10] Schummer, J.: "Influencing Waiting Lists" (2015) *mimeo* Northwestern.

[11] Unver, U.: "Dynamic Kidney Exchange", *Review of Economic Studies*, (2010) 77, pp. 372-414.