Pareto optimal, strategy proof, and non-bossy matching mechanisms

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Abstract

The set of all Pareto optimal, strategy proof and non-bossy mechanisms is characterized as the set of trading and braiding mechanisms. Fix a matching problem with more than three houses and a profile of preferences. At the start of a trading and braiding mechanism at most one house is brokered; all other houses are owned. In the first trading round, owners point to their most preferred houses, the broker - if there is one - points to his most preferred owned house, and houses point to the agents who control them. At least one cycle forms. The agents in such a cycle are matched to the houses they point to. The process is repeated with the remainder. Once there are only three houses left the mechanism might turn into a braid, a device that avoids a particular matching.

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1 Introduction

I characterize the set of all Pareto optimal, strategy proof, and non-bossy matching mechanisms. There are finitely many agents and objects, called houses. Agents strictly rank any two different houses and prefer any house to homelessness. Mechanisms map profiles of preferences to matchings. A matching maps agents to houses in such a way that each agent obtains at most one house and no house is matched to two different agents. A mechanism is Pareto optimal if it maps any profile of preferences to a Pareto optimal matching. It is strategy proof if no agent can improve his match by submitting a false preference. In a non-bossy mechanism, an agent's change of preference can change some other agent's match only if it also changes the agent's own match. I call a mechanism that satisfies these three criteria *good*.

The main result of the paper (Theorem 1) is that a mechanism is good if and only if it is a *trading and braiding mechanism*.

At the start of a trading and braiding mechanism all houses are either owned or brokered. An owner might own multiple houses. There is at most one broker and he brokers at most one house. Matchings are determined through a sequence of trading cycles. To construct cycles, any house points to the agent who owns or brokers it. Any owner points to his most preferred house. The broker, if there is one, points to his most preferred house among the owned ones. (A broker may not point to the house he brokers.) At least one cycle forms. Each agent in such a cycle is matched with the house he is pointing to. All matched agents and houses exit the mechanism.

A new round of pointing ensues. A fixed rule determines the ownership and brokerage rights over all unmatched houses. This rule is such that any owner who remains unmatched continues to own (at least) the houses he owned at the start of the preceding round. Moreover any unmatched broker continues to broker the same house if at least two owners from the preceding round remain unmatched. As long as at least four houses remain unmatched, the mechanism continues as described so far.

If following some round there are exactly three houses left, the mechanism may turn into a braid. A braid is defined via an arbitrary matching that I call an *avoidance-matching*. For any profile of preferences the braid selects a Pareto optimum that minimally coincides with the avoidance matching. The name *braid* derives from the intricate and symmetric relationship between all three agent-house pairs (strands) in the avoidance matching. The trading and braiding process terminates once a matching is reached.

A mechanism is group-strategyproof if no group of agents can misstate their preferences to make at least one of its members better off while making no member worse off. Combining Papai's [14] result that a matching mechanism is group-strategyproof if and only if it is non-bossy and strategyproof with Theorem 1 it follows that the set of group-strategyproof and Pareto optimal mechanisms coincides with the set trading and braiding mechanisms.

Although the current paper is most closely related to Pycia and Ünver [16] let me first review the long tradition that has studied good mechanisms. This literature starts with the definition of two canonical mechanisms. According to serial dictatorship one agent, the first dictator, gets to choose a house out of the grand set. Then another agent, the second dictator, gets to choose out of the remainder and so forth. A second mechanism, Gale's top trading cycles (Shapley and Scarf [18]), applies to the case of equally many agents and houses: each agent starts out owning exactly one house and points to his most preferred house. At least one cycle forms. All agents in all such cycles are matched with the houses they point to. The procedure is repeated until a matching is obtained. Svensson [19] characterized serial dictatorship as the unique mechanism that satisfies strategy proofness, non-bossiness and neutrality in the sense that the outcome of the mechanism does not depend on the names of the houses. Ma [13] characterized Gale's top trading cycles as the unique Pareto optimal and strategy proof mechanism that satisfies individual rationality in the sense that every agent is matched to a house that he likes at least as much as the house he is endowed with. Svensson [19] does not not impose Pareto optimality, whereas Ma [13] does not impose non-bossiness. In the two characterizations, these two properties arise as a consequence of the respective other three criteria imposed.

Papai [14] showed a mechanism is good and reallocation proof if and only if it can be represented as a hierarchical exchange mechanism. A mechanism is reallocation proof if no two individuals can gain by misrepresenting preferences and then swapping the houses they are matched to under the mechanism. The difference between Gale's top trading cycles mechanisms and hierarchical exchange mechanisms lies in the fact that agents may own multiple houses in hierarchical exchange mechanisms. The set of hierarchical exchange mechanisms coincides with the set of trading and braiding mechanisms without brokers or braids.[1] Brokers entered the matching literature with Pycia and Ünver [16]'s trading cycles mechanisms.

The present paper would not have been possible without the work by Papai [14] and by Pycia and Ünver [16]. My proof directly employs a Lemma from Papai [14] and various steps of my proof mirror similar chapters in the proofs of Papai [14] and of Pycia and Ünver [16]. More importantly my characterization relies on the basic and ingenious ideas of constructing new good mechanisms by extending Gale's top trading cycles mechanisms to allow for the ownership of multiple houses (Papai [14]) and brokerage (Pycia and Ünver [16]).

Pycia and Ünver [16] set out with the same goal as the current paper. They defined *trading cycles mechanisms* with the aim to to characterize the set of all good mechanisms. However, the characterization by Pycia and Ünver [16] is not correct as they overlooked braids. Braids are good - but they are not representable in the framework of Pycia and Ünver [16]. So the class of Pycia and Ünver's [16] trading cycles mechanisms is strictly nested between Papai's [14] hierarchical exchange mechanisms and the set of all good mechanisms.

A variety of papers investigates how the grand set of all good mechanisms is restricted by some additional assumptions. Che, Kim and Kojima [8] for example contrast the non-existence of an ex post incentive compati-

[1] Abdulkadiroglu and Sönmez [3] defined the set of you-request-my-house-I-get-your-turn mechanisms. There mechanisms have been characterized by Sönmez and Ünver [17] as the set of all Pareto optimal, strategy proof, individually rational, weakly neutral and consistent mechanisms. Abdulkadiroglu and Sönmez [4] defined the set of all top trading cycles mechanisms, which were then characterized by Abdulkadiroglu and Che [1]. Serial dictatorships and Gale's top trading cycles mechanism are both you-request-my-house-I-get-your-turn mechanisms. The set of top trading cycles mechanisms is strictly nested between the sets of you-request-my-house-I-get-your-turn mechanisms and hierarchical exchange mechanisms.

ble and Pareto optimal mechanism when agents have interdependent values with the large set of Pycia and Ünver's trading cycles mechanisms. Velez [20] uses trading cycles mechanisms to put into relief the restrictions that arise when one imposes consistency upon good mechanisms. Ehlers and Westkamp [9] highlight the limited scope for strategyproofness when allowing for indifferences by a comparison with the large set trading cycles mechanism - all of which are strategyproof under the assumption that preferences are linear orders. In some cases, knowing the grand set, convoluted as it may be, allows us to draw stronger conclusions about simple mechanisms. Bade [6], for example, argues that the uniform randomization of any good mechanism results in a random serial dictatorship. The proof of this result heavily relies on the characterization of the set of all good mechanisms. In this case, the characterization of all good mechanisms strengthens the appeal of one - very simple - mechanism: random serial dictatorship. Similarly Bade [5] claims that with endogenous information acquisition there is exactly one ex ante Pareto optimal, strategy proof, and non-bossy mechanism: serial dictatorship. Once again the proof starts by considering the grand set of all mechanisms that satisfy these three criteria under the assumption that information is exogenously given.

The characterization of the grand set of all good mechanisms allow us to precisely identify goals that are achievable in addition to Pareto optimality, strategyproofness and non-bossiness. Braids and brokerage are design tools to avoid certain matchings. Suppose that a designer wishes to use a good mechanism to match a set of administrators to a set of tasks via good mechanism with the additional caveat that some secretary, say Amy, should not proctor exams. By letting Amy broker the task of proctoring, the designer ensures that she will only be matched to this task if all other secretaries prefer their respective matches to proctoring. Similarly if there are exactly three tasks and three secretaries there exists a good mechanism that allows the designer to specify some matching which is to be avoided: the designer can use a braid. While trading and braiding mechanisms map out a clear path as to how to incorporate avoidance goals into good mechanism, they at the same time show that not many of such goals can be accommodated simultaneously, given that braids are only defined when there are exactly three houses and given that there is at most one brokered house in any trading round that is not a braid.

2 Concepts

A housing problem consists of a set of agents $N\colon = \{1, \cdots, n\}$, a finite set of houses H and a profile of preferences $R = (R_i)_{i=1}^n$. The option to stay homeless $\emptyset$ is always available: $\emptyset \in H$. An agent's preference R_i is a linear order[2] on H and each agent prefers any house to homelessness, so $hR_i\emptyset$ holds for all $i, h \in H$. The set of all profiles of preferences is $\mathcal{R}$. The notation hR_iH' means that agent i prefers h to each house in H'. If an agent i holds a preference $eR_igR_iH \setminus \{e, g\}$, I write $R_i : e\ g$. For any fixed profile R, the preferences of a group of agents $G \subset N$ is denoted R_G, similarly R_{-G} denotes the preferences of all agents not in this group, so we have $R = (R_G, R_{-G}) = (R_i, R_{-i})$. The profile $\overline{R}$ is the **restriction of** R **to** some $N' \subset N$ and $H' \subset H$ if $eR_ig \Leftrightarrow e\overline{R}_ig$ for $g, h \in H'$ and $i \in N'$. Two preferences R_i, R'_i **coincide on** $H' \subset H$ if $eR_ig \Leftrightarrow eR'_ig$ holds for all $e, g \in H'$. So R_i, R'_i coincide on H' if and only if their restrictions to H' are identical.

Submatchings match subsets of agents to at most one house each. A **submatching** is a function $\nu : N \to H$ such that $\nu(i) = \nu(j)$ and $i \neq j$ imply $\nu(i) = \emptyset$. The sets of agents and houses matched under ν are $N_\nu\colon = N \setminus \nu^{-1}(\emptyset)$ and $H_\nu\colon = \nu(N_\nu)$. When $\nu(i) \neq \emptyset$ then $\nu(i)$ is the house that agent i is matched to under ν; $\overline{N}_\nu\colon = N \setminus N_\nu$ and $\overline{H}_\nu\colon = H \setminus H_\nu$ are the sets of agents and houses not matched by ν. When convenient I interpret a submatching ν as the set of agent-house pairs $\{(i, h) : \nu(i) = h \neq \emptyset\}$. For two submatchings ν and ν' with $N_\nu \cap N_{\nu'} = \emptyset = H_\nu \cap H_{\nu'}$ the submatching $\nu \cup \nu' : N_\nu \cup N_{\nu'} \to H_\nu \cup H_{\nu'}$ is defined by $(\nu \cup \nu')(i) = \nu(i)$ if $i \in N_\nu$ and $(\nu \cup \nu')(i) = \nu'(i)$ otherwise. A submatching μ is a **matching** if $H_\mu = H$ or $N_\mu = N$ (or both) hold, the set of all matchings is $\mathcal{M}$.

A **mechanism** is a function $M : \mathcal{R} \to \mathcal{M}$. The outcome of M at R, $M(R)$, matches agent i to house $M(R)(i)$. A mechanism M is **Pareto opti-**

[2]So hR_ih' and $h'R_ih$ together imply $h = h'$.

mal if for no R there exists a matching $\mu' \neq M(R)$ such that $\mu'(i) R_i M(R)(i)$ for all i.[3] A mechanism M is **strategy proof** if $M(R)(i) R_i M(R'_i, R_{-i})(i)$ holds for all $R \in \mathcal{R}$, all R'_i and all agents i: declaring one's true preferences is a weakly dominant strategy in a strategy proof mechanism. A mechanism M is **non-bossy** if $M(R)(i) = M(R'_i, R_{-i})(i)$ implies $M(R)(j) = M(R'_i, R_{-i})(j)$ for all R, R'_i and all $i, j \in N$, so an agent can only change someone else's match if he also changes his own match. A Pareto optimal, strategy proof and non-bossy mechanism is a **good**.

3 Braids

A **braid** $B : \mathcal{R} \to \mathcal{M}$ is a mechanism for a problem with exactly three houses and at least as many agents. It is fully defined through an **avoidance matching** ω. Matchings $B(R)$ are chosen to avoid matching i to $\omega(i)$ while keeping the set of matched agents equal to the set of agents matched under ω. For any R let $\overline{PO}(R)$ be the set of Pareto optima μ with $N_\omega = N_\mu$. If $\min_{\mu \in \overline{PO}(R)} \mid \{i : \mu(i) = \omega(i)\} \mid$ is attained at a unique μ^* then let $B(R) = \mu^*$. If not, at least two agents in N_ω must rank some house $h^* = \omega(i^*)$ at the top and the pair (i^*, h^*) is decisive in the following sense. If only one agent $j \neq i^*$ ranks h^* at the top then $B(R)$ is the unique minimizer that matches j to h^*. If both agents $i \neq i^*$ rank h^* at the top, then $B(R)$ is the unique minimizer preferred by i^*.

To concretely illustrate braids let $H = \{e, g, g'\}$ and $\mid N \mid \geq 3$. Since any agent i stays unmatched under the avoidance matching ($\omega(i) = \emptyset$) if and only if he stays unmatched at any outcome of the braid ($B(R)(i) = \emptyset$ for all R) and since B is non-bossy it is w.l.o.g to assume that $N = \{1, 2, 3\}$. Given $\mid H \mid = \mid N \mid = 3$ it is convenient to denote matchings as vectors with the understanding that the i-th component of such a vector represents agent i's match. Moreover given that there are only three agents, the requirement $N_\omega = N_\mu$ for any $\mu = B(R)$ is automatically satisfied and can therefore be ignored. Let $\omega \colon = (e, g', g)$. There are exactly two matchings $\omega' \neq \omega''$ with $\omega'(i) \neq \omega(i) \neq \omega''(i)$ for all i. Let $\omega' \colon = (g, e, g')$, and $\omega'' \colon = (g', g, e)$. If

[3] Since all R_i are linear at least one agent must strictly prefer $\mu'(i)$ to $M(R)(i)$ if $\mu' \neq M(R)$.

$R_i : e\ g$ for $i = 1, 2, 3$ then $\min_{\mu \in PO(R)} \mid \{i : \omega(i) = \mu(i)\} \mid$ is attained both at ω' and ω''. Since at least two agents rank house $e = \omega(1)$ at the top, agent 1 is decisive. Agent 1's preference of g over g' implies $B(R)(1) = g$ and $B(R) = \omega'$. Under (R'_2, R_{-2}) with $R'_2 : g\ g'$ (and R_i as above) there are four Pareto optima: $\omega, (e, g, g'), (g, g', e)$ and ω''. Since ω'' is the unique Pareto optimum with $\mid \{i : \omega(i) = \mu(i)\} \mid = 0$, $B(R)$ equals ω''.

Lemma 1 *The braid B is good.*

Proof W.l.o.g fix ω, ω' and ω'' as in the preceding paragraph and assume that $N = \{1, 2, 3\}$.

(*) If R is such that for some $j \neq j'$, $R_j : \omega'(j)$, $\omega'(j') R_{j'} \omega''(j')$ and $\omega(j') = \omega'(j)$ hold, then $B(R) = \omega'$. To prove (*) assume w.l.o.g that $j = 2$, so $\omega'_j(2) = e$, $\omega(1) = e$, $j' = 1$ and $\omega'(1) = g$. Since ω' is Pareto optimal at R $B(R) = \omega'$ holds if $\omega'' \notin PO(R)$. If $\omega'' \in PO(R)$ then $R_3 : e$ has to hold. But then agent 1 is decisive and $B(R) = \omega'$ holds since $g = \omega'(1) R_1 \omega''(1) = g'$

(**) If $B(R)(i) = \omega(i)$, then i is the only agent who under R ranks $\omega(i)$ at the top.

Any profile R is mapped to a Pareto optimum according to $B(R)$, so B is Pareto optimal.

To see that B is strategy proof fix a profile R an agent i and a deviation R'_i. If there is a unique Pareto optimum at R each agent obtains his most preferred house and we have $B(R)(i) R_i B(R'_i, R_{-i})$. Now consider the case that at least two agents rank the same house at the top. W.l.o.g let this house be $e = \omega(1)$.

Case 1: $g R_1 g'$ and $R_2 : e$. Then (*) implies $B(R) = \omega'$ and (**) implies that g is the best attainable house for agent 1 given R_{-1}. Agent 2 is matched to his most preferred house under $B(R)$. Finally (*) implies $B(R) = B(R'_3, R_{-3})$ for all R'_3. So $B(R)(i) R_i B(R'_i, R_{-i})$ holds. Mutatis mutandis, the same reasoning implies that $B(R)(i) R_i B(R'_i, R_{-i})$ also holds in Case 2: $g' R_1 g$ and $R_3 : e$.

Case 3: $g R_1 g'$ but not $R_2 : e$. Since at least two agents rank e at the top $R_1 : e\ g$ and $R_3 : e$ must hold. If $R_2 : g'$, then $(g, g', e) \in PO(R) \subset \{\omega, (e, g, g'), (g, g', e)\}$, where the latter two matchings minimize

$| \{i : \omega(i) = \mu(i)\} |$. Observation (**) implies $B(R) = (g, g', e)$. Since agents 2 and 3 obtain their most preferred house, they have no incentive to deviate. Moreover Observation (**) implies that $B(R_1', R_{-1})(1) \neq e$, and consequently agent 1 has no incentive to deviate. If $R_2 : g$, $\omega'' \in PO(R)$ but $\omega' \notin PO(R)$, so $B(R) = \omega''$. Once again agents 2 and 3 obtain their most preferred house and have no incentive to deviate. Observation (**) implies that $B(R_1', R_{-1})(1) \neq e$. On the one hand we have $\omega'' \in PO(R_1', R_{-1})$ for all R_1'. On the other hand ω' is only an element of $PO(R_1', R_{-1})$ if $R_1 : g$. But then agent 3 is decisive as $\omega(g) = 3$. Since $g' R_3 e$ $B(R_1', R_{-1})(3) = g'$ and therefore $B(R_1', R_{-1}) = \omega'$. So $B(R)(i) R_i B(R_i', R_{-i})$ holds when $g R_1 g'$ but not $R_2 : e$. The same arguments apply mutatis mutandis to Case 4: $g' R_1 g$ but not $R_3 : e$. In sum B is strategy proof.

To see that B is non-bossy, let $B(R)(1) = B(R_1', R_{-1})(1) :\ = h$, so $B(R) = \{(1, h)\} \cup \nu^*$ and $B(R_1', R_{-1}) = \{(1, h)\} \cup \nu^\circ$ for some submatchings ν^*, ν°. Since B is Pareto optimal $\nu^*, \nu^\circ \in PO(\overline{R})$ where $\overline{R}$ is the restriction of agent 2 and 3's preference to $H \setminus \{h\}$. If $PO(\overline{R})$ is a singleton then $\nu^* = \nu^\circ$ and we are done. So suppose $PO(\overline{R})$ is not a singleton. If $h = \omega(1)$, then $PO(\overline{R}) = \{\nu, \nu'\}$ where $\nu(2) = \omega(2)$ and $\nu'(2) = \omega(3)$. So $\{(1, h)\} \cup \nu = \omega$ and therefore $\min_{\mu \in PO(R)} | \{i : \mu(i) \neq \omega(i)\} |$ can only be attained at $\{(1, h)\} \cup \nu'$, implying $\nu' = \nu^* = \nu^\circ$. If $h \neq \omega(1)$, then $PO(\overline{R}) = \{\nu, \nu'\}$ is such that $\{(1, h)\} \cup \nu' \in \{\omega', \omega''\}$ and $\{(1, h)\} \cup \nu \notin \{\omega', \omega''\}$. Once again $\min_{\mu \in PO(R)} | \{i : \mu(i) \neq \omega(i)\} |$ can only be attained at $\{(1, h)\} \cup \nu'$, implying $\nu' = \nu^* = \nu^\circ$. Having covered all cases we can conclude that $B(R)(1) = B(R_1', R_{-1})(1)$ implies $B(R) = B(R_1', R_{-1})$ and B is non-bossy. $\square$

4 Trading and braiding mechanisms

A **control rights function** at some submatching ν $c_\nu : \overline{H}_\nu \to \overline{N}_\nu \times \{o, b\}$ assigns control rights over any unmatched house to some unmatched agent and specifies a type of control. If $c_\nu(h) = (i, x)$, then agent i **controls** house h at ν. If $x = o$, then i **owns** h; if $x = b$ he **brokers** h. Control rights functions satisfy the following three criteria:

(C1) If more than one house is brokered, then there are exactly three houses and they are brokered by three different agents.
(C2) If exactly one house is brokered then there are at least two owners.
(C3) No broker owns a house.

A **general control rights structure** c maps a set of submatchings ν to control rights functions c_ν. For now just assume that c is defined for sufficiently many submatchings to ensure that the following algorithm is well defined for any fixed R.

Initialize with $r = 1$, $\nu_1 = \emptyset$

Round r: only consider the remaining houses and agents $\overline{H}_{\nu_r}$ and $\overline{N}_{\nu_r}$.

Braiding: If more than one house is brokered under c_{ν_r} let B be the braid defined by the avoidance matching ω with $c_{\nu_r}(\omega(i)) = (i, b)$. Terminate the process with $M(R) = \nu_r \cup B(\overline{R})$ where $\overline{R}$ is the restriction of R to $\overline{H}_{\nu_r}$ and $\overline{N}_{\nu_r}$. If not, go on to the next step.

Pointing: Each house points to the agent who controls it, so $h \in \overline{H}_{\nu_r}$ points to $i \in \overline{N}_{\nu_r}$ with $c_{\nu_r}(h) = (i, \cdot)$. Each owner points to his most preferred house, so owner $i \in \overline{N}_{\nu_r}$ points to house $h \in \overline{H}_{\nu_r}$ if $hR_i\overline{H}_{\nu_r}$. Each broker points to his most preferred owned house, so broker $j \in \overline{N}_{\nu_r}$ with $c_{\nu_r}(h_b) = (j, b)$ points to house $h \in \overline{H}_{\nu_r} \setminus \{h_b\}$ if $hR_j\overline{H}_{\nu_r} \setminus \{h_b\}$.

Cycles: Select at least one cycle. Define ν° such that $\nu^\circ(i) :\, = h$ if i points to h in one of the selected cycles.

Continuation: Define $\nu_{r+1} :\, = \nu_r \cup \nu^\circ$. If ν_{r+1} is a matching terminate the process with $M(R) = \nu_{r+1}$. If not, continue with round $r + 1$.

The trading and braiding process starts with the initial submatching $\nu_1 = \emptyset$. If c does not call for a braid at ν_1, each house points to the agent who controls it according to c_{ν_1}. Each owner points to his most preferred house, and the only broker (if there is one) points to his most preferred house among the owned houses. At least one cycle forms. All agents and houses in some chosen cycles are matched, yielding the submatching ν_2. A new round of either braiding or pointing cycles ensues. Once a matching is reached the process terminates.

A submatching ν is **reachable under** c at R if some round of a trading and braiding process can start with ν. A submatching ν is c**-relevant** if it is reachable under under c at some R.[4] A submatching ν' is a **direct** c**-successor of** of some c-relevant ν if there exists a profile of preferences R such that ν is reachable under $c(R)$ and ν' arises out of matching a single cycle at ν. A **control rights structure** c maps any c-relevant submatching ν to a control rights function c_ν and satisfies requirements (C4), (C5), and (C6).

Fix a c-relevant submatching ν° together with a direct c-successor ν.

(C4) If $i \notin N_\nu$ owns h at ν° then i owns h at ν.
(C5) If at least two owners at ν° remain unmatched at ν and if $i \notin N_\nu$ brokers h at ν° then i brokers h at ν.
(C6) If i owns h at ν° and ν and if $i' \notin N_\nu$ brokers h' at ν° but not at ν, then i owns h' at ν and i' owns h at $\nu \cup \{(i, h')\}$.

In a braid no agent owns any house. Consequently, c can only map a submatching ν to a braid (a control right function c_ν with three brokered houses), if (C4), (C5) and (C6) do not require any ownership at ν. The trading and braiding mechanism defined by the control rights structure c is also denoted c and $c(R)$ is the outcome of the trading and braiding mechanism c at the profile of preferences R.

Any c-relevant submatching ν defines a **submechanism** $c[\nu]$ that maps restrictions $\overline{R}$ (of some $R \in \mathcal{R}$ to $\overline{N}_\nu$ and $\overline{H}_\nu$ to submatchings) μ' with the feature that $\nu \cup \mu'$ is a matching in the original problem. The control rights structure defining $c[\nu]$ is such that $\nu^\circ = \nu \cup \nu'$ is c-relevant if and only if ν'

[4] Consider a control rights structure c with three agents $\{1, 2, 3\}$ and 4 houses $\{h, g, k, h'\}$, where agent 1 starts out owning house h and g and agent 2 starts out owning the remaining houses. Suppose the profile of preferences R is such that 1 most prefers h, and 2 most prefers h'. Then the following submatchings involving agents 1 and 2 are reachable under $c(R)$: $\{(1, h)\}, \{(2, h')\}$ and $\{(1, h), (2, h')\}$. The submatching $\{(1, g)\}$ is c-relevant since agent 1 could appropriate house g, but not it is not reachable under $c(R)$ given that 1 prefers h to g. The submatching $\{(3, h)\}$ is not c-relevant since 3 does not own any house at the start of the mechanism.

is $c[\nu]$-relevant. For any such pair ν°, ν' we have $c[\nu]_{\nu'} = c_{\nu^\circ}$. It is easy to check that $c[\nu]$ also defines a trading and braiding mechanism. Fixing any c, R, and ν that is reachable under $c(R)$, the definition of the trading-cycles process implies $c(R) = \nu \cup c[\nu](\overline{R})$, where $\overline{R}$ is the restriction of the profile of preferences R to the set of agents $\overline{N}_\nu$ and the houses $\overline{H}_\nu$.

5 Theorem

Theorem 1 *Any good mechanism has a unique representation as a trading and braiding mechanism. Any trading and braiding mechanism is good.*

A trading and braiding mechanism is a hierarchical exchange mechanism following Papai [14] if there are no brokers or braids. Just as in hierarchical exchange mechanisms, owners in trading and braiding mechanisms are free to either appropriate or trade any house he owns. A trading a braiding cycles mechanism is a trading cycles mechanism following Pycia and Ünver [16] if it does not have any braids. So Pycia and Ünver's [16] notion of brokerage is identical to the present one. A broker may not appropriate the house he brokers. He is however free to exchange this house as he pleases. Moreover when ν is not mapped to a braid there is at most one broker at any given round and he brokers at most one house.

There is one important formal difference between the definition of trading and braiding mechanisms and its predecessors: Trading and braiding mechanisms only require that at least one cycle is matched in any one round. All other definitions based on sequences of trading cycles, that I am aware of, require the matching of all cycles at any given round. The disadvantage of the current definition is that it is a priori not clear that all different orders of elimination of trading cycles lead to the same outcome. I therefore need to show that trading and braiding mechanisms are well-defined (see Section 6.1). But once this proof is out of the way the proofs that trading and braiding mechanisms are strategy proof and non-bossy turn out to be much easier.[5]

Pycia and Ünver [16] introduced the notion of control rights functions into the matching literature. The approach rendered the definition of any mechanism that use sequences of trading rounds significantly simpler. With my definition of trading and braiding mechanisms I follow their lead. However, I only require that control rights functions c_ν be defined on c-relevant submatchings while Pycia and Ünver's [16] control rights structures are defined on *all* submatchings that are not themselves matchings. Moreover the present requirement (C2) that there are at least two owners when there is a broker, strengthens Pycia and Ünver's [16] requirement (R2) that an agent must own all unmatched houses if he is the only unmatched agent. The latter two differences allow for the uniqueness statement in Theorem 1. If one was to replace (C2) as stated here with Pycia and Ünver's [16] (R2) the class of mechanisms described would not change. However the uniqueness result in Theorem 1 would no longer hold.

6 Proof

I prove Theorem 1 by induction over the number of agents n.

Start: If $n = 2$ then any good mechanism is either a serial dictatorship or Gale's top trading cycles mechanism, both of which are trading and braiding mechanisms.

Suppose that trading and braiding mechanisms with n agents are well-defined. Suppose that any good mechanism with n agents can be represented as a unique trading and braiding mechanism and that any trading and braiding mechanism with n agents is good. Fix a control rights structure c with

[5] Bade's [6] proof that the symmetrization of any good mechanism is equivalent to random serial dictatorship relies on the flexibility to eliminate trading cycles in any which order. Carroll [7] also uses this flexibility in his proof that the symmetrization of any top trading cycles mechanism is equivalent to random serial dictatorship. Since the definition of top trading cycles mechanism requires the immediate elimination of all trading cycles at any stage, Carroll [7] provides a proof of the equivalence of all orders of elimination of trading cycles in top trading cycles mechanisms.

$n+1$ agents that is not itself a braid and a profile of preferences R. In Section 6.1 I show that the order of the elimination of trading cycles does not matter for the outcome $c(R)$, so c indeed defines a mechanism. Section 6.2 shows that c is good. Section 6.3 shows that the representation is unique.

Section 6.4 lists a set of arguments that are repeatedly used in the upcoming proof that any good mechanism M can be represented as a trading and braiding mechanism. For M to be representable as a trading and braiding mechanism c one needs to pinpoint for any house an agent who controls this house at $\emptyset$. Here I follow the proof of Pycia and Ünver [16] and define a function $c_\emptyset : H \to N \times \{o, b\}$ in Section 6.5. Section 6.6 shows that this function $c_\emptyset$ shares a set of useful properties with control rights functions as defined in Section 4. Section 6.7 shows that braids are the only alternative to trading rounds, so $c_\emptyset$ satisfies (C1). In the same section I show that $c_\emptyset$ also satisfies (C2) and (C3), implying that $c_\emptyset$ is indeed a control rights function. Section 6.8 shows that the outcome of $M(R)$ when M is not a braid is consistent with the submatching achieved in a first trading round under $c_\emptyset$ at that R. The inductive hypothesis is used to show that any submatching that can be reached via the formation of a single cycle at $c_\emptyset$ under any R is followed by a well-defined trading and braiding (sub-)mechanism $c[\nu]$. These submechanisms $c[\nu]$ together with $c_\emptyset$ define a control rights structure c. Section 6.9 finishes the proof by showing that c satisfies the remaining requirements (C4), (C5) and (C6) that link control rights between rounds.

6.1 Trading and braiding mechanisms are well-defined

Fix a control rights structure c for $n+1$ agents and a profile of preferences R. To see that the order of the elimination of trading cycles does not matter let ν_s and ν_a arise out of matching respectively an arbitrary (non-empty) subset or all of the cycles at $\emptyset$. It suffices to show that ν_a is reachable after the agents and houses in some chosen cycles are matched to obtain ν_s. The hypothesis of the induction implies that the order of elimination does not matter in $c[\nu_a]$. If there is exactly one cycle at $\emptyset$, $\nu_s = \nu_a$ must hold and we are done. So suppose there are at least two cycles at $\emptyset$ and let $\nu_s \subsetneq \nu_a$. Since no broker may point to the house he brokers at least one agent in $N_{\nu_a} \setminus N_{\nu_s}$ is an owner at $\emptyset$. Given (C4) this agent must also be an owner at ν_s. Therefore c cannot map ν_s to a braid.

Case I: $c_\emptyset(h) = c_{\nu_s}(h)$ holds for all $h \in H_{\nu_a} \setminus H_{\nu_s}$. So any $h \in H_{\nu_a} \setminus H_{\nu_s}$ points to the same agent at $\emptyset$ and at ν_s. Moreover any agent $i \in N_{\nu_a} \setminus N_{\nu_s}$ points to the same house at $\emptyset$ and at ν_s (since $h^* R_i H$ implies $h^* R_i \overline{H}_{\nu_s}$ and since $h^* R_i H \setminus \{h_b\}$ implies $h^* R_i \overline{H}_\nu \setminus \{h_b\}$, which is relevant if i brokers h_b). So any cycle at $\emptyset$ that is not immediately eliminated remains a cycle at ν_s. By the inductive hypothesis the order of the elimination of cycles does not matter in the submechanism $c[\nu_s]$. We can start by matching all cycles that already existed at $\emptyset$ and ν_a is reachable.

Case II: $c_\emptyset(h) \neq c_{\nu_s}(h)$ holds for some $h \in H_{\nu_a} \setminus H_{\nu_s}$. By (C4) h cannot be owned at $\emptyset$ so $c_\emptyset(h) = (i_b, b)$ holds for some i_b. As a broker i_b must point to some owned house h^* at $\emptyset$. Let $c_\emptyset(h^*) = (i^*, o)$. By (C4) agent i^* also own h^* at ν_s. By (C5) i^* is the only agent who owns houses at $\emptyset$ and ν_s. In sum, the cycle $i_b \to h^* \to i^* \to h_b$ forms at $\emptyset$ and $\nu_a = \nu_s \cup \{(i^*, h), (i_b, h^*)\}$. By (C6) i^* owns h at ν_s. Since i^* points to h at $\emptyset$ we have $h R_{i^*} H$ and consequently $h R_{i^*} \overline{H}_{\nu_s}$. So at ν_s, h points to i^* and i^* to h. By the inductive hypothesis the order of the elimination of cycles does not matter in $c[\nu_s]$, so $\nu_s \cup \{(i^*, h)\}$ is reachable. (C6) implies that at $\nu_s \cup \{(i^*, h)\}$ agent i_b owns all houses owned by i^* at $\emptyset$, in particular house h^*. Given that $h^* R_{i_b} H \setminus \{h\}$ implies $h^* R_{i_b} \overline{H}_{\nu_s} \setminus \{h\}$ a cycle just involving i_b and h^* forms. By the inductive hypothesis we may eliminate this one cycle and $\nu_a = \nu_s \cup \{(i^*, h), (i_b, h^*)\}$ is reachable.

6.2 Trading and braiding mechanisms are good

Let $\mathcal{N}(c_\emptyset)$ be the the set of direct c-successors to $\emptyset$ and let $\mathcal{N}(c_\emptyset)(R)$ be the subset of direct c-successors to $\emptyset$ that are reachable under $c(R)$. Since these sets coincide for any two different trading and braiding mechanisms c, c' with $c_\emptyset = c'_\emptyset$ only the control rights function $c_\emptyset$ is used to define $\mathcal{N}(c_\emptyset)$ and $\mathcal{N}(c_\emptyset)(R)$. Fix an agent i and a preference R'_i. For any $\nu \in \mathcal{N}(c_\emptyset)$ let $\overline{R}$ and $\overline{R}'_i$ be the restrictions of R and R'_i to $\overline{N}_\nu, \overline{H}_\nu$. If $\nu \in \mathcal{N}(c_\emptyset)(R)$ the definition of the trading process implies that $c(R) = \nu \cup c[\nu](\overline{R})$ and $c(R'_i, R_{-i}) = \nu \cup c[\nu](\overline{R}'_i, \overline{R}_{-i})$ if $i \notin N_\nu$. The proof that c is strategyproof and non-bossy of c is split into two cases.

Case I: There exists some $\nu \in \mathcal{N}(c_\emptyset)(R)$ with $i \notin N_\nu$. Fix such a ν. Since $c[\nu]$ involves at most n agents, it is strategy proof by the inductive hypothesis and $c[\nu](\overline{R})(i)\overline{R}_i c[\nu](\overline{R}_i', \overline{R}_{-i})(i)$ holds. Given that R_i and $\overline{R}_i$ coincide on $\overline{H}_\nu$, $c[\nu](\overline{R})(i) = c(R)(i)$, and $c[\nu](\overline{R}_i', \overline{R}_{-i})(i) = c(R_i', R_{-i})(i)$ we have $c(R)(i)R_i c(R_i', R_{-i})(i)$. By the inductive hypothesis $c[\nu]$ is non-bossy and $c[\nu](\overline{R})(i) = c[\nu](\overline{R}_i', \overline{R}_{-i})(i)$ implies $c[\nu](\overline{R}) = c[\nu](\overline{R}_i', \overline{R}_{-i})$. Consequently $c(R)(i) = c(R_i', R_{-i})(i)$ (which holds if and only if $c[\nu](\overline{R})(i) = c[\nu](\overline{R}_i', \overline{R}_{-i})(i)$) implies $c(R) = \nu \cup c[\nu](\overline{R}) = \nu \cup c[\nu](\overline{R}_i', \overline{R}_{-i}) = c(R_i', R_{-i})$.

Case II: The only $\nu \in \mathcal{N}(c_\nu)(R)$ is such that $i \in N_\nu$. Suppose there existed a cycle not involving i at $\emptyset$ under $c(R_i', R_{-i})$. Since the preferences of all agents other than i are identical under R and (R_i', R_{-i}) this cycle would also exist at $\emptyset$ under $c(R)$ contradicting the assumption that $\{\nu\} = \mathcal{N}(c_\emptyset)(R)$. Since there must be some cycle at $\emptyset$ under $c(R_i', R_{-i})$, i is part of such a cycle and his match $c(R_i', R_{-i})(i)$ is the house that he points to at $\emptyset$ under $c(R_i', R_{-i})$. If $c(R_i', R_{-i})(i) = c(R)(i)$ then the same cycle forms at $\emptyset$ under $c(R_i', R_{-i})$ and under $c(R)$ and we obtain $c(R_i', R_{-i}) = \nu \cup c[\nu](\overline{R}) = c(R)$. So c is non-bossy. Since $c(R)(i)$ is the R_i-best house among all houses that i may point to at $\emptyset$ under c, $c(R)(i)R_i c(R_i', R_{-i})(i)$ must hold and c is strategyproof.

To see that c is Pareto optimal fix any $\nu \in \mathcal{N}(c_\emptyset)(R)$. By (C1) $\nu(i)R_i H$ is violated for at most one agent in N_ν. This agent is a broker. Let $c_\emptyset(h_b) = (i_b, b)$. Since the brokered house h_b is the only house that the broker may not point $R_{i_b} : h_b\ \nu(i_b)$ must hold. Since $i_b \in N_\nu$, $\nu(i^*) = h_b$ holds for some other agent $i^* \in N_\nu$. Since i^* is an owner $h_b R_{i^*} H$ holds. So we cannot make i_b any better off without making i^* worse off, implying that we cannot make any agent in N_ν any better off without making some agent in N worse off. Since $c(R) = \nu \cup c[\nu](\overline{R})$ and since $c[\nu]$ is Pareto optimal by the inductive hypothesis, there is no μ that Pareto dominates $c(R)$.

6.3 Uniqueness

Let c' be another control rights structure that defines the same mechanisms as c. If c and c' are both braids derived from different avoidance matchings or if only one of the two mechanisms is a braid, then c and c' define different mechanisms. So suppose that according to $c_\emptyset$ and $c'_\emptyset$ there is at most one brokered house. Fix any $e \in H$. If $c_\emptyset(e) = (j, o)$ and $c'_\emptyset(e) = (j', o)$ holds for some $j \neq j'$, then $c(R^*)(e) = j \neq j' = c'(R^*)(e)$ holds for $R^*_i : e$ for all i (in each case the owner of the universally most preferred house e points to it in the first round).

Now let $c_\emptyset(e) = (j, b) \neq c'_\emptyset(e)$. If $c'_\emptyset(e) = (j', o)$ (possibly $j = j'$), then (C2) implies that $c_\emptyset(g) = (k, o)$ holds for some $g \neq e$ and $k \notin \{j, j'\}$. Letting $R^g_i : e\ g$ for all i we obtain $c(R^g)(k) = e$ and $c'(R^g)(j') = e$. If $c'_\emptyset(e) = (j', b)$ with $j \neq j'$, then $c(R^g)(j) = g = c'(R^g)(j')$ holds. In sum we obtain that $c_\emptyset = c'_\emptyset$ must hold for c and c' to define the same mechanism. Since $c_\emptyset = c'_\emptyset$, ν is a direct c-successor of $\emptyset$ if and only if it is a direct c'-successor of $\emptyset$. The hypothesis of the induction implies that $c[\nu]$ is identical to $c'[\nu]$ for any such $\nu \in \mathcal{N}(c_\emptyset)$.

6.4 A collection of arguments

Fix an arbitrary good mechanism M, a profile of preferences R, and a deviation R'_i. Let $M(R)(i) = e$. The following arguments are used throughout the next sections. Strategy proofness implies that nothing changes for agent i when he ranks $e = M(R)(i)$ at least as high under R'_i as under R_i:

SP-I If $eR_ih \Rightarrow eR'_ih$ for all $h \in H$, then $M(R'_i, R_{-i})(i) = e$.

Since M is non-bossy we additionally obtain:

SP-NB If $eR_ih \Rightarrow eR'_ih$ for all $h \in H$, then $M(R'_i, R_{-i}) = M(R)$.

If R'_i and R_i differ only on the relative ranking of two houses, we obtain:

SP-II If eR_ig, gR'_ie and R'_i coincides with R_i on $H \setminus \{e, g\}$ for some $g \neq e$, then $M(R'_i, R_{-i})(i) \in \{e, g\}$.

In combination with Pareto optimality the preceding observation yields

SP-PO If eR_ig, gR'_ie and R'_i coincides with R_i on $H \setminus \{e, g\}$ for some $g \neq e$ and if $M(R)$ is not Pareto optimal at (R'_i, R_{-i}), then $M(R'_i, R_{-i})(i) = g$.

Finally I also use Lemma 5 from Papai [14]

L5-Papai If j is such that $M(R)(j)R_iM(R)(i)$ then $M(R)(j)R_jM(R'_i, R_{-i})(j)$.

6.5 The Definition of $c_\emptyset$

Define a function $c_\emptyset : H \to N \times \{o, b\}$ as follows. Fix some house e and for any $g \neq e$ define R^g such that $R^g_i : e\ g$ holds for all i. If $M(R^g)(i) = e$ holds for all R^g with $g \neq e$ then let $c_\emptyset(e) = (i, o)$. If not let $c_\emptyset(e) = (j, b)$ where j is such that $M(R^g)(j) = g$. The goal of the current section is to show that $c_\emptyset$ is well-defined. Lemma 2 shows that if $M(\hat{R}^g)(i) = e$ holds for a particular $\hat{R}^g$ with $\hat{R}^g : e\ g$ then $M(R^g)(i) = e$ holds for any R^g, implying that we either have $c_\emptyset(e) = (\cdot, o)$ or $c_\emptyset(e) = (\cdot, b)$, not both. If $c_\emptyset(e) = (\cdot, o)$ then there is a unique agent i for whom $M(R^g)(i) = e$ holds for all g and $c_\emptyset(e) = (i, o)$ is well-defined. Finally Lemma 3 shows that there exists a unique agent i_b who obtains the second best house in any profile R^g when $c_\emptyset(e) = (\cdot, b)$ and $c_\emptyset(e) = (i_b, b)$ is well-defined. Consequently $c_\emptyset$ is a well-defined function.

Lemma 2 *Fix a set of $m+2$ different houses $\{h_1, \cdots, h_m, e, g\}$. Let R and R^* be such that $R_i : h_1\ \cdots h_m\ e\ g$ and $R^*_i : h_1\ \cdots h_m\ e\ g$ for all i. Then $M(R)(i^*) = e$ implies $M(R^*)(i^*) = e$.*

Proof Suppose we had $M(R)(2) = e$ and $M(R'_1, R_{-1})(3) = e$ for some $R'_1 : h_1\ \cdots h_m\ e\ g$. Since $M(R) \neq M(R'_1, R_{-1})$ and since M is non-bossy, $g' := M(R)(1)$ must differ from $h' := M(R'_1, R_{-1})(1)$. Since $R_1 : h_1\ \cdots h_m\ e\ g$ and $R'_1 : h_1\ \cdots h_m\ e\ g$ strategy proofness implies that $g', h' \notin \{h_1 \cdots h_m, e, g\}$. By SP-NB it is w.l.o.g to assume that $R_1 : h_1\ \cdots h_m\ e\ g\ g'\ h'$ and $R'_1 : h_1\ \cdots h_m\ e\ g\ h'\ g'$. Letting $M(R)(i) = g$ observe that $i \neq 1, 2$ since $M(R)(2) = e \neq g \neq g' = M(R)(1)$. If $i = 3$ a violation of L5-Papai would arise, in that case we would have $g = M(R)(3) R_1 M(R)(1) = g'$ and $e = M(R'_1, R_{-1})(3) R_3 M(R)(3)$. Assume that $i = 4$ and $R_4 : h_1\ \cdots h_m\ e\ g\ g'$ (which is by SP-NB, w.l.o.g.). Define $R'_i : h_1\ \cdots h_m\ g\ e$ for $i = 2, 3$ and $R'_4 : h_1\ \cdots h_m\ e\ g'\ g$ to coincide with R_i on $H \setminus \{h_1, \cdots, h_m, e, g\}$ and respectively $H \setminus \{h_1, \cdots, h_m, e, g, g'\}$.

SP+NB implies

$$(A) : M(R) = M(R'_3, R_{-3}).$$

SP-PO together with $M(R'_3, R_{-3})(2) = e$ imply

$$(B) : M(R'_{\{2,3\}} R_{\{1,4\}})(2) = g.$$

Since $M(R'_{\{2,3\}}, R_{\{1,4\}})(4) \neq g$, SP-NB implies $M(R'_{\{2,3\}}, R_{\{1,4\}}) = M(R'_{-1}, R_1)$ and particularly

$$(C) : M(R'_{-1}, R_1)(2) = g.$$

SP-PO and (A) yield $M(R'_{\{3,4\}}, R_{\{1,2\}})(4) = g'$ which together with SP-NB implies

$$(D) : M(R'_{\{3,4\}}, R_{\{1,2\}}) = M(R'_{-2}, R_2).$$

SP+NB also implies $M(R'_1, R_{-1}) = M(R'_{\{1,2\}}, R_{\{3,4\}})$, particularly $M(R'_{\{1,2\}}, R_{\{3,4\}})(3) = e$, which - given SP-II - implies

$$(E) : M(R'_{-4}, R_4)(3) \in \{e, g\}.$$

SP-PO and the initial assumption that $M(R'_1, R_{-1})(3) = e$ imply

$$(F) : M(R'_{\{1,3\}}, R_{\{2,4\}})(3) = g.$$

Since $M(R'_{\{1,3\}}, R_{\{2,4\}})(4) \neq g$, SP-NB implies $M(R'_{\{1,3\}}, R_{\{2,4\}}) = M(R'_{-2}, R_2)$. So we may extend (D) to

$$(D') : M(R'_{\{3,4\}}, R_{\{1,2\}}) = M(R'_{-2}, R_2) = M(R'_{\{1,3\}}, R_{\{2,4\}}).$$

It follows from (C) and SP-II that $M(R'_{\{3,4\}}, R_{\{1,2\}})(2) \in \{e, g\}$. Given (D') and (F) $M(R'_{\{3,4\}}, R_{\{1,2\}})(2)$ cannot equal g, and we obtain $M(R'_{\{3,4\}}, R_{\{1,2\}})(2) = e$. Using (D') once again we obtain $M(R'_{\{1,3\}}, R_{\{2,4\}})(2) = e$. By SP-II

$$(G) : M(R'_{-4}, R_4)(2) \in \{e, g\}.$$

Together (E) and (G) imply that either

$$(H) : M(R'_{-4}, R_4)(2) = e \text{ and } M(R'_{-4}, R_4)(3) = g \text{ or}$$
$$(I) : M(R'_{-4}, R_4)(2) = g \text{ and } M(R'_{-4}, R_4)(3) = e \text{ hold.}$$

Suppose that (H) did hold. Given (B) we would then have $M(R'_{-4}, R_4)(2) = eR'_1 M(R'_{-4}, R_4)(1) \notin \{e, g\}$ and $M(R'_{\{2,3\}}, R_{\{1,4\}})(2) = gR'_2 M(R'_{-4}, R_4)(2) = e$ a contradiction to Papai-L5. So (I) must hold. Applying SP-NB to (I) we obtain $M(R'_{-4}, R_4) = M(R'_{\{1,2\}}, R_{\{3,4\}})$ and in particular $M(R'_{\{1,2\}}, R_{\{3,4\}})(2) = g$. SP-II implies that $M(R'_1, R_{-1})(2) \in \{e, g\}$. Our initial assumption that

$M(R'_1, R_{-1})(3) = e$, then implies $M(R'_1, R_{-1})(2) = g$. A contradiction with Papai-L5 arises since $M(R'_1, R_{-1})(2) = g R'_1 M(R'_1, R_{-1})(1) = h'$ and $M(R)(2) = e R_2 M(R'_1, R_{-1})(2) = g$.

In sum we obtain that $M(R'_1 R_{-1})(2) = e$ holds for any $R'_1 : h_1 \cdots h_m\ e\ g$. Since 1 was chosen arbitrarily, we can inductively obtain the above argument to obtain that $e = M(R)(2) = M(R^*_1, R_{-1})(2) = M(R^*_{\{1,2\}}, R_{-\{1,2\}})(2) = \cdots = M(R^*)(2)$. □

Lemma 3 *Let $M(R^g)(1) = e = M(R^{g'})(2)$ for some $R^g, R^{g'}$. Then there exists an agent i_b such that $M(R^h)(i_b) = h$ for any $h \neq e$.*

Proof Fix $\hat{R}^{g'}_i : e\ g'\ g$ for all i and let $M(\hat{R}^{g'})(i_b) = g'$. To show that $M(R^g)(i_b) = g$ let let $\tilde{R}^{g'}_i : e\ g'\ g$ coincide with R^g_i on $H \setminus \{g'\}$ for all i. Since $\hat{R}^{g'}_i$ and $\tilde{R}^{g'}_i$ agree on $e\ g'\ g$, Lemma 2 implies $M(\hat{R}^{g'})(i_b) = g' = M(\tilde{R}^{g'})(i_b)$. Inductively switching $\tilde{R}^{g'}_i$ to R^g_i for all $i \neq i_b$ SP-NB implies that $M(\tilde{R}^{g'}_{i_b}, R^g_{-i_b}) = M(\tilde{R}^{g'})$. Let $\hat{R}^g_{i_b} : e\ g\ g'$ coincide with $R^g_{i_b}$ on $H \setminus \{g'\}$. By SP-II $M(\hat{R}^g_{i_b}, R^g_{-i_b})(i_b) \in \{g, g'\}$. If $M(\hat{R}^g_{i_b}, R^g_{-i_b})(i_b) = g'$, then non-bossiness implies that $M(\hat{R}^g_{i_b}, R^g_{-i_b}) = M(\tilde{R}^{g'}_{i_b}, R^g_{-i_b}) = M(\tilde{R}^{g'})$. Lemma 2 and $M(R^{g'})(2)$ imply $M(\tilde{R}^{g'})(2) = e$ and therefore $M(\hat{R}^g_{i_b}, R^g_{-i_b})(2) = e$. On the other hand Lemma 2 and $M(R^g)(1) = e$ imply $M(\hat{R}^g_{i_b}, R^g_{-i_b})(1) = e$, a contradiction. So we must have $M(\hat{R}^g_{i_b}, R^g_{-i_b})(i_b) = g$. SP-I then implies that $M(R^g)(i_b) = g$.

Using the same arguments as above switching the roles of g and g' (using that $M(R^g)(i_b) = g$ holds for any R^g, particularly a profile in which all agents rank g' third) we obtain that $M(R^{g'})(i_b) = g'$ holds for any $R^{g'}$. Now fix any $h \neq e$. Apply the above arguments to R^h and $R^{h'}$ with $R^{h'}_i : e\ h'\ h$ for all i where $h' = g'$ if $M(R^h)(e) = 2$ and $h' = g$ otherwise to obtain that $M(R^h)(i_b) = h$ holds for all R^h. □

6.6 Properties of $c_\emptyset$

In the following Lemmas 4, 5, and 6 show that $c_\emptyset$ satisfies a range of properties. Lemma 4, which is used in the proofs of nearly all upcoming Lemmas, shows that $\{(i, e), (j, g)\} \subset M(R')$ holds for any R' with $R'_i : e$ and $R'_j : g$

or $R'_j : e\ g$ if $c_\emptyset(e) = (j, b)$ and $e = M(R^g)(i)$. Lemma 5 shows that i is matched to e under $M(R)$ if he owns e according to $c_\emptyset$ ($c_\emptyset(e) = (i, o)$) and if i prefers e to all other houses ($R_i : e$). Lemma 6 shows that any broker only brokers a single house and does not own any house: $c_\emptyset(e) = (i, b)$ implies $c_\emptyset(h) \neq (i, \cdot)$ for any $h \neq e$. If all agents rank some e with $c_\emptyset(e) = (i, b)$ at the top, then the owner of the second most preferred house of i is matched to e. The lemmas in the current section condense the lemmas of the preceding section. The lemmas of the preceding section are not (directly) used after the current section.

Lemma 4 *Let $c_\emptyset(e) = (1, b)$. If $M(R^g)(2) = e$ holds for some R^g, then $M(R)(2) = e$ and $M(R)(1) = g$ holds for all R such that $R_2 : e$ and either $R_1 : g$ or $R_1 : e\ g$.*

Proof Fix any profile R such that $R_2 : e$ and either $R_1 : g$ or $R_1 : e\ g$. For all i let $\hat{R}^g_i : e\ g$ and R_i coincide on $H \setminus \{e, g\}$. Lemma 2 and the assumption that $M(R^g)(2) = e$ implies $M(\hat{R}^g)(2) = e$. Lemma 3 and the and $c_\emptyset(e) = (1, b)$ imply $M(\hat{R}^g)(1) = g$. Inductively apply SP-NB, dropping the houses e and g in the rankings of all agents $i \neq 1, 2$ to obtain $M(\hat{R}^g_{1,2}, R_{-\{1,2\}}) = M(\hat{R}^g)$. Drop house g in agent 2's ranking and house e in agent 1's ranking if $R_1 : g$ to obtain $M(\hat{R}^g_{1,2}, R_{-\{1,2\}}) = M(R)$ in particular $M(R)(1) = g$ and $M(R)(2) = e$. □

Lemma 5 *If $c_\emptyset(e) = (1, o)$ and $R_1 : e$, then $M(R)(1) = e$.*

Proof Showing $M(R)(1) = e$ if $R_i : e$ holds for all i is sufficient. To see this fix any R° with $R^\circ_1 : e$ let $R_i : e$ and R°_i coincide on $H \setminus \{e\}$ for all i. If $M(R)(1) = e$, then $M(R^\circ)(1) = e$ follows from the inductive application of SP-NB.

Now suppose $M(R)(1) \neq e$ held for some $R_i : e$ for all i and $R_1 : e\ g$. For all i, let R^g_i coincide with R_i on $H \setminus \{g\}$, $M(R^g)(j) = g$ and $M(R)(j) = g'$. If gR_jg' then SP-NB yields $M(R) = M(R^g_j, R_{-j})$. Given that Lemma 4 implies $M(R^g_j, R_{-j}) = M(R^g)$ we obtain $M(R) = M(R^g)$ and the contradiction $e \neq M(R)(1) = M(R^g)(1) = e$ since $c_\emptyset(e) = (1, o)$. So $g'R_jg$ and $g' \neq g$ must hold. SP-NB implies $M(R) = M(R'_j, R_{-j})$ for $R'_j : e\ g'\ g$. To obtain a contradiction consider agent 1's match under $M(R^{g'}_1, R'_j, R_{-\{1,j\}})(1)$.

Let $R_i^{g'} : e\ g'$ coincide with R'_j and with R_i for $i \neq 1, j$ on $H \setminus \{g'\}$ and let $R_1^* : g'$. Since $c_\emptyset(e) = (1, o)$ we have $M(R^{g'})(1) = e$. SP-PO implies $M(R_1^*, R_{-1}^{g'})(1) = g'$, the inductive application of SP-NB implies $M(R_1^*, R'_j, R_{-\{1,j\}})(1) = g'$. Combining the last statement with SP-PO yields $M(R_1^{g'}, R'_j, R_{-\{1,j\}})(1) \in \{e, g'\}$. If $M(R_1^{g'}, R'_j, R_{-\{1,j\}})(1) = e$ then agent 1 gains by misrepresenting his preference at (R'_j, R_{-j}) and M is not strategyproof. If $M(R_1^{g'}, R'_j, R_{-\{1,j\}})(1) = g'$ then $M(R_1^{g'}, R'_j, R_{-\{1,j\}})(j) \neq g'$ and SP-NB imply $M(R_1^{g'}, R_j^g, R_{-\{1,j\}}) = M(R_1^{g'}, R'_j, R_{-\{1,j\}})$ which yields a contradiction to Lemma 4 which requires that $M(R_1^{g'}, R_j^g, R_{-\{1,j\}})(1) = e \neq g'$. □

Lemma 6 *Let $c_\emptyset(e) = (1, b)$. Then*

a) $c_\emptyset(h) \neq (1, o)$ holds for all $h \in H$.

b) $c_\emptyset(h) \neq (1, b)$ holds for all $h \in H \setminus \{e\}$.

c) If $c_\emptyset(g^*) = (j, o)$ then $M(R^{g^*})(j) = e$.

Proof Since $c_\emptyset(e) = (1, b)$, it is w.l.o.g to assume that $M(R^g)(2) = e$, $M(R^{g'})(3) = e$, for some $g \neq e \neq g'$.

a) Since $M(R^g)(e) \neq 1$ even though $R_1^g : e$, Lemma 5 implies $c_\emptyset(e) \neq (1, o)$. Suppose $c_\emptyset(g) = (1, o)$ for some $g \neq e$. Let $R_2^* : g\ e$. SP-PO implies $M(R_2^*, R_{-2}^g)(2) = g$. Since $c_\emptyset(g) = (1, o)$ Lemma 5 and strategyproofness imply $M(R_2^*, R_{-2}^g)(1) R_1^g g$. According to R_1^g only e and g are (weakly) preferred to g. Since g is matched to 2 under (R_2^*, R_{-2}^g) we obtain $M(R_2^*, R_{-2}^g)(1) = e$. SP-I then implies $M(R_1^{g'}, R_2^*, R_{-\{1,2\}}^g)(1) = M(R_2^*, R^g)(1) = e$. But Lemma 4 together with $M(R^{g'})(e) = 3$, $R_1^{g'} : e\ g'$, and $R_3^g : e$ implies $M(R_1^{g'}, R_2^*, R_{-\{1,2\}}^g)(3) = e$.

b) Suppose $c_\emptyset(g) = (1, b)$ held for some $g \neq e$. So there exists some house h such that $M(R^\circ)(j) = g$, with $j \notin \{1, 2\}$ and $M(R^\circ)(1) = h$ where $R_i^\circ : g\ h$ holds for all i. For all i let $R_i^* : e\ g\ h$ coincide with R_i° on $H \setminus \{e\}$. The assumptions $M(R^g)(2) = e$ and $M(R^g)(1) = g$ together with Lemmas 2 and 3 imply that $M(R^*)(2) = e$ and $M(R^*)(1) = g$. Given that neither 1 nor $j \neq 2$ is matched to e under $M(R^*)$ the inductive application of SP-NB implies $M(R^*)(1) = M(R_{\{1,j\}}^\circ, R_{-\{1,j\}}^*)(1) = g$. But Lemma 4 requires the contradiction $M(R_{\{1,j\}}^\circ, R_{-\{1,j\}}^*)(j) = g$.

c) The definition of $c_\emptyset$ and the assumption $c_\emptyset(e) = (1, b)$ imply $M(R^g)(1) = g$. Since $c_\emptyset(g^*) = (j, o)$ Lemma 5 implies that agent j is matched to a house $M(R^{g^*})(j) R_j^{g^*} g^*$. Since e is the only house other than g^* that satisfies this relation $M(R^{g^*})(j) = e$ must hold. □

6.7 Braids, (C1), (C2), and (C3)

In Lemma 7 I show that M is a braid if $c_\emptyset$ calls for at least two houses to be brokered, implying that $c_\emptyset$ satisfies (C1). If there is at most one brokered house according to $c_\emptyset$, let i, e be such that $c_\emptyset(e) = (i, b)$. Part a) of Lemma 6 then implies $c_\emptyset(h) \neq (i, o)$ as required by (C3). Finally part c) of Lemma 6 together with the definition of $c_\emptyset$ implies that there must be at least two owners under $c_\emptyset$ for there to be a broker under $c_\emptyset$ as required by (C2). In sum, $c_\emptyset$ satisfies (C1), (C2) and (C3).

Lemma 7 *Let $c_\emptyset(e) = (1, b)$ and $c_\emptyset(g) = (k, b)$ for $e \neq g$. Then $\mid H \mid= 3$ and M is a braid.*

Proof W.l.o.g. assume that $M(R^g)(2) = e = M(R^{g'})(3)$ for some $R^g, R^{g'}$.

Claim 1: Fix R^* with $R_i^* : g\ e$ for all i. Then $M(R^*)(2) = g$ must hold.

Fix $R_i^g : e\ g$ to coincide with R_i^* on $H \setminus \{g\}$ for all i. Inductively dropping house e in the rankings R_i^g of all agents $i \neq 2$, SP-NB yields $M(R^g) = M(R_2^g, R_{-2}^*)$. If $M(R_2^g, R_{-2}^*) = M(R^*)$ then $2 = k$ and $M(R^*)(1) = g$ must hold. Lemma 4 implies $M(R_{\{1,2\}}^*, R_{-\{1,2\}}^g)(1) = g$. SP-II applied to agent 1's choice yields $M(R_2^*, R_{-2}^g)(1) \in \{e, g\}$. Since $M(R_2^*, R_{-2}^g)(2) = g$ holds due to SP-PO and $M(R^g)(2) = e$ $M(R_2^*, R_{-2}^g)(1)$ must equal e. SP-NB implies $M(R_2^*, R_1^{g'}, R_{-\{1,2\}}^g) = M(R_2^*, R_{-2}^g)$ in particular $M(R_2^*, R_1^{g'}, R_{-\{1,2\}}^g)(1) = e$. Lemma 4 and the assumption $M(R^{g'})(3) = e$ imply $M(R_2^*, R_1^{g'}, R_{-\{1,2\}}^g)(3) = e$, a contradiction. So $M(R_2^g, R_{-2}^*)$ must differ from $M(R^*)$. Non-bossiness and SP-II then imply $e = M(R_2^g, R_{-2}^*)(2) \neq M(R^*)(2) = g$.

Claim 2: $c_\emptyset(g) = (k, b)$, so $k = 3$.

Since $c_\emptyset(e) = (1, b)$ part b) of Lemma 6 implies that $k \neq 1$. Since $M(R^*)(2) = g$ as established in Claim 1 $k \neq 2$. Claim 2 follows since the assumption that

$k > 3$ together with Lemma 4, $M(R^{g'})(1) = e$ and $M(R^*)(2) = e$ leads to the contradiction that

$$M(R^*_{\{2,k\}}, R^{g'}_{-\{2,k\}})(k) = M(R^*_{\{2,k\}}, R^{g'}_{-\{2,k\}})(1) = e.$$

Claim 3: There is no $h \in H$ and $j > 3$ such that $c_\emptyset(h) = (j, o)$.

Suppose $c_\emptyset(h) = (j, o)$ held for some $j > 3$ and $h \in H$. Fix some R^h. Since $c_\emptyset(e) = (1, b)$, $M(R^h)(1) = h$ holds by the definition $c_\emptyset$. Since $c_\emptyset(h) = (j, o)$ part c) of Lemma 6 implies $M(R^h)(j) = e$. By Lemma 4 together with $M(R^h)(j) = e$, $M(R^*)(2) = g$, $M(R^*)(3) = e$ (as established in Claims 1 and 2) we obtain the contradiction

$$M(R^h_{\{1,j\}}, R^*_{-\{1,j\}})(j) = M(R^h_{\{1,j\}}, R^*_{-\{1,j\}})(3) = e.$$

Claim 4: There is no $h \in H$ and $j > 3$ such that $c_\emptyset(h) = (j, b)$.

Suppose $c_\emptyset(h) = (j, b)$ held for some $j > 3$ and $h \in H$. Fix $R^\alpha_i : h\ e$ for all i and let $M(R^\alpha)(j') = h$. Mutatis mutandis Claim 1 implies that $M(R^h)(j') = e$. Since $c_\emptyset(e) = (1, b)$, j' cannot equal 1. If $j' \neq 2$ then Lemma 4 and $M(R^g)(2) = e$ imply

$$M(R^g_{\{1,2\}}, R^\alpha_{-\{1,2\}})(2) = M(R^g_{\{1,2\}}, R^\alpha_{-\{1,2\}})(j) = e$$

a contradiction since $j > 3$. If $j' = 2$ a similar contradiction is obtained replacing agent 2 with 3 and house g with g'.

Claim 5: $c_\emptyset(g') = (2, b)$ and $H = \{e, g, g'\}$.

If $c_\emptyset(h) = (i, \cdot)$ holds for some $h \in H$ then $i \leq 3$ follows from Claims 3 and 4. Parts a) and b) of Lemma 6 applied to $c_\emptyset(e) = (1, b)$ and $c_\emptyset(g) = (3, b)$ (which follow from the initial assumption and Claim 2) yield $c_\emptyset(g') \neq (1, \cdot)$ and $c_\emptyset(g') \neq (3, \cdot)$ Since $M(R^{g'})(2) \notin \{e, g'\}$ $c_\emptyset(g') \neq (2, o)$. The only remaining option is $c_\emptyset(g') = (2, b)$. Since for any $i \in \{1, 2, 3\}$ there is a house $h \in \{e, g, g'\}$ such that $c_\emptyset(h) = (i, b)$ parts a) and b) of Lemma 6 imply that $c_\emptyset(h) \neq (i, \cdot)$ holds for all $h \notin \{e, g, g'\}$ and $i \leq 3$. In sum we obtain $H = \{e, g, g'\}$.

Claim 6: $M(R) = B(R)$.

I first show that M and B match the the top ranked houses to the same agents if R is such that $R_i = R_j$ for all i, j. For $R \in \{R^g, R^{g'}, R^*\}$ the initial assumption as well as Claims 2 and 3 determine the agents that are matched with the two top ranked houses. Since $c_\emptyset(g) = (3, b)$ there must exist a house $h \notin \{e, g\}$ and an agent $j \notin \{2, 3\}$ such that $M(R^\circ)(j) = g$ for $R_i^\circ : g\ h$ for all i. Since there are only three houses h must equal g'. Moreover $j = 1$, since otherwise Lemma 4 implies the contradiction

$$M(R^g_{\{1,2\}}, R^\circ_{-\{1,2\}})(1) = M(R^g_{\{1,2\}}, R^\circ_{-\{1,2\}})(3) = g.$$

Let $\hat{R}_i : g'\ g$ and $\tilde{R}_i : g'\ e$ for all i. By Claim 4 $c_\emptyset(g') = (2, b)$ and we have $M(\hat{R})(2) = g$ and $M(\tilde{R})(2) = e$. Mutatis mutandis Claim 1 then implies that $M(\hat{R})(1) = g' = M(\tilde{R})(3)$.

At R^g ω' and ω'' are both Pareto optimal. Since at least two agents rank e at the top under R^g, since $\omega(e) = 1$ and $g = \omega'(1) R_1^g \omega''(1) = g''$ $B(R^g) = \omega'$ must hold, in particular we have $B(R^g)(1) = g$ and $B(R^g)(2) = e$, so B and M match the two top ranked houses under R^g to the same agents. The proof this statement holds for all $R \in \{R^{g'}, R^*, R^\circ, \hat{R}, \tilde{R}\}$ follows from the same arguments mutatis mutandis.

To see that $M(R^g)(3) = g'$, observe that SP-NB implies $M(R^g) = M(R_1^*, R_2^g, R_3^\circ)$. Lemma 4 together with $M(R^\circ)(1) = g$ and $M(R^\circ)(3) = g'$ (as established in the preceding paragraph) then implies $M(R_1^*, R_2^g, R_3^\circ)(3) = g'$. Mutatis mutandis the same arguments prove that $M(R)(i) \in \{e, g, g'\}$ holds for all $R \in \{R^{g'}, R^*, R^\circ, \hat{R}, \tilde{R}\}$ and $i \leq 3$. So $M(R) = B(R)$ holds for all $R \in \{R^g, R^h, R^*, R^\circ, \hat{R}, \tilde{R}\}$.

Fix R with $R_j : \omega'(j)$, $\omega'(j') R_{j'} \omega''(j')$ and $\omega(j') = \omega'(j)$ (or $R_j : \omega''(j)$, $\omega''(j') R_{j'} \omega'(j')$ and $\omega(j') = \omega''(j)$) for some $j \neq j'$. Let $j' = 1$ and $j = 2$, so $R_2 : e$, $\omega'(1) = g R_1 g' = \omega''(1)$ and $\omega(1) = \omega'(2) = e$. We know from (**) in the proof of Lemma 1 that $B(R) = \omega'$. The arguments in the preceding two paragraphs yield $M(R^g) = B(R^g) = \omega'$. If $R^g \neq R$, then the inductive application of SP-NB implies $M(R) = M(R^g) = B(R^g) = B(R)$. For all other combinations of agents j, j' the $M(R) = B(R)$ is established mutatis mutandis.

Fix R with $R_j : h$ with $h \neq \omega'(j)$, $\omega'(j') R_{j'} \omega''(j')$, $\omega(j') = \omega'(j)$ and $R_{j''} : \omega'(j)$ where $\{j, j', j''\} = \{1, 2, 3\}$. Let $j' = 1$ and $j = 2$, so $R_2 : h$,

$\omega'(1) = gR_1g' = \omega''(1)$ and $\omega(1) = \omega'(2) = e$ and $R_3 : e$. Let $R_2' : e$. Since $M(R_1^{g'}, R_{-1})(1) = g'$, SP-I implies that $M(R)(1) \neq e$. Since $M(R_2^g, R_{-2})(2) = g$ and since M is strategy proof $M(R)(2) \in \{e, g, g'\}$. Since M is Pareto optimal $M(R)(2) \neq e$. So $M(R)(3) = e$. Since $M(R_2', R_{-2})(2) = e$ agent 2 must prefer $M(R)(2)$ to e. Since we only specified agent 2 ranks $h \neq e$ at the top under R_2 this would have to apply in particular if agent 2 ranks e in second place, implying $M(R)(2) = h$. Since $M(R_1^{g'}, R_{-1})(1) = g'$ strategyproofness implies that agent 1 is matched with some house at R. Given that $M(R)(2) = h$ and $M(R)(3) = e$, $M(R)(1)$ is the one remaining house in $\{e, g, g'\}$. The case that $R_j : h$ with $h \neq \omega''(j)$, $\omega''(j')R_{j'}\omega'(j')$, $\omega(j') = \omega''(j)$ and $R_{j''} : \omega''(j)$ where $\{j, j', j''\} = \{1, 2, 3\}$ and all alternative combinations of agents j, j' and a house h can be dealt with with the same arguments mutatis mutandis.

Only once case remains to be considered R is such that $R_i : \omega(i)$ for all i (as in this case i is the only agent to rank $\omega(i)$ at the top). The two preceding paragraphs imply that for any $i \leq 3$ there exists a $R_i' : h$ with $\omega(i) \neq h$ such that $M(R_i', R_{-i})(i) \in \{e, g, g'\}$. Since M is strategyproof $M(R)(i) \neq \emptyset$ must hold for all $i \leq 3$. But there is only one Pareto optimum at R with this feature $M(R) = \omega$ and we also obtain $M(R) = B(R) =$ in this last case. $\square$

6.8 Pointing

Assume that M is not a braid and fix some R°. Here I show that the outcome $M(R^\circ)$ is consistent with submatching achieved under R° in the first round of any trading and braiding mechanism with $c_\emptyset$ the control rights function at $\emptyset$. Recall the definitions of $\mathcal{N}(c_\emptyset)$ and $\mathcal{N}(c_\emptyset)(R)$ in Section 6.2 as the set of direct c-successors to $\emptyset$ and the subset of direct c-successors to $\emptyset$ that are reachable under the profile of preferences R. Since only $c_\emptyset$ is used in determining the set of direct c-successors to $\emptyset$, $\mathcal{N}(c_\emptyset)$ and $\mathcal{N}(c_\emptyset)(R)$ describe the same sets for any trading and braiding mechanism c that prescribes $c_\emptyset$ at $\emptyset$. In Lemma 8 I show that $\nu \subset M(R^\circ)$ holds for any $\nu \in \mathcal{N}(c_\emptyset)(R)$. In Lemma 9 I show that the calculation of $M(R^\circ)$ can be split into the calculation of a submatching $\nu \in \mathcal{N}(c_\emptyset)(R)$ and the outcome of a well-defined submechanism. The submechanism inherits the property of being good from M and can by the inductive hypothesis be represented as a trading and braiding mechanism $c[\nu]$. For the next two Lemmas fix a submatching $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$. W.l.o.g. assume $N_\nu = \{1, \cdots, m\}$ and $c^*_\emptyset(h_i) = (i, \cdot)$ for all $i \leq m$.

Lemma 8 *Any submatching that arises out of matching one cycle under $c_\emptyset$ at R° is part of the outcome $M(R^\circ)$, $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$ implies $\nu \subset M(R^\circ)$.*

Proof Case 1: $m = 1$. So $\nu = \{(i, h)\}$ holds for some i, h and $c_\emptyset(h) = (i, \cdot)$ must hold since h points to i at $\emptyset$. Since i may only point to h if $c_\emptyset(h) \neq (i, b)$, $c_\emptyset(h) = (i, o)$ must hold. Since i, as an owner may point to any house, $R^\circ_i : h$ must hold. Lemma 5 then implies $M(R^\circ)(i) = h$ and therefore $\nu \subset M(R^\circ)$.

Case 2: $m > 1$ and $c_\emptyset(h_i) = (i, o)$ for all $i \leq m$. W.l.o.g assume that $\nu(i) = h_{i+1}$ for all $i < m$ and $\nu(m) = h_1$. For all $i \leq m$ let $R^*_i : \nu(i)\ h_i$ coincide with R_i on $H \setminus \{h_i\}$ and let $R^*_i = R^\circ_i$ for $i > m$. Suppose we had $M(R^*)(i^*) \neq \nu(i^*)$ for at least one $i^* \leq m$, say $i^* = m$. Since $c_\emptyset(h_m) = (m, o)$ Lemma 5 implies $M(R'_m, R^*_{-m})(m) = h_m$ for $R'_m : h_m$. SP-II implies that $M(R^*)(m) \in \{h_1, h_m\}$. The assumption $M(R^*)(m) \neq \nu(m) = h_1$ then implies $M(R^*)(m) = h_m$. So $M(R^*)(m-1)$ differs from $\nu(m-1) = h_m$. Inductively applying these arguments to all agents in the cycle we obtain that $M(R^*)(i) = h_i$ for all $i \leq m$. This contradicts the Pareto optimality of M since any $i \leq m$ strictly prefers $\nu(i)$ to h_i. So $\nu \subset M(R^*)$ must hold. Inductively applying SP-NB to drop h_i in the rankings of all agents $i \leq m$ we obtain $M(R^*) = M(R^\circ)$.

Case 3: $c_\emptyset(h_m) = (m, b)$. Since m, as a broker, may not point to h_m we have $m > 1$. As in the preceding case assume that $\nu(i) = h_{i+1}$ for all $i < m$ and $\nu(m) = h_1$. Define $R^*_{m-2} : h_{m-1}\ h_1$, $R^*_m : h_m\ h_1\ h_{m-1}$ and $R^*_i : \nu(i)\ h_i$ for all other $i \leq m$. For all preference statements that have not been explicitly mentioned let R^* and R° coincide. Under $M(R'_{m-2}, R^*_{-(m-2)})$ with $R'_{m-2} : h_1$ the owners $\{1, \cdots, m-2\}$ form a pointing cycle. Applying the result of Case 2 we obtain $M(R'_{m-2}, R^*_{-(m-2)})(m-2) = h_1$. Lemma 4 and part c) of Lemma 6 imply that $M(R'_m, R^*_{-m})(m) = h_{m-1}$ for $R'_m : h_m\ h_{m-1}$. Strategyproofness implies $M(R^*)(m-2) R^*_{m-2} h_1$, $M(R^*)(m) R^*_m h_{m-1}$, and $M(R^*)(m) \neq h_m$. So then agents $\{m-2, m\}$ must be matched to the houses $\{h_1, h_{m-1}\}$ under R^*. Pareto optimality requires that $M(R^*)(m-2) = h_{m-1}$ and $M(R^*)(m) = h_1$. The last observation in turn implies that $M(R^*)(1) \neq$

h_1. Since $c_\emptyset(h_1) = (1, o)$, Lemma 5 together with strategyproofness implies that $M(R^*)(1) R_1^* h_1$. So $M(R^*)(1)$ must equal h_2. We can now inductively apply the same arguments to all other agents to obtain that $\nu \subset M(R^*)$. Inductively applying SP-NB to drop h_1 in R^*_{m-2}, h_{m-1} and h_m in R^*_m, and h_i in all other rankings R^*_i with $i \leq m$ we obtain $M(R^*) = M(R^\circ)$. □

Keeping $\nu \in \mathcal{N}(c_\emptyset)(R)$ fixed as above, let $\overline{R}$ be the restriction of R to the agents $\overline{N}_\nu$ over houses $\overline{H}_\nu$. Let $\overline{\mathcal{R}}$ be the set of all such restrictions. Define a trading and braiding mechanism $c[\nu]$ that maps any $\overline{R} \in \overline{\mathcal{R}}$ to $c[\nu](\overline{R}) :\, = M(R) \setminus \nu$.

Lemma 9 *The mechanism $c[\nu]$ is well-defined.*

Proof Fix another profile R^* such that $\nu \in \mathcal{N}(c_\emptyset)(R^*)$. Only the preferences of agents $i \leq m$ matter for the formation of ν under $M(R^*)$ and $M(R^\circ)$. So the mechanisms M^* and M° that respectively map any $\overline{R} \in \overline{\mathcal{R}}$ to $M^*(\overline{R}) :\, = M(R^*_{N_\nu}, R_{-N_\nu}) \setminus \nu$ and $M^\circ(\overline{R}) :\, = M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu$ are well-defined.

To see that $M^* = M^\circ$, fix some some $\overline{R} \in \overline{\mathcal{R}}$ and note

$$
\begin{aligned}
\nu \cup M^*(\overline{R}) = M(R^*_{N_\nu}, R_{-N_\nu}) = \\
M(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) = \cdots = M(R^\circ_{\{1,\cdots,m\}}, R^*_m, R_{-N_\nu}) = \\
M(R^\circ_{N_\nu}, R_{-N_\nu}) = \nu \cup M^\circ(\overline{R}).
\end{aligned}
$$

The first and the last equality follow from the definitions of M^* and M°. Since $\nu \in \mathcal{N}(c_\emptyset)(R^*)$ and $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$, $R^*_i : \nu(i)$, $R^\circ_i : \nu(i)$ must hold for all owners $i \leq m$ (that is all agents $i \leq m$ with $c_\emptyset(h_i) = (i, o)$) and $\nu_i(i) R^*_i H \setminus \{h_i\}$, $\nu_i(i) R^\circ_i H \setminus \{h_i\}$ must hold for a broker $i \leq m$ (if there is one) where $c_\emptyset(h_i) = (i, b)$ and h_i is the house he brokers. So the cycle also arises under $c_\emptyset$ at any of the intermediate profiles $(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) \cdots (R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu})$. Lemma 8 implies that $\nu \subset M(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}), \cdots, \nu \subset M(R^\circ_{\{1,\cdots,m\}}, R^*_m, R_{-N_\nu})$. Given that agent $i \leq m$ is matched to $\nu(i)$ for any $(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) \cdots (R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu})$ the non-bossiness of M the implies the intermediate equalities.

To see that M° is strategy proof fix an agent $i \in \overline{N}_\nu$ and a deviation $\overline{R}'_i$ for agent i. Let R'_i and R be such that $\overline{R}'_i$ and $\overline{R}$ are the restrictions of R'_i

and R to $\overline{H}_\nu$, $\overline{N}_\nu$. The definition of M° together with the strategyproofness of M then imply

$$M^\circ(\overline{R})(i) = M(R^\circ_{N_\nu}, R_{-N_\nu})(i) R_i M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})(i) = M^\circ(\overline{R}'_i, \overline{R}_{-i})(i).$$

To see that M° is non-bossy assume that $M^\circ(\overline{R})(i) = M^\circ(\overline{R}'_i, \overline{R}_{-i})(i)$. Since $i \notin N_\nu$ we obtain $M(R^\circ_{N_\nu}, R_{-N_\nu})(i) = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})(i)$. The non-bossiness of M then implies $M(R^\circ_{N_\nu}, R_{-N_\nu}) = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})$ and therefore

$$M^\circ(\overline{R}) = M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})}) \setminus \nu = M^\circ(\overline{R}'_i, \overline{R}_{-i}).$$

Finally M° is Pareto optimal since $M(R^\circ_{N_\nu}, R_{-N_\nu}) = \nu \cup M^\circ(\overline{R})$ is Pareto optimal for any R. In sum M° is a good mechanism. By the hypothesis of the induction M° can be represented as a trading and braiding mechanism $c[\nu]$. □

6.9 Requirements that link control rights functions: (C4), (C5) and (C6)

The definition of $c_\emptyset$ in Section 6.5 together with Lemma 9 on the submechanisms $c[\nu]$ define a control rights structure c via $c_{\nu^\circ} = c[\nu]_{\nu'}$ for any $\nu^\circ = \nu \cup \nu'$ with $\nu \in \mathcal{N}(c_\emptyset)$ and ν' $c[\nu]$-relevant. Since a submatching ν° is c-relevant if and only if it can be split into a submatching $\nu \in \mathcal{N}(c_\emptyset)$ and a submatching ν' that is $c[\nu]$-relevant, c is defined on all c-relevant submatchings.

We know from Section 6.7 that $c_\emptyset$ satisfies (C1), (C2) and (C3). Since $c[\nu]$ is a trading and braiding mechanism for any $\nu \in \mathcal{N}(c_\emptyset)$, (C1), (C2) and (C3) are satisfied for any c-relevant $\nu^\circ \neq \emptyset$. Moreover (C4), (C5) and (C6) are satisfied by any pair of a c-relevant $\nu^\circ \neq \emptyset$ with a direct c-successor ν of ν°. In the following Lemma 10 I show that (C4), (C5) and (C6) are also satisfied for $\nu^\circ = \emptyset$ and any $\nu \in \mathcal{N}(c_\emptyset)$. Keep R°, $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$ and the definition of $\overline{R}$ fixed as above.

Lemma 10 *(C4), (C5) and (C6) hold for $\emptyset$ and $\nu \in \mathcal{N}(c_\emptyset)$.*

Proof **(*)** For any trading and braiding mechanism c° we have $c^\circ_\emptyset(e) = (i,o)$ holds if and only if $c^\circ(R)(i) = e$ for all R such that $R_i : e$. If $c^\circ_\emptyset(e) \neq (\cdot, o)$ then $c^\circ_\emptyset(e) = (i,b)$ if and only if $c^\circ(R^g)(i) = g$.

(C4) Let $i^* \notin N_\nu$ be such that $c_\emptyset(e) = (i^*, o)$. Fix any $\overline{R}$ with $\overline{R}_{i^*} : e$. Define R such that $R_i = R^\circ_i$ for all $i \in N_\nu$ and $R_{i^*} : e$. Since i^* owns* e Lemma 5 implies $M(R)(i^*) = e$. The results in the preceding section imply that $M(R) = \nu \cup c[\nu](\overline{R})$, in particular $M(R)(i^*) = e = c[\nu](\overline{R})(i^*)$. By (*) i^* owns e under $c[\nu]_\emptyset = c_\nu$.

(C5) Let $i_b, j, j' \notin N_\nu$ be such that $c_\emptyset(e) = (i_b, b)$ and agents j and j' respectively own houses g and g' under $c_\emptyset$. By the preceding paragraph on (C4), j and j' own g and g' under c_ν. By Lemma 8 we have $M(R^\circ_{N_\nu} R^g_{-N_\nu})(i_b) = g$, $M(R^\circ_{N_\nu} R^g_{-N_\nu})(j) = e = M(R^\circ_{N_\nu} R^{g'}_{-N_\nu})(j')$. By the preceding section we have $M(R^\circ_{N_\nu} R^g_{-N_\nu}) = \nu \cup c[\nu](\overline{R}^g)$ and $M(R^\circ_{N_\nu} R^{g'}_{-N_\nu}) = \nu \cup c[\nu](\overline{R}^{g'})$. In sum we get that $c[\nu](\overline{R}^g)(j) = e = c[\nu](\overline{R}^{g'})(j')$ and by (*) e is not owned at $c[\nu]_\emptyset$. Since $M(R^\circ_{N_\nu} R^g_{-N_\nu})(i_b) = g = c[\nu](\overline{R}^g)(i_b)$ (*) also implies that i_b brokers e under $c[\nu]_\emptyset = c_\nu$.

(C6) Let $c_\emptyset(g) = c_\nu(g) = (j,o)$, $c_\emptyset(e) = (i_b, b) \neq c_\nu(e)$ and $i_b \notin N_\nu$. Let $R_i = R^g_i$ for all $i \in \overline{N}_\nu$ and let $R_i = R^\circ_i$ for all $i \in N_\nu$. Let $R'_i : h^*\ e\ g$ for some fixed house $h^* \in H_\nu$. By Lemma 8 $M(R)(i_b) = g$ and $M(R)(j) = e$. If i_b did own e under $c[\nu]_\emptyset$ we obtain $M(R'_{i_b}, R_{-i_b})(i_b) = c[\nu](\overline{R}'_{i_b}, \overline{R}_{-i_b})(i_b) = e$ contradiction to strategyproofness.

Now suppose an agent $k \notin \{i_b, j\}$ controls e at $\emptyset$ under $c[\nu]$. If $c[\nu]_\emptyset(e) = (k,b)$ we obtain a contradiction to strategyproofnes given that R'_{i_b} ranks $g = M(R)(i_b)$ above $M(R'_{i_b}, R_{-i_b})(i_b) = c[\nu](\overline{R}'_{i_b}, \overline{R}_{-i_b})(i_b)$ since under $c[\nu](\overline{R}'_{i_b}, \overline{R}_{-i_b})$ k, the (new) broker of e is matched with g and j the owner of g is matched with e. If $c[\nu]_\emptyset(e) = (k,o)$ we obtain a similar contradiction to strategyproofness: $e = M(R)(j) R'_j M(R'_j, R_{-j})(j) = c[\nu](\overline{R}'_j, \overline{R}_{-j})(j)$ since $c[\nu](\overline{R}'_j, \overline{R}_{-j})(k) = e$. So $c_\nu(e) = (j, \cdot)$ must hold. Since j owns g at $c[\nu]_\emptyset = c_\nu$, and since by (C3) no broker owns a house, j must own e under $c[\nu]_\emptyset = c_\nu$.

When $c[\nu]_\emptyset(e) = (j,o)$ the submatching $\{(j,e)\}$ is $c[\nu]$-relevant. To see that i_b must own g at this submatching fix any profile of preferences $\tilde{R}$ for the agents $\overline{N}_\nu \setminus \{j\}$ over the houses $\overline{H}_\nu \setminus \{e\}$ with $\tilde{R}_{i_b} : g$. Define R such that $R_i = R^\circ_i$ for all $i \in \overline{N}_\nu$, $R_j : e$, $R_{i_b} : g$ and $\tilde{R}_i$ is restriction of R_i to

$\overline{H}_\nu \setminus \{e\}$ for all $i \in \overline{N}_\nu \setminus \{j\}$. Lemma 8 implies that $M(R)(i_b) = g$. The preceding section implies that $M(R) = \nu \cup c[\nu](\overline{R})$. Since $c[\nu]$ is a trading and braiding mechanism with $c[\nu]_\emptyset(e) = (j, o)$ and since $\overline{R}_j : e$ we obtain $c[\nu](\overline{R}) = \{(j, e)\} \cup c[\nu \cup \{(j, e)\}](\tilde{R})$. In sum we have that $M(R)(i_b) = g = c[\nu \cup \{(j, e)\}](\tilde{R})(i_b)$. So (*) implies that i_b owns g under $c[\nu]_{\{(j,e)\}} = c_{\nu \cup \{(j,e)\}}$.
□

7 Conclusion

Let me highlight three features of the proof of Theorem 1. I started by showing that the order of elimination of trading cycles does not matter in the definition of trading and braiding cycles mechanisms. The freedom to eliminate trading cycles in any order allows me to conveniently structure the inductive proof that trading and braiding mechanisms are good. The calculation of any outcome $c(R)$ can in particular be split into finding one direct c-successor ν of $\emptyset$ that is reachable under $c(R)$ and the outcome of the submechanism following ν: $c[\nu](\overline{R})$. Since any such $c[\nu]$ involves less than $n+1$ agents the inductive hypothesis applies: so any such $c[\nu]$ is good. Only very few cases need to be considered to show that c itself is good. These cases all pertain to cycles that form at the start of a trading and braiding mechanism c.

Secondly, while induction over the number of agents turns out to be a major simplifier in the proof that any trading and braiding mechanism is good, it plays only less important role in the proof that any good mechanism can be represented as a trading and braiding mechanism. The core of this part of the proof lies in definition of a control rights function $c_\emptyset$ and in showing that braids can only arise with exactly three houses. Once this groundwork has been laid the proof unfurls with far greater ease. The inductive structure over the number of agents in the mechanism only matters in the proof of Lemma 9. In this lemma I show that the calculation of any $M(R)$ (when M is not a braid) can be split into a first round of trading cycles and a submechanism. This submechanism inherits the properties of Pareto optimality, strategy proofness and non-bossiness from its parent M. Since the submechanism involves fewer agents than its parent, the submechanism has - by the inductive hypothesis - a representation as a trading and braiding cycle.

Thirdly, all three assumptions: strategy proofness, Pareto optimality and non-bossinss are repeatedly used in the proof that any good mechanism can be represented as a trading and braiding mechanism. Already the very first lemma in this proof (Lemma 2) requires all three properties. Given that strategyproofness and Pareto optimality are better founded than non-bossiness as principles of mechanism design one might wonder about a characterization of the set of all strategy proof and Pareto optimal mechanisms. But, given that my proof extensively relies on all three properties a simple extension of the same proof to the grand set of all strategy proof and Pareto optimal mechanisms is out of the question.

However, the representation of good mechanisms as trading and braiding mechanisms can be used to construct a class of Pareto optimal and strategy proof mechanisms. Simply modify control rights structures insofar as that the inheritance of houses not only depends on submatchings ν but also on the preferences of the matched agents. Such a modified control rights structure c maps any combination of a c-relevant ν and a profile of preferences R to a control rights function. Keeping (C1)-(C6) intact we would have to additionally impose that a c-relevant ν and two profiles of preferences R and R' can only be mapped to two different control rights functions if $R_i \neq R'_i$ holds for some $i \in N_\nu$. A serial dictatorship in which the second dictator depends on the first dictators preferences over houses he did not choose is the simplest example of such a bossy mechanism. The present proof that any trading and braiding mechanism is good can easily be extended to show that any mechanism, defined through such a modified control rights structure, is strategyproof and Pareto optimal. The question whether any Pareto optimal and strategyproof mechanisms can be represented by such a modified control rights structure awaits some new ideas and techniques of proof.

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