Stochastic Independence with Maxmin Expected Utilities*

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May 27, 2008

Abstract

I develop a behavioral notion of stochastic independence. This notion builds on the idea that a decisionmaker who is indifferent between a bet on some event G and a constant payment x, should be indifferent between x and an act according to which the bet is played only when a stochastically independent event E occurs and x is being paid otherwise. Considering only preferences that can be represented by maxmin expected utility functions following Gilboa and Schmeidler [10], I show that an event E is independent of an event G according to this notion, if and only if the set of prior beliefs on G is equal to the set of all Bayesian posterior beliefs on G when updating with respect to E. Refinements of this notion are discussed and characterized. I compare this notion to an alternative notion of independence that builds on the idea that E is independent of G if and only if updating with respect to G does not change the agent's evaluation of E. I argue that the criterion that learning an event E that is independent of G should not change the preference over any two bets on G should be added to the list of desiderata for updating rules.

*I thank Jürgen Eichberger, Paolo Ghirardato, Ed Green, Vijay Krishna, Alexey Kushnir, Massimo Marinacci, Maria Wihardja and the seminar audiences at Heidelberg, Paris, Cornell and Queens University for helpful comments.

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1 Introduction

Two events E and G are considered independent, if the event G occurs no more or less often when E occurs as well, as it does unconditionally. In probability theory this condition translates without much philosophizing to the condition that the two events are independent if and only if the probability of both occurring can be calculated as the product of the probabilities of E and G occurring, formally $\pi(E \cap G) = \pi(E)\pi(G)$. Independence is a simple concept as long as we assume that decision makers base their decisions on probabilities. This changes dramatically once we leave the world of probabilistic sophistication.

Consider the following example of two Ellsberg urns, the first of which containing 10 blue and yellow balls, the second containing 7 red and green balls. Label the event that a blue ball is drawn from the first urn and the event that a green ball is drawn from the second urn E and G respectively. Consider a decisionmaker that has to choose among different payoff schemes that are all based on draws from the two urns. If the decisionmaker's preferences have an expected utility representation with a probability π on the state space, then we should expect that the decision maker assigns probability $\pi(E)\pi(G)$ to the event that a blue ball is drawn from the first urn while a green ball is drawn from the second urn. This seems reasonable as the two urns represent a prototypical example of (physical) independence.

The assumption that the decisionmaker acts as an expected utility maximizer in the given context, is questionable. Camerer and Weber [2] summarize a large body of empirical violations of probabilistic sophistication when considering bets that are defined over Ellsberg urns. But how should a decisionmaker that violates expected utility theory think about the events E, G and $E \cap G$? Given the physical setup of the example the decisionmaker should probably consider the events E and G independent. But what does independence mean for the case of a decisionmaker that is not probabilistically sophisticated? Behavioral axioms of independence are needed to answer this question.

The idea behind the basic notion of independence proposed here goes as follows. Consider a bet on G and a constant payment x such that the decision maker is indifferent between the bet on G and x, the "security equivalent" of the bet. If the event E is independent of G then the decisionmaker should be indifferent between the bet on G and a more complicated act which yields the bet on G if E occurs and x otherwise. Conversely, if the decisionmaker exhibits this indifference for any such bet on G, and the corresponding more complicated act that is constructed out of the bet and its security equivalent, then we should say that the decisionmaker considers E to be independent of G. Suppose E was dependent of G according to this notion. In that case there exists a bet on G, paying α when G occurs and β otherwise, such that the decisionmaker is not indifferent between the bet and the more complicated act that pays the bet if E occurs and the security equivalent otherwise. For concreteness suppose that the decisionmaker prefers α to β, and the more complicated act to the original bet. How could this latter preference be explained? If E does not happen the complicated act pays the security equivalent of the bet, so a strict preference of the more complicated act must derive from the payments in case that E happens. But if E happens the bet is played. The only reason for this preference is that the uncertainty of receiving the good payoff α is lower in the case that E occurs. In other words, if such a preference exists we are forced to conclude that there is an underlying association between E and G. Going back to the example of the two urns, suppose that the decisionmaker is indifferent between betting a dollar on a green ball being drawn form the second urn and not betting. The independence of the event that a green ball is being drawn from the second urn and the event that a blue ball is being drawn from the first would then imply that the decisionmaker should also be indifferent between betting on green and the bet taking place only if a blue ball is drawn form the first urn.

This basic idea is captured in the notion of "weak independence" (Section 5). I strengthen this notion in two different ways by imposing symmetry and separation requirements (Sections 5 and 6). I generalize all these notions for independent events to notions for independent algebras (Section 8).

I restrict attention to preferences that can be represented with a maxmin expected utility following Gilboa and Schmeidler [10]. In this case the representation of preferences does not entail one but a convex and compact set of probabilities. I discuss some "natural" generalizations of the probabilistic notion of independence to the case of a set of multiple priors C. In particular, I relate the condition that $\pi(E \cap G) = \pi(E)\pi(G)$ for *all* probabilities in the set C and the condition that updating with respect to an event G does not change the set of probabilities assigned to another event E to behavioral axioms of independence. I fully characterize belief sets C under different assumptions on the independence between algebras on the state space.

In the literature I could discern only one application of a purely behavioral definition of independence to the case of maxmin expected utilities. Klibanoff's [16] definition of an independent randomization device corresponds to the weakest definition proposed here. His characterization result of a stochastically independent randomization device differs from the one derived here since he requires that preferences over acts that are measurable with respect to the randomization device have an expected utility maximization in addition to the requirement of stochastic independence.

The lions share of notions of stochastic independence for the case of agents that violate the independence axiom build on the idea that the belief on an event $E \cap G$ should be calculated as the product of beliefs on E and G, when these two events are stochastically independent. Gilboa and Schmeidler [10], Hendon, Jacobsen, Sloth and Tranæs [12], Ghirardato [7] all use this criterion to define their respective notions of stochastically independent events. Not all alternatives to expected utility theory focus on generalizing the independence axiom. Brandenburger, Blume and Dekel [1], for instance, focus on relaxing the Archimedean axiom in their "Lexicographic Probabilities and Choice under Uncertainty". This article is explicitly geared towards game theory and therefore also contains some observations on stochastic independence. Brandenburger, Blume and Dekel provide two definitions of independence and one axiom. Just like the works cited above, the definitions focus on generalizing the condition $\pi(E \cap G) = \pi'(E)\pi''(G)$ to the environment of lexicographic probability systems. The behavioral axiom, uses the idea that updating with respect to an independent event should not change conditional preferences to define stochastic independence. I will say more about this approach below.

One would expect that the issue of stochastic independence appears in the literature on games with uncertainty averse agents and in the literature on updating uncertainty averse preferences. Within the literature on games three different approaches can be distinguished. Some authors avoid the issue by focussing on two player games exclusively (see for example Mariacci [21] and Dow and Werlang [3]). Others assume that the player's strategies are independent according to notions that are based on the idea that the belief on an event $E \cap G$ should be calculated as the product of beliefs on E and G, Lo [20] for example uses Gilboa and Schmeidler's [10] notion of stochastic independence in his article on game theory, Lehrer's [18] notion of stochastic independence for partially-specified probabilities also derives from a similar idea. Finally, a third group of authors, acknowledges that stochastic independence would be an interesting requirement on beliefs over the strategies of other players, but proceed without any such assumption of stochastic independence (see for example Klibanoff [15] and Eichberger and Kelsey [4]).

The concept of independence is intimately related to the concept of updating. In standard probabilistic theory two events E and G are independent if and only if the posterior with respect to G does not change when learning whether E is true. In Section 7 I shall use this idea to say that E is independent of G if a decisionmaker's preference over two bets on G is never reverted by learning the event E. This does not yield one but many notions of independence since there is not one but many notions of updating in the literature on maxmin expected utilities. I will discuss some of the main pretenders in Section 7. Furthermore, I will establish relationships between the behavioral notions of independence and the notions of independence generated by using the above idea on updating. There, I will argue that given the lack of agreement on how to update maxmin expected utilities we should not use updating rules to define stochastic independence, but should rather use a behavioral notions of stochastic independence as another selection criterion for updating rules. A good updating rule should satisfy the criterion that a decisionmaker should not revert his preference over bets on G when learning an event E that is independent of G.

2 Preliminaries

2.1 State Spaces, Algebras, Acts

Throughout the text I fix a finite state space Ω. The algebra of all subsets of Ω is called σ^*. Events are denoted by capital letters E, G, H, the complement of an event E is denoted by $\overline{E}$. The algebra generated by two subalgebras σ_1 and σ_2, denoted by $\sigma_1\sigma_2$, is defined as $\sigma_1\sigma_2 = \{H : H = E \cap G \text{ for some } E \in \sigma_1,\ G \in \sigma_2\}$. The algebra induced by an event $E \in \sigma^*$ is denoted by σ_E, so $\sigma_E = \{\emptyset, E, \overline{E}, \Omega\}$.

Preferences are defined over all σ^*-measurable acts $f : \Omega \to \mathcal{P}(X)$, where X is a finite set of outcomes and $\mathcal{P}(X)$ the space of all lotteries on X. An act f is called a *bet* on event G if the act assumes one value on G and another on $\overline{G}$. A constant act is an act $f : \Omega \to \mathcal{P}(X)$ with $f(\omega) = x$ for all $\omega \in \Omega$, as a shorthand such an act is denoted by x. A constant act x_f with $f \sim x_f$ is called the *security equivalent* of f.

The act $f_E g$ which combines acts f and g is defined by $(f_E g)(\omega) = f(\omega)$ for $\omega \in E$ and $(f_E g)(\omega) = g(\omega)$ otherwise. An event E is considered *null* if $f_E g \sim h_E g$ holds for all acts $f, g, h : \Omega \to \mathcal{P}(X)$. The notation $f(G)$ is sometimes used as a shorthand for $f(\omega)$ with $\omega \in G$ when $f(\omega) = f(\omega')$ for all $\omega, \omega' \in G$.

2.2 Preference Representation

I assume throughout the text that preferences have a maxmin expected utility representation following Gilboa and Schmeidler [10].

Definition 1 *The preferences $\succsim$ can be represented by a* maxmin expected utility *if there exists an affine utility functional $u : \mathcal{P}(X) \to \mathbb{R}$ and a convex and compact set of priors C such that the utility of act f, $U(f)$ can be calculated as follows:*

$$U(f) = \min_{\pi \in C} \sum_{\omega \in \Omega} u(f(\omega))\pi(\Omega).$$

The maxmin utility representation has been axiomatized by Gilboa and Schmeidler [10]. In the special case that the set of beliefs is a singleton ($C = \{\pi\}$ for some $\pi \in \mathcal{P}(\Omega)$) the maxmin expected utility reduces to a standard expected utility. Following Nehring [17] and Ghirardato, Maccheroni and Marinacci [9], I call an event E *ambiguous* if there exist $\pi, \pi' \in C$ such that $\pi(E) \neq \pi'(E)$. Observe that only the extrema of the set of beliefs C, called $Ext(C)$ will matter in calculating actual utility values (as the minimum is always attained at an extreme point).

2.3 Stochastic Independence According to Gilboa/Schmeidler

Gilboa and Schmeidler [10] propose a notion of stochastic independence that applies to maxmin expected utilities. To understand their condition of stochastic independence remember that an event E is independent of another event G according to the probabilistic definition of independence if and only if $\pi(E \cap G) = \pi(E)\pi(G)$. This notion is standardly extended to a notion of independent algebras through the requirement that any two events in the algebras under consideration should be independent. Formally, two subalgebras $\sigma_1, \sigma_2 \subset \sigma^*$ are called independent if $\pi(E \cap G) = \pi(E)\pi(G)$ for all $E \in \sigma_1$, $G \in \sigma_2$. This condition can be restated as $\pi|_{\sigma_1 \sigma_2} = \pi|_{\sigma_1} \times \pi|_{\sigma_2}$, where $\pi|_\sigma$ denotes the restriction of π to some subalgebra σ of σ^* and $(\pi|_{\sigma_1} \times \pi|_{\sigma_2})(H)$ is defined as $\pi|_{\sigma_1}(E)\pi|_{\sigma_2}(G)$ for $H = E \cap G$ with $E \in \sigma_1, G \in \sigma_2$. Conversely for two algebras σ_1, σ_2 with $\sigma_1 \cap \sigma_2 = \{\emptyset, \Omega\}$ and two priors $\pi_i : \sigma_i \to [0,1]$ for $i = 1,2$ another a prior $\pi_1 \times \pi_2 : \sigma_1\sigma_2 \to [0,1]$ can be defined by $(\pi_1 \times \pi_2)(H) = \pi_1(E)\pi_2(G)$ for $E \in \sigma_1$, $G \in \sigma_2$ and $H = E \cap G$. Using the notation $C|_{\sigma'} = \{\pi|_{\sigma'} : \pi \in C\}$ for the restriction of the set of priors C to a particular subalgebra σ' of σ^* we are now ready to state Gilboa and Schmeidler's [10] definition of stochastic independence.

Definition 2 *Two algebras* σ_1, σ_2 *are called* GS-independent *if* $\pi|_{\sigma_1\sigma_2} \in Ext(C|_{\sigma_1\sigma_2})$ *holds if and only if there exist* $\pi|_{\sigma_i} \in Ext(C|_{\sigma_i})$ *for* $i = 1,2$ *such that* $\pi|_{\sigma_1\sigma_2} = \pi|_{\sigma_1} \times \pi|_{\sigma_2}$.

Let $C|_{\sigma_1} \times C|_{\sigma_2}$ denote convex hull of the set of all priors that can be generated as products of priors in $\pi|_{\sigma_i}$ in $Ext(C|_{\sigma_i})$, formally $C|_{\sigma_1} \times C|_{\sigma_2} := co(\{\pi|_{\sigma_1} \times \pi|_{\sigma_2} : \pi|_{\sigma_i} \in Ext(C|_{\sigma_i})$ for $i = 1,2\})$. Call this set the *product set* of $C|_{\sigma_1}$ and $C|_{\sigma_2}$. Then the above definition of GS-independence can be restated as $C|_{\sigma_1\sigma_2} = C|_{\sigma_1} \times C|_{\sigma_2}$.

3 An Example

Consider the following "picture book case" of stochastically independent events already mentioned in the introduction: There are two Ellsberg urns, the first containing blue and yellow balls the second containing red and green balls. Assume that the decision maker considers the draws from the two urns "independent". Assume, furthermore, that the decision maker's preferences have a maxmin expected utility representation with a set of priors C, such that the set of priors that blue is being drawn from the first urn (event E) satisfies $\{\pi(E)|\pi \in C\} = [1/2, 3/4]$ and the set of priors that green is being drawn from the second urn (event G) satisfies $\{\pi(G)|\pi \in C\} = [1/4, 1/2]$.

The independence axioms proposed in this paper are based on the idea that an individual's preferences over bets on an event (say green being drawn from the second urn) should not depend on the occurrence of an independent event (blue being drawn from the second urn). Formally take a bet f on green and its security equivalent x_f. Then construct an act according to which the bet f is played if and only if a blue ball is drawn from the first urn, otherwise the security equivalent is being paid. If the draws from the two urns are independent we would expect these two acts to be indifferent, formally

$f_E x_f \sim x_f$. This notion of independence can be strengthened to require that the decision maker should be indifferent between all acts that either pay the bet f or it's security equivalent x_f if E occurs while paying the same constant payoff x if E does not occur, formally $f_E x \sim {x_f}_E x$ for all constant acts x and all bets f on G. Note that the former requirement can be obtained from the latter letting $x = x_f$, so the latter requirement is strictly stronger.

I will next provide 4 different maximin utility functions $U^0, ..., U^3$ that differ only in their belief sets $C^0, ..., C^3$ on the events $E \cap G, E \cap \overline{G}, \overline{E} \cap G$ and $\overline{E} \cap \overline{G}$ (they share the same expected utility functional u over lotteries) to illustrate the potential meanings of the statement that the decision maker considers the draws from the two urns stochastically independent. I will show first that GS-independence stronger than the two requirements discussed above. Next, I provide an example of a set of beliefs in which the weaker condition mentioned above ($f_E x_f \sim x_f$ for all bets f on G) holds while the stronger one fails. Finally I show that the conditions mentioned here do not imply symmetry, I construct a set of beliefs in which $g_G x_g \sim x_g$ holds for all bets g on E while there exists a bet f on G such that $f_E x_f \nsim x_f$. The sets $C^0, ..., C^3$ are defined using priors $\pi_1, ..., \pi_6$, where each of the following matrices should be read as

$$\pi = \begin{matrix} \pi(E \cap G) & \pi(\overline{E} \cap G) \\ \pi(E \cap \overline{G}) & \pi(\overline{E} \cap \overline{G}). \end{matrix}$$

$$\pi_1 = \begin{matrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{matrix} \quad \pi_2 = \begin{matrix} 3/16 & 1/16 \\ 9/16 & 3/16 \end{matrix} \quad \pi_3 = \begin{matrix} 3/8 & 1/8 \\ 3/8 & 1/8 \end{matrix} \quad \pi_4 = \begin{matrix} 1/8 & 1/8 \\ 3/8 & 3/8 \end{matrix}$$

$$\pi_5 = \begin{matrix} 1/6 & 1/6 \\ 1/2 & 1/6 \end{matrix} \quad \pi_6 = \begin{matrix} .1 & .15 \\ .4 & .35. \end{matrix}$$

Define

$$C^0 = co(\pi_1, \pi_2, \pi_3, \pi_4), \quad C^1 = co(C^0, \pi_5),$$
$$C^2 = co(\pi_1, \pi_2), \qquad C^3 = co(\pi_1, \pi_2, \pi_6).$$

Observe that $C^0 = [1/2, 3/4] \times [1/4, 1/2]$, this implies that E and G are GS-independent if and only if $C = C^0$. I will show next that the GS-independence of E and G is sufficient for E to be independent of G according to the stronger condition mentioned above, to see this I need to show that $U^0(f_E x) = U^0(x_{f_E} x)$ for any bet f on G and any constant act x. So fix two such acts f and x and observe that

$$U^0(f_E x) = \min_{\pi \in C^0} \left[\pi(E)(\pi(G|E)f(G) + \pi(\overline{G}|E)f(\overline{G})) + \pi(\overline{E})u(x)\right] =$$

$$\min_{\pi(E)\in[1/2,3/4]} \left[\pi(E) \min_{\pi'(G)\in[1/4,1/2]} \left[\pi'(G)f(G) + \pi'(\overline{G})f(\overline{G})\right] + \pi(\overline{E})u(x)\right] =$$

$$\min_{\pi \in C^0} \left[\pi(E)U(f) + \pi(\overline{E})u(x)\right] = U(x_{f_E} x).$$

To see that GS-independence is indeed a strictly stronger condition than the two conditions discussed above, observe that the set C^1 does not satisfy GS-independence while one the other hand $f_E x \sim x_{f_E} x$ holds for all bets f on G and all constant acts x according to U^1. To see this observe first of all that $U^0(x_{f_E} x) = U^1(x_{f_E} x)$ for any two constant acts x_f and x as $\{\pi(E)|\pi \in C^0\} = \{\pi(E)|\pi \in C^1\} = [1/2, 3/4]$. Secondly, observe that $U^0(x_{f_E} x) \geq U^1(x_{f_E} x)$ as $C^0 \subset C^1$. The inverse inequality also holds, as

$$U^1(f_E x) = \min_{\pi \in C^1} \left[\pi(E)(\pi(G|E)f(G) + \pi(\overline{G}|E)f(\overline{G})) + \pi(\overline{E})u(x)\right] \geq$$

$$\min_{\pi \in C^1} \left[\pi(E) \min_{\pi' \in C^1} \left[\pi'(G|E)f(G) + \pi'(\overline{G}|E)f(\overline{G})\right] + \pi(\overline{E})u(x)\right] =$$

$$\min_{\pi(E)\in[1/2,3/4]} \left[\pi(E) \min_{\pi(G)\in[1/4,1/2]} \left[\pi(G)f(G) + \pi(\overline{G})f(\overline{G})\right] + \pi(\overline{E})u(x)\right] = U^0(f_E x).$$

The inequality holds as forcing $\pi(E)$ and $\pi(G|E)$ to vary jointly can only increase the the minimal value attained. The equality follows from the fact that $\{\pi(G|E)|\pi \in C^1\} = [1/4, 1/2]$. The desired conclusion ($U^1(f_E x) = U^1(x_{f_E} x)$) follows from the above observation that $U^0(f_E x) = U^0(x_{f_E} x)$ holds for all bets f on G and all constant acts x.

It remains to be shown that E and G are not GS-independent, for this it is sufficient to show that $\pi_5 \notin C^0$. Suppose we were able to represent π_5 as a convex combination of the corners of C^0, formally $\pi_5 = \lambda_1\pi_1 + \lambda_2\pi_2 + \lambda_3\pi_3 + \lambda_4\pi_4$. Since $\pi_5(E \cap G) = \pi_5(\overline{E} \cap \overline{G})$ it has to be true that $\lambda_3 = \lambda_4$, so we can rewrite the above equation as $\pi_5 = \lambda_1\pi_1 + \lambda_2\pi_2 + \lambda_3'\pi^*$ with $\pi^* = \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4$. Now observe that $\pi_1(E \cap G), \pi_2(E \cap G), \pi^*(E \cap G) > \pi_5(E \cap G)$ and therefore no such convex combination can exist, so $\pi_5 \notin C^0$. It can be concluded that GS-independence is not necessary for the stronger of the above mentioned conditions to hold.

Next, take C^2 as an example of a set in which the stronger independence condition mentioned above is violated, whereas the weaker one holds. To see that the stronger condition is violated define a bet f on G such that $u(f(G)) = 1$ and $u(f(\overline{G})) = 0$ and observe that $U^2(f) = 1/4$. Choose x such that $u(x) = 0$, so $U^2(x_{f_E}x) = \min_{\pi\in C^2} \pi(E)1/4 = 1/8$ and $U^2(f_E x) = \min_{\pi\in C^2} \pi(E \cap G) = 3/16$ implying that $f_E x \nsim x_{f_E}x$ for U^2. To see on the other hand that $f_E x_f \sim x_f$ holds for all bets f on G, observe that

$$U^2(f_E x_f) =$$
$$\min_{i=1,2} \left[\pi_i(E \cap G)u(f(G)) + \pi_i(E \cap \overline{G})u(f(\overline{G})) + \pi_i(\overline{E})U^2(f)\right] =$$
$$\min_{i=1,2} \left[\pi_i(E)(\pi_i(G)u(f(G)) + \pi_i(\overline{G})u(f(\overline{G}))) + \pi_i(\overline{E})U^2(f)\right] = U^2(f).$$

The crucial last step follows from the observation that

$$\pi_i(G)u(f(G)) + \pi_i(\overline{G})u(f(\overline{G})) \geq$$
$$\min_{i=1,2} \left[\pi_i(G)u(f(G)) + \pi_i(\overline{G})u(f(\overline{G}))\right] = U(f).$$

Finally let me cite C^3 as an example to show that the independence conditions introduced above need not be symmetric. Consider the act f as defined above and calculate $U^3(f) = 1/4$ together with $U^3(f_E x_f) = \min_{\pi\in C^3} \pi(E\cap G) + (1 - \pi(E))1/4 = 9/40 \neq 1/4$. Therefore $f_E x_f \sim f$ does not hold according to C^3. I omit the proof that $g_G x_g \sim g$ holds for all bets g on E as it follows along very similar lines as the arguments given in the discussion of the prior example.

To summarize observe that $[1/2, 3/4]\times[1/4, 1/2] \subset C$ held whenever E and G satisfied the stronger condition of independence mentioned above. Next, note that the appealing condition that E and G should be independent according to all priors $\pi \in Ext(C)$ turns out not to be necessary for the requirements discussed above: It was shown that the stronger condition of the two is satisfied for U^1, even though $\pi_5 \in Ext(C^1)$ and $\pi_5(E)\pi_5(G) = 2/3 \times 1/3 \neq 1/6 = \pi_5(E \cap G)$. The set C^2 shows that requiring standard independence for all priors $\pi \in Ext(C)$ is not sufficient for the stronger independence requirement. I will show in the sequel that this condition is sufficient for weaker requirement of independence. Next it is important to note that that $\{\pi(E)|\pi \in C\} = [1/2, 3/4] = \{\pi(E|G)|\pi \in C\}$ held if and only if G was considered independent of E according to the weaker notion of independence. This leads to the conjecture that the weaker requirement coincides with the requirement that the set of priors on an event G is equal to the set of updated priors on that event G - when updating with respect to the (independent) event E.

4 Sets of Beliefs

The goal of this study is to relate different axioms of stochastic independence to different conditions on the set C. Some more technical vocabulary is needed to state all the relevant conditions on C.

It should not come as a big surprise that Bayesian updates matter a great deal to the present analysis. An equivalent definition of stochastic independence for the probabilistic context states that two events E and G are independent if $\pi(G|E) = \pi(G)$, where the Bayesian update of π with respect to E with $\pi(E) \neq 0$ is defined as $\pi(\cdot|E) = \frac{\pi(\cdot \cap E)}{\pi(E)}$. Also remember that it was conjectured at the end of preceding section that sets of conditionals would play a major role in characterizing the stochastic independence.

To ensure that sets of Bayesian updates of priors in C are always well-defined I assume throughout the text that all priors in C are *mutually absolutely continuous*, formally $\pi(E) > 0$ for some $\pi \in C$ implies that $\pi'(E) > 0$ for all $\pi' \in C$. This guarantees that $\pi(\cdot|E)$ (as defined in the following paragraph) is well-defined for all $\pi \in C$ if and only if it is well-defined for one $\pi' \in C$. The assumption of mutual absolute continuity of all priors has been axiomatized by Klibanoff [15] and by Epstein and Marinacci [5]. The imposition of the assumption of mutual absolute continuity allows me to define $C(\cdot|E) := \{\pi(\cdot|E)|\pi \in C\}$ as the set of Baysian updates of all priors in C with respect to E for any set of priors C and any event E with $\pi'(E) > 0$ for some $\pi' \in C$.

Given that the relation between sets of restricted priors $C|_{\sigma_i}$ and restrictions of updates $C(\cdot|H)|_{\sigma_i}$ for H in σ_j proves to be central to the analysis of the independence between two algebras σ_i and σ_j it is useful to characterize sets of beliefs C for which $C(\cdot|H)|_{\sigma_i} \subset C|_{\sigma_i}$ for all $H \in \sigma_j$ and $i \neq j$. To this end I define the *square set* $C_1 \Box C_2$ for any two sets of priors $C_i = \{\pi_i : \sigma_i \to [0,1]\}$ with $i = 1,2$ as the maximal set of beliefs $\pi : \sigma_1\sigma_2 \to [0,1]$ such that $\pi(\cdot|H)|_{\sigma_i} \in C_i$ for $H \in \sigma_j$, $i \neq j$.

For the case that C is a singleton the algebras σ_1 and σ_2 are independent if and only if any of the following 4 conditions holds: (i) $C(\cdot|H)|_{\sigma_1} = C|_{\sigma_1}$ for all $H \in \sigma_2$ (ii)

$C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$, (iii) $\pi \in Ext(C)$ implies $\pi(E \cap G) = \pi(E)\pi(G)$ for $E \in \sigma_1$ and $G \in \sigma_2$ (iv) $C|_{\sigma_1\sigma_2} = C|_{\sigma_1} \times C|_{\sigma_2}$. This equivalence does not hold for the case that the decision maker exhibits ambiguity aversion (C not a singleton). If the set of beliefs C is not a singleton conditions (i) through (iv) can be associated with some different axiomatic notions of stochastic independence. In the following Lemma I relate the named conditions to each other, the proof of the Lemma can be found in the Appendix.

Lemma 1 *Let C be a set of beliefs and let σ_1, σ_2 be two subalgebras of σ^* such that $\sigma_1 \cap \sigma_2 = \{\emptyset, \Omega\}$. If $C|_{\sigma_1\sigma_2} = C|_{\sigma_1} \times C|_{\sigma_2}$ then $\pi|_{\sigma_1\sigma_2} = \pi|_{\sigma_1}\pi|_{\sigma_2}$ holds for any $\pi \in Ext(C)$ and any $E \in \sigma_1$ and $G \in \sigma_2$. If this latter condition holds then $C|_{\sigma_1\sigma_2}$ is a subset of $C|_{\sigma_1} \times C|_{\sigma_2}$. The subset relation $C|_{\sigma_1} \times C|_{\sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$ always holds. None of the inverse implications hold.*

Let me conclude this section by the introduction of some more shorthand notation that will come in handy when studying independent events and facilitates an illustration of Lemma 1 at the hand of the examples provided in Section 3. As a shorthand for $C|_{\sigma_E}$ I write $C(E)$, where $C(E)$ can be thought of as the set $\{\pi(E) : \pi \in C\}$, similarly I write $C(E|G)$ as a shorthand for $C(\cdot|G)|_{\sigma_E}$, where again $C(E|G)$ can be thought of as the set $\{\pi(E|G) : \pi \in C\}$. Now let $C_1 = C(E)$ and $C_2 = C(G)$ for some events E and G. The square set $C(E)\Box C(G)$ is the maximal set in which $C(E|G), C(E|\overline{G})$ are subsets of $C(E)$ and $C(G|E), C(G|\overline{E})$ are subsets of $C(G)$. Alternatively the product set $C(E) \times C(G)$ can be constructed as the convex hull of four different priors, each of which assigns either $\min_{\pi \in C} \pi(E)$ or $\max_{\pi \in C} \pi(E)$ to the event E and similarly for the event G.

In terms of the examples in Section 3 we can say that $C^0(H) = C^i(H)$ for $H = G, E$ and for all $i = 1, 2, 3$. For the set C^0 we have that $C^0(E) \times C^0(G) = C^0$. The set C^2 satisfies $\pi|_{\sigma_1\sigma_2} = \pi|_{\sigma_1}\pi|_{\sigma_2}$ for all $\pi \in Ext(C)$, it can therefore be read as an example that the inverse conclusion of the first claim in Lemma 1 is not valid. At the same time $C^2 \subset C^0$ can serve as an illustration of the second claim in the Lemma. The fact that $C^0 \subset C^1 \neq C^0$ shows that $C|_{\sigma_1} \times C|_{\sigma_2}$ does not generally equal $C|_{\sigma_1} \Box C|_{\sigma_2}$ need not hold as $C^1 \subset C^1(E) \Box C^1(G)$.

5 Weakly Independent Events

The weakest notion of independence presented in this study builds on the idea that E is independent of G if the agent is indifferent between any bet on G and an act under which the bet is played if E occurs and that pays the security equivalent of the bet otherwise.

Definition 3 *An event E is called weakly independent of an event G if and only if*

$$f_E x_f \sim f$$

holds for all bets f on G.

Klibanoff's [16] definition of independence in his characterization of stochastically independent randomization devices is somewhat more general as it applies to independent algebras. Once I generalize my definitions of independence to algebras the definition of weak independence coincides with Klibanoff's. Note that the axiom coincides with the standard definition of stochastic independence for an expected utility maximizing agent. The following equation holds for all bets on G f if and only if $\pi(E \cap G) = \pi(E)\pi(G)$ (and $\pi(E) > 0$):

$$\begin{aligned} U_\pi(f_E x_f) &= \\ \pi(E)\left(u(f(G))\pi(G|E) + u(f(\overline{G}))\pi(\overline{G}|E)\right) + (1-\pi(E))U_\pi(f) &= \\ \pi(E)\left(u(f(G))\pi(G) + u(f(\overline{G}))\pi(\overline{G})\right) + (1-\pi(E))U_\pi(f) &= U_\pi(f). \end{aligned}$$

In the following proposition I show that E is weakly independent of G, if Bayesian updating with respect to E does not change the minimal and maximal probability assigned to G according to C. This proves the conjecture developed in Section 3. Propositions 1, 2 and 3 are generalized in Theorems 3, 4 and 5. The proofs for the latter results can be found in the Appendix.

Proposition 1 *An event E is weakly independent of another event G if and only if $C(G|E) = C(G)$.*

If C is a singleton $\{\pi\}$ the condition that $C(G|E) = C(G)$ translates to the condition that $\pi(G|E) = \pi(G)$, which defines stochastic independence in the probabilistic context.

Note that the standard probabilistic definition of independence implies three kinds of symmetry. According to this definition an event E is independent of an event G if and only if any of the three following conditions holds: (i) E is independent of $\overline{G}$, (ii) $\overline{E}$ is independent of G and (iii) G is independent of E. The discussion of the set of beliefs C^3 in Section 3 already revealed that condition (iii) need not hold for weakly independent events. In the following Lemma I show that weak independence always satisfies the first kind of symmetry and that neither one of the other two kinds of symmetry needs to be satisfied. To this end, let me say that a relation of independence exhibits *complementary symmetry* if and only if H independent of H' implies that $\overline{H}$ independent of H', the relation exhibits *mutual symmetry* if and only if H independent of H' implies that H' independent of H, for $H, H' \in \{E, G, \overline{E}, \overline{G}\}$.

Lemma 2 *Let the event E be weakly independent of the event G. Then E is weakly independent of $\overline{G}$. Weak independence need neither exhibits mutual or complementary symmetry. Mutual symmetry implies complementary symmetry, the converse does not hold.*

Proof To see the first claim observe that $f_E x_f \sim f$ holds for all bets f on G if and only if it holds for all bets f' on $\overline{G}$ as an act is a bet on G if and only if it is a bet on $\overline{G}$. Now assume that E is weakly independent of G and that the relation exhibits mutual symmetry, so that G is weakly independent of E. Applying the first part of the lemma we obtain that E is weakly independent of $\overline{G}$ and G is weakly independent of $\overline{E}$. Using once again mutual symmetry we conclude that $\overline{G}$ is weakly independent of E and $\overline{E}$ weakly independent of G. Finally the initial observation implies that $\overline{E}$ is weakly independent of $\overline{G}$ and $\overline{G}$ is weakly independent of $\overline{E}$, so complementary symmetry also holds for the relation. Next consider belief set C^3 of Section 3 and define an additional belief set $C^4 = co(\pi_1, \pi_2, \pi_7)$ with

$$\pi_7 = \begin{matrix} 2/16 & 1/16 \\ 9/16 & 4/16. \end{matrix}$$

Observe that

$$C^i(E) = C^4(E|G) = C^i(E|\overline{G}) = [1/2, 3/4] \text{ for } i = 3, 4 \text{ and } C^3(E|G) = [2/5, 3/4],$$
$$C^4(G) = [3/16, 1/2] \text{ but } C^4(G|E) = [2/11, 1/2], \text{ and } C^4(G|\overline{E}) = [1/5, 1/2]$$
$$C^3(G) = C^3(G|\overline{E}) = [1/4, 1/2] \text{ but } C^3(G|E) = [1/5, 1/2].$$

Consequently the relation of weak independence according to C^3 exhibits neither mutual nor complementary symmetry, whereas the relation of weak independence according to C^5 exhibits complementary symmetry without exhibiting mutual symmetry. $\square$

According to the motivation of the definition of independence we should conclude from $C^4(E) = C^4(E|G)$ that G is not correlated with E. If we do so we have to ask, how this can be consistent with that $C^4(G) \neq C^4(G|E)$, which should imply that E is correlated with G. Shouldn't correlation be a property that exhibits mutual symmetry? The reason why the latter need not hold is that knowing E has potentially two effects. If there is a correlation between E and G knowing E should tell me something about possible distribution of G. Knowing E could also increase or decrease of my uncertainty. It is this latter effect that need not be symmetric: If knowing E reduces my uncertainty about the likelihood of G happening, it need not be true that knowing G reduces my uncertainty about the likelihood of E happening. These two effects cannot be disentangled within the framework of Gilboa and Schmeidlers [10] maxmin expected utilities. Avoiding to take sides in the discussion around symmetry let me provide an alternative definition of weak independence that mandates symmetry in addition to the notion of weak independence developed above.

Definition 4 *Two events E, G are called* weakly independent *if E is weakly independent of G and if the relation exhibits mutual symmetry.*

Observe that if two events E and G are weakly independent, this relation also exhibits complementary symmetry (Lemma 2). It turns out that two events are weakly independent if the set of beliefs on the occurrence of the two events is a subset of the square set of beliefs generated by the beliefs on E and G alone $C(E)$ and $C(G)$.

Proposition 2 *Two events E and G are weakly independent if and only if $C|_{\sigma_E \sigma_G} \subset C(E) \square C(G)$*

It might not always be easy to check whether this condition holds. A set of four ranges of conditional probabilities have to be calculated to ascertain whether two events are independent. To see that E and G are weakly independent according to belief set C^1 in Section 3 the four probability ranges $C^1(E|G)$, $C^1(E|\overline{G})$, $C^1(G|E)$ and $C^1(G|\overline{E})$ have to be calculated. Fortunately Lemma 1 allows us to conclude that $C|_{\sigma_E \sigma_G} \subset C(E) \times C(G)$

and $\pi(E \cap G) = \pi(E)\pi(G)$ for all $\pi \in Ext(C|_{\sigma_E \sigma_G})$ are two sufficient conditions for $C|_{\sigma_E \sigma_G} \subset C(E)\square C(G)$ to hold. The latter observation allows us to immediately conclude that E and G are independent for the belief sets C^0 and C^2 in Section 3. The former observation allows us that E and G would also be independent for a belief set such as $co(\pi_1, \pi_2, 1/3\pi_3 + 2/3\pi_4)$ which is a subset of $C^0 = (\pi_1, \pi_2, \pi_3, \pi_4)$, even though E and G are not independent according to the prior $1/3\pi_3 + 2/3\pi_4$.

6 Strongly Independent Events

Independence can be interpreted as a separation property. In fact assume that E is weakly independent of G. This implies that the evaluation of the uncertainty embodied in the some bets on G yields the same result whether the decisionmaker considers the bets unconditionally or whether she considers more complicated acts that pay bets on G in case that E occurs and the security equivalent otherwise. In a sense it does not matter whether the decision maker evaluates the unconditional bet or only considers the separate case that E occurs. In the next definition I strengthen weak independence by requiring a stronger property of separation. The following definition requires that a decisionmaker should be indifferent between a bet on G and its security equivalent conditional on event E, when normalizing the payoff in the alternative event $\overline{E}$ to any constant payoff. This separates the uncertainty whether E occurs from the uncertainty that G occurs in the sense that the bet on E $x_{f_E} x$ is equivalent to the more complicated act $f_E x$ which can be seen as the composition of f a bet on G and $x_{f_E} x$ a bet on E.

Definition 5 *Events E and G are called* strongly independent *if and only if*

$$f_E x \sim x_{f_E} x \text{ and } x_E f \sim x_E x_f$$
$$h_G x \sim x_{h_G} x \text{ and } x_G h \sim x_G x_h.$$

holds for all acts $f, h : \Omega \to \mathcal{P}(X)$ that are bets on G and E respectively.

Observe that strong independence has been defined as a symmetric property. Mutual and complementary symmetry are implied by the definition of strong independence. To see that strong implies weak independence simply let $x = x_f$ or $x = x_h$ in the definition of strong independence. The next Lemma formalizes the argument that weak and strong independence differ by a separation requirement.

Lemma 3

Let E and G be weakly independent. If

$$f_E x \succsim f'_E x \Rightarrow f_E x' \succsim f'_E x', \quad x_E f \succsim x_E f' \Rightarrow x'_E f \succsim x'_E f'$$
$$h_G x \succsim h'_G x \Rightarrow h_G x' \succsim h'_G x', \quad x_G h \succsim x_G h' \Rightarrow x'_G h \succsim x'_G h'$$

for all $f, f', h, h' : \Omega \to \mathcal{P}(X)$ that are bets on G and E respectively and all constant acts x, x', then E and G are strongly independent.

The proof is easy and therefore omitted. The condition named in the Lemma restricts the validity of the Sure Thing Principle to a particular set of acts, namely the bets on E and G and constant acts x, x'. Based on a different restriction of the Sure Thing principle, Zhang [24] calls an event E unambiguous if $f_E x \succsim f'_E x \Rightarrow f_E x' \succsim f'_E x'$ and $x_E f \succsim x_E f' \Rightarrow x'_E f \succsim x'_E f'$ holds for *all* acts f, f'. Using this definition it can be said that two weakly independent events E and G are strongly independent if they are unambiguous following Zhang when considering only $C|_{\sigma_E \sigma_G}$.[1] I show in the next Proposition that strong independence holds if $C(E) \times C(G) \subset C|_{\sigma_E \sigma_G}$ holds in addition to the requirement of weak independence.

Proposition 3 *Let $E, G \in \sigma^*$. Then E and G are strongly independent if and only if $C(E) \times C(G) \subset C|_{\sigma_E \sigma_G} \subset C(E) \square C(G)$.*

This proposition allows us to immediately conclude that any GS-independent events E and G are also strongly independent. In particular E and G are strongly independent according to C^0 in the example in Section 3. Since $C^1(E) \times C^1(G) \subset C^1 \subset C^1(E) \square C^1(G)$ E and G are also strongly independent according to C^1. None of the other sets defined satisfies the property that $C^i(E) \times C^i(G) \subset C^i$ and Proposition 3 tells us that E and G cannot be strongly independent according to any of these sets of beliefs.

[1] It should be remembered that an event E is called unambiguous in the present study if $C(E)$ is a singleton, see Section 2.2. Any event that is unambiguous according to this definition is unambiguous according to Zhang's definition. The converse does not hold: for $C = C(E) \times C(G)$, the event E unambiguous following Zhang whether $C(E)$ is a singleton or not.

7 Conditional Expectations

Compared to the literature on independence without expected utility maximization the literature on updating with non-expected utility is large (see for example: Gilboa and Schmeidler [11], Epstein and Schneider [6], Pires [22], Siniscalchi [23], Hanany and Klibanoff, [13]). This is surprising given the intimate link between the two concepts: In the standard probabilistic case two events E and G are independent if and only if the unconditional probability of E is equal to the conditional probability of E with G as the conditioning event, formally, $\pi(E) = \pi(E|G)$. An alternative route to define independence would not proceed via the imposition of some behavioral axiom of independence but rather state that two events E and G are independent if learning E does not make the a bet f on G any more or less attractive than its (unconditional) security equivalent x_f. This implies that the preference over two bets f and g on G is never reversed by learning an independent event E. In this section I will show that there exists a strong link between this kind of a definition of independence and the definitions proposed here. Let me denote the agents E-conditional preferences when using updating rule X as $\succsim|_E^X$. Independence can now be defined as follows.

Definition 6 *An event E is called X-independent of G, if $f \sim|_E^X x_f$ for all bets f on G. Two events E, G are called X-independent if E as well as $\overline{E}$ are X-independent of G and if G as well as $\overline{G}$ are X-independent of E.*

Without any knowledge of the updating rule X this definition is just an empty shell. To generate meaningful concepts we need to substitute different updating rules into the placeholder X. I will first discuss full Bayesian and maximum likelihood updating. Further below I will discuss dynamically consistent updating.

7.1 Bayesian and Maximum Likelihood Updating

In this section I show that the independence concept generated using full Bayesian updating ($X = B$) is equivalent to the concept of weak independence. I also show that any two events that are strongly independent are independent according to the independence concept generated by maximum likelihood updating ($X = M$), however, the converse does not generally hold. To do so I define these two updating rules.

Definition 7 *Let $E \subset \Omega, \pi(E) > 0$ for some $\pi \in C$. The conditional preference $\succsim |_E^X$ can be represented by a maxmin expected utility U_E^X using the original utility functional $u_E^X = u$ over lotteries and sets of beliefs $C_E^B = \{\pi(\cdot|E) : \pi \in C\}$ and $C_E^M = \{\pi(\cdot|E) : \pi \in C$ and $\pi(E) = \max_{\pi \in C} \pi(E)\}$ for* full Bayesian *and* maximum likelihood updating *respectively.*

Full Bayesian updating ($\succsim |_E^B$), in which every prior in the set C is updated with respect to the conditioning event E was axiomatized by Pires [22]. Maximum likelihood updating only uses updates of the priors π in the set C that assign the conditioning event E maximal probability. Gilboa and Schmeidler [11] axiomatized the Dempster-Shafer updating rule for maxmin expected utilities and showed that it coincides with the maximum likelihood updating rule defined above.

Theorem 1 *An event E is weakly independent of an event G if and only if E is B-independent of G. Two events are weakly independent if and only if they are B-independent. If two events E, G are strongly independent then they are M-independent. Conversely two M-independent events need not even be weakly independent.*

Proof Let E be weakly independent of an event G. Following Proposition 1 this is equivalent to $C(G) = C(G|E)$, which in turn holds if and only if $f \sim |_E^B x_f$ for all bets f on G, the definition of B-independence. The second claim follows directly from the first.

For the proof of the third claim observe that E and G are strongly independent if and only if $C(E) \times C(G) \subset C|_{\sigma_E \sigma_G} \subset C(E) \Box C(G)$. Let $C^* = \{\pi \in C : \pi(E) = \max_{\pi \in C} \pi(E)\}$. Since $C(E) \times C(G) \subset C|_{\sigma_E \sigma_G}$ we have that $\{C(G) \times \{\pi^*(E)\} : \pi^*(E) = \max_{\pi \in C} \pi(E)\} \subset C^*$ and consequently $C(G) \subset C^*(G|E)$. On the other hand since $C^* \subset C|_{\sigma_E \sigma_G} \subset C(E) \Box C(G)$ and $C(E) \Box C(G)$ was defined such that $\pi(G|E) \subset C(G)$ for all $\pi \in C(E) \Box C(G)$ we can conclude that $C(G) = C^*(G|E)$. Consequently we have that $f \sim |_E^M x_f$ for all bets f on G, and E, G are M-independent.

To see that the converse does not hold observe that the definition of independence through maximum likelihood updating only restricts the conditional priors for probabilities π^* such that $\pi^*(H) \in \{\min(\pi(H)), \max(\pi(H))\}$ for $H = E, G$. For any other priors $\pi \in C$ this concept does not at all restrict the updates. Referring back to section 3 define belief set $C^5 = co(C^0, \pi_8)$ with

$$\pi_8(E \cap G) = 1/8 \quad \pi_8(\overline{E} \cap G) = 1/4$$
$$\pi_8(E \cap \overline{G}) = 1/2 \quad \pi_8(\overline{E} \cap \overline{G}) = 1/8.$$

Observe first of all that E and G are not weakly independent as $\pi_8(G|E) = 1/5 \notin C^5(G) = [1/4, 1/2]$. On the other hand E and G are M-independent. To see this observe that $\max \pi(H) \neq \pi_8(H)$ for $H \in \{E, G, \overline{E}, \overline{G}\}$. So π_8 will not appear in the set $C^*(H) = \{\pi \in C^5 : \pi(H) = \max_{\pi \in C^5} \pi(H)\}$ for $H \in \{E, G, \overline{E}, \overline{G}\}$. The problem is thus equivalent to the updating problem with the set of priors C^0 according to which E and G are strongly independent. But we have shown already that this implies that E and G are M-independent. □

Pires [22] axiomatization of full Bayesian updating links updated to unconditional preferences through an axiom which states that $f \sim_E x$ implies that $f_E x \sim x$ for all non-null events E, where $\succsim_E$ denotes the conditional preference. This axiom can be used to give an elementary and simple proof of the fact that any two events that are B-independent are weakly independent. To see this suppose that E was B-independent of G. This would imply that $f \sim |_E^B x_f$ for all bets f on G. Pires axiom then implies in turn that $f_E x_f \sim x_f$ for all bets f on G, which is none other than the definition of E being weakly independent of G. While the converse is not as obvious it has to be said that the Pires proof of her axiomatization of Full Bayesian Updating and the proof that E is weakly independent of G if and only if $C(G) = C(G|E)$ are very similar.

Gilboa and Schmeidler [11] show that if there exists an $x^* \in X$ such that $x^* \succsim f$ for all acts $f : \Omega \to \mathcal{P}(X)$ the maximum likelihood updating rule can be defined by $f \succsim |_E^M g :\Leftrightarrow f_E x^* \succsim g_E x^*$. This observation yields again an elementary and simple proof that any two events that are strongly independent are M-independent. To see this observe that if $f_E x \sim x_{f_E x}$ holds for all bets f on G and all x, then it holds in particular for $x = x^*$. The above observation then implies that $f|_E^M x_f$ for all bets f on G, which in turn implies that E is M-independent of G. It should also be apparent that there is no reason for the opposite conclusion: without any further assumptions $f_E x^* \succsim g_E x^*$ does not generally imply $f_E x \succsim g_E x$ for any other constant act x. The maximum likelihood updating rule requires that the subject discards all priors that do not maximize the probability of the event E when updating with respect to E. The failure to generate a sensible notion of independence relates to the "extremism" of the maximum likelihood updating rule, which lies in the fact that all measures that do not maximize the probability assigned to the event E are discarded from consideration when updating with respect to E. This feature of maximum likely hood updating entails that the definition of independence through updating does not place any restriction on the priors in C which don't assign maximal probability to the "independent" events E and G.

7.2 Dynamic Consistency

Dynamic consistency requires that learning E should be irrelevant for the choice among acts that are identical for the case that E does not happen. Formally, an updating rule X is dynamically consistent if for any three acts f, g, h and for any non-null event E we have that $f_E h \succsim g_E h$ implies $f_E h \succsim |_E^X g_E h$. For expected utility maximizers dynamic consistency implies Bayesian updating and vice versa. In the case of multiple prior preferences considered here neither full Bayesian updating nor maximum likelihood updating are dynamically consistent. These updating rules satisfy two other criteria: consequentialism which requires that the preference $\succsim |_E^X$ should only depend on values that the acts attain on E, and reduction which requires that preferences are defined over acts. It has been shown that these three criteria, dynamic consistency, reduction and consequentialism imply expected utility when requiring some additional standard axioms (see Karni and Schmeidler [14] and Ghirardato [8]).

To obtain dynamically consistent updating rules some other features have to sacrificed. I will next discuss two ways out of this dilemma. Epstein and Schneider [6] single out a particular class of events for which full Bayesian updating is dynamically consistent. Hanany and Klibanoff [13] drop the requirement of consequentialism and weaken the requirement of dynamic consistency somewhat to obtain a set of appealing updating rules for the case of maxmim expected utilities.

The following definition of relative rectangularity generalizes the class of events singled out by Epstein and Schneider [6].

Definition 8 *The algebra σ' is* relatively rectangular *with respect to $E, \overline{E}$ if for all $\pi^1, \pi^2, \pi^3 \in C$ there exists a $\pi^4 \in C$ such that*

$$\pi^4|_{\sigma'} = \pi^3(E)\pi^1(\cdot|E)|_{\sigma'} + \pi^3(\overline{E})\pi^2(\cdot|\overline{E})|_{\sigma'}.$$

The set C is rectangular with respect to $E, \overline{E}$ according to the original definition by Epstein and Schneider [6] if σ^* is relatively rectangular with respect to $E, \overline{E}$ according to generalized definition proposed here. Epstein and Schneider's result that full Bayesian updating with with respect to E is dynamically consistent iff C is rectangular with respect to $E, \overline{E}$ can be extended somewhat using the definition of relative rectangularity. When considering only choices among σ'-measurable acts, Bayesian updating with respect to E implies dynamic consistency if σ' is relatively rectangular with respect to $E, \overline{E}$. The reason for generalizing the notion of rectangularity to relative rectangularity is that my notion of weak independence is such that if E and $\overline{E}$ are weakly independent of σ' then σ' is relatively rectangular with respect to $E, \overline{E}$. In that case weak independence implies that $\pi^1(\cdot|E)|_{\sigma'}$ and $\pi^2(\cdot|\overline{E})|_{\sigma'}$ are both contained in $C|_{\sigma'}$, so the convex combination of the two is also contained in $C|_{\sigma'}$. This observation should not come as a big surprise after the proof that the notion of independence implied by full Bayesian updating is the notion of weak independence. This proof implies in particular that full Bayesian updating is dynamically consistent for the case of updating with respect to independent events. The results of Epstein and Schneider imply that C must be (relatively) rectangular with respect to E and $\overline{E}$. Observe, however, that independence does not imply rectangularity: it is not true that full Bayesian updating is consistent only for independent events.

Hanany and Klibanoff's [13] commitment to have a dynamically consistent updating rule for *all* events E forces them to let the updated preferences depend on more than just the updating event. They derive a set of dynamically consistent updating rules that depend on the conditioning event E, the set of choices B (which is assumed to be compact and convex) and the act h which is chosen from B unconditionally. In addition they only require dynamic consistency on the optimal path. I next define the set of all dynamically consistent updating rules $\succsim|_{(E,h,B)}$ following Hanany and Klibanoff and in this set I single out $\succsim|^*_{(E,h,B)}$ as the rule in which learning minimally reduces ambiguity.

Definition 9 *Let $E \in \sigma^*, \pi(E) > 0$ for some $\pi \in C$ and let f, g, h be any σ^*-measurable acts. Let B be a convex and compact set of acts containing f, g, h, such that $h \succsim h'$ for all $h' \in B$. Updating is dynamically consistent if $f \succsim|_{(E,h,B)} g$ holds if*

$$\min_{\pi \in C'} \sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|E) \geq \min_{\pi \in C'} \sum_{\omega \in \Omega} u(g(\omega))\pi(\omega|E) \tag{1}$$

for some convex and compact set $C' \subset C$ that contains a belief π^* such that

$$\min_{\pi \in C'} \sum_{\omega \in \Omega} u(h(\omega))\pi(\omega|E) = \sum_{\omega \in \Omega} u(h(\omega))\pi^*(\omega|E) \tag{2}$$

and

$$\sum_{\omega \in E} u(h(\omega))\pi^*(\omega) \geq \sum_{\omega \in E} u(h'(\omega))\pi^*(\omega) \tag{3}$$

for all $h' \in B$ with $h(\omega) = h'(\omega)$ for $\omega \notin E$.

The ambiguity maximal dynamically consistent updating rule $\succsim|^*_{(E,h,B)}$ requires that the set C' used in equation 1 is the maximal set that satisfies the conditions named in equations 2 and 3.

The updating rule $\succsim|^*_{(E,h,B)}$ is defined such as to minimally reduce ambiguity while maintaining dynamic consistency. Hanany and Klibanoff show that an ambiguity maximal updating rule exists and that it is unique. Observe that Hanany and Klibanoff's updating rule shares the feature that only Bayesian updates are used to calculate the conditional preferences with with full Bayesian and maximum likelihood updating. At the same time the rule proposed by Hanany and Klibanoff is less "extreme" than either full Bayesian Updating which uses updates of *all* priors or maximum likelihood updating which uses *only* updates of priors that *maximize* the probability of the conditioning event. Given that Hanany and Klibanoffs dynamically consistent updating rules depend on the feasible set B and the unconditionally chosen act h, I need to revise my definition of X-independence somewhat.

Definition 10 *An event E is HK-*independent *of an event G, if $f \succsim|_{(E,f,B)} g$ holds for all $g \in B := co(f, x_f)$, all bets f on G and all dynamically consistent updating rules. An event E is HK^*-*independent *of an event G, if $f \succsim|^*_{(E,f,B)} g$ holds for all $g \in B := co(f, x_f)$ and all bets f on G.*

Note that the definition of HK-independence is strong insofar as that it requires that $f \succsim|_{(E,f,B)} g$ for *all* dynamically consistent updating rules. Any events that are independent according to this notion must also be HK^* independent as this notion builds on the idea that $f \succsim |_{(E,f,B)} g$ has to hold for *one particular* dynamically consistent updating rule, namely the ambiguity maximizing one. It turns out that the notion of HK-independence is so strong that no two ambiguous events can be HK-independent. On the other hand the notion of independence generated by the ambiguity maximizing dynamically consistent updating rule corresponds to the notion of weak independence advocated in the present paper.

Theorem 2 *There is no set of beliefs C such that G is ambiguous and E is HK-independent of G. An event E is weakly independent of an event G if and only if it is HK^*-independent of G. Two events are weakly independent if and only if they are HK^*-independent.*

Proof Let C be a set of beliefs such that E is independent of the ambiguous event G. So $C(G) = [x, y]$ with $0 < x < y < 1$ (the inequalities $0 < x,\ y < 1$ follow from the assumption of mutual absolute continuity, $x < y$ follows from the assumption that G is ambiguous). Define a bet f on G with $u(f(G)) = 1/x$, $u(f(\overline{G})) = 0$. Observe that $u(x_f) = 1$. Observe that for any $g \in co(f, x_f)$ with $f(\omega) = g(\omega)$ for all $\omega \notin E$ it must hold that $f = g$ Conditions (2) and (3) named in the Hanany and Klibanoff's construction of dynamically consistent updating rules never bind when considering choice sets $B = co(f, x_f)$. The set C' needs to fulfill only one condition such as to yield a dynamically consistent updating rule: it needs to be a convex and compact subset of C. Let $z \in C(G|E)$, observe that either $x < z$ or $y > z$ or both. Assume that the former inequality holds. Pick $C' = \{\pi\}$ such that $\pi(G|E) = z$. So for this particular updating rule it follows that $f \succsim |_{(E,f,B)} x_f$ is equivalent with

$$\sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|E) \geq u(x_f) \Leftrightarrow \frac{z}{x} \geq 1$$

a contradiction. The case in which $y > z$ holds follows by similar arguments changing the definition of f such that $u(f(G)) = 0$, $u(f(\overline{G})) = \frac{1}{1-y}$.

Now let E be HK^*-independent of G, so $f \succsim |^*_{(E,f,B)} g$ holds for all $g \in co(f, x_f)$ for all bets f on G. The conditional preference $f \succsim |^*_{(E,f,B)} g$ holds if and only if the following

$$\min_{\pi \in C'} \sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|E) \geq \min_{\pi \in C'} \sum_{\omega \in \Omega} u(g(\omega))\pi(\omega|E)$$

for the maximal subset C' of C that satisfies conditions (2) and (3) in Hanany and Klibanoff's definition of dynamically consistent updating rules. But we observed already above that these conditions have no bite in the current case. Therefore the maximal subset of C that satisfies these conditions is C itself. So $f \succsim |^*_{E,f,B} g$ holds if and only if

$$\min_{\pi \in C} \sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|E) \geq \min_{\pi \in C} \sum_{\omega \in \Omega} u(g(\omega))\pi(\omega|E) \Leftrightarrow$$
$$\min_{\rho \in C(\cdot|E)} \sum_{\omega \in \Omega} u(f(\omega))\rho(\omega) \geq \min_{\rho \in C(\cdot|E)} \sum_{\omega \in \Omega} u(g(\omega))\rho(\omega)$$

So E is HK^* independent of G if and only if $C(G|E) = C(G)$. But this conditions is of course equivalent to E being weakly independent of G as was shown in Proposition 1. □

Hanany and Klibanoff clearly delineate how much dynamic consistency we can hope for in the context of updating multiple prior preferences. They describe the set of all dynamically consistent updating rules when using a rather weak concept of dynamic consistency. They go on to show that most stronger requirements of dynamic consistency trigger the non-existence of such rules. As a conclusion I would like to suggest to complement their requirement of dynamic consistency by the requirement that no preference reversals should occur when learning about an independent event. If I prefer steak to broccoli unconditionally and if the quality of either dish is unrelated to the event of "rain in Hamburg", then my learning of the weather in Hamburg should not influence my preferences over the two dishes. With the above theorem I have shown that such a requirement does not lead to the non-existence of dynamically consistent updating rules. If defining independence as weak independence then Hanany and Klibanoff's ambiguity maximal dynamically consistent updating rule, satisfies the requirement.

8 Independent Algebras

In this section I extend the notions of independence introduced above to the question when two subalgebras $\sigma_1, \sigma_2 \subset \sigma^*$ are independent. This is important insofar as that one of the main applications of the notion of stochastic independence in economics is game theory. In game theory we need to be able to say when two strategies are independent. Two strategies are called independent if they are measurable with respect to two independent algebras on Ω. Given the above motivation of weak independence it seems reasonable to require that $f_E x_f \sim x_f$ should hold for any σ_2-measurable act for E to be independent of σ_2. Consider again the case in which this indifference is violated, say $f_E x_f \succ x_f$, since x_f and $f_E x_f$ pay the same in case that E does not happen the strict preference for $f_E x_f$ should derive from the case that E happens. But the act f restricted to E can only yield a higher utility than the unconditional act f if knowing E reduces the uncertainty about the better outcomes of f happening. I interpret such a reduction of uncertainty as a violation of independence.

Definition 11 *An event E is called weakly independent of an algebra σ_2 if and only if*

$$f_E x_f \sim f$$

holds for all σ_2-measurable acts $f : \Omega \to \mathcal{P}(X)$.

The following extension of Proposition 1 is obtained for the present case:

Theorem 3 *Let $E, G \in \sigma^*$. Then E is weakly independent of σ_2 if and only if $co(C(\cdot|E)|_{\sigma_2}) = C|_{\sigma_2}$.*

Imposing weak independence of σ_i of all events in $E \in \sigma_j$ for $i \neq j$ a more demanding definition for σ_1 and σ_2 being weakly independent is obtained:

Definition 12 *Two algebras σ_1 and σ_2 are called weakly independent if and only if*

$$f_E x_f \sim f \qquad \text{and} \qquad h_G x_h \sim h$$

holds for all σ_2-measurable acts $f : \Omega \to \mathcal{P}(X)$, all σ_1-measurable acts $h : \Omega \to \mathcal{P}(X)$, all $E \in \sigma_1$ and all $G \in \sigma_2$.

Observe that this notion of weak independence for algebras generalizes the notion of weakly independent events. Two events E and G are weakly independent if and only if σ_E and σ_G are independent. Of course Proposition 2 for the case of weakly independent events can be generalized to a theorem on weakly independent algebras.

Theorem 4 *Two algebras σ_1 σ_2 are weakly independent if and only if $C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$.*

Applying Lemma 1 to the above theorem it can be concluded that $\pi|_{\sigma_1\sigma_2} = \pi_{\sigma_1} \times \pi_{\sigma_2}$ for all $\pi|_{\sigma_1\sigma_2} \in Ext(C|_{\sigma_1\sigma_2})$ is a sufficient condition to σ_1 and σ_2 to be independent. In short, if $C|_{\sigma_1\sigma_2}$ can be constructed as the convex hull of a set of probability measures for which σ_1 and σ_2 are independent according to the standard definition of stochastic independence, then σ_1 and σ_2 are weakly independent. A weaker (and less operational) sufficient condition for σ_1 and σ_2 to be weakly independent is that $C|_{\sigma_1\sigma_2}$ is a subset of $C|_{\sigma_1} \times C|_{\sigma_2}$.

In the section on independent events strong independence was derived from weak independence imposing the requirement that a decisionmaker can consider uncertainty about G in separation from uncertainty about E. This separation implies that the bet on E $x_{f_E}x$ is equivalent to the more complicated act $f_E x$ which can be seen as the composition of f a bet on G and $x_{f_E}x$ a bet on E. It was observed above in Lemma 3 that weak implies strong independence for two events E and G if

$$f_E x \succsim f'_E x \Rightarrow f_E x' \succsim f'_E x', \quad x_E f \succsim x_E f' \Rightarrow x'_E f \succsim x'_E f'$$
$$h_G x \succsim h'_G x \Rightarrow h_G x' \succsim h'_G x', \quad x_G h \succsim x_G h' \Rightarrow x'_G h \succsim x'_G h'$$

holds for all $f, f', h, h' : \Omega \to \mathcal{P}(X)$ that are bets on G and E respectively and all constant acts x, x'.

In the same vein the strong independence of two algebras is obtained from the weak independence of two algebras imposing the additional requirement that the decisionmaker considers uncertainty in the dimension of σ_1 separately from uncertainty on σ_2. Formally, this requirement can be stated as

$$f_E g \succsim f'_E g \Rightarrow f_E g' \succsim f'_E g', \quad g_G f \succsim g'_G f \Rightarrow g_G f' \succsim g'_G f'$$

for all $f, f' : \Omega \to \mathcal{P}(X)$ that are σ_2-measurable, all $g, g' : \Omega \to \mathcal{P}(X)$ that are σ_1-measurable and all $E \in \sigma_1$, $G \in \sigma_2$.

Definition 13 *An algebra σ_1 is called strongly independent of an algebra σ_2 if and only if*

$$f_E g \sim x_{f_E} g \qquad \text{and } g_G f \sim x_{g_G} f$$

for all σ_2 measurable acts $f : \Omega \to \mathcal{P}(X)$ and all σ_1-measurable acts g and all $E \in \sigma_1, G \in \sigma_2$.

This axiom, again, reflects a stronger property of separation between the uncertainty on σ_1 and σ_2. It says that if uncertainty in the dimension of σ_1 occurs only for one event

$G \in \sigma_2$ then this uncertainty can be considered separately in the sense that an indifferent act can be constructed by replacing f with its security equivalent on G. Observe that this definition is equivalent to the definition of strong independence for events when $\sigma_1 = \sigma_E$ and $\sigma_2 = \sigma_G$.

Theorem 5 *Two algebras σ_1 σ_2 are strongly independent if and only if $C|_{\sigma_1} \times C|_{\sigma_2} \subset C|_{\sigma_1 \sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$.*

In the standard case two algebras σ_1, σ_2 are independent according to if and only if any two events $E \in \sigma_1$ and $G \in \sigma_2$ are independent. This relation of equivalence no longer holds for the case of maxmin expected utilities.

Lemma 4 *If σ_1, σ_2 are weakly (strongly) independent then $E \in \sigma_1$ and $G \in \sigma_2$ are weakly (strongly) independent. The converse does not hold in either case.*

9 Conclusion

In this paper I proposed various behavioral axioms to capture the notion of independent events and algebras. The basic axiom (weak independence) states that an event E is independent of another event G if the decisionmaker is for all bets on G indifferent between that bet and a more complicated act that pays the bet only if E occurs and pays the security equivalent of the bet otherwise. This notion was strengthened by some requirements of symmetry and separation. I applied the behavioral axioms to maxmin expected utilities following Gilboa and Schmeidler [10]. The behavioral notions of independence were compared to a different notion of independence which is derived from the idea that conditional preferences should not differ from unconditional preferences over a bet on an event G if the conditioning event E is independent of G. I argued that we should not use concepts of updating to generate notions of independence via the requirement just mentioned but that we should rather demand that our theories of updating satisfy this criterion.

Ultimately the question whether agents consider any two events stochastically independent and what they mean by independence is an empirical one. I hope to test the axioms proposed in the present study in laboratory experiments. Next it would be important to discover the implications of the axioms of independence proposed here when assuming different representations of uncertainty averse preferences. The combination of the two could, in turn, yield important empirical insights about the validity of some of these theories: say it turns out that a particular representation of uncertainty averse preferences X admits weakly independent events if and only if the representation reduces to an expected utility representation. Say the laboratory experiments find that people do consider draws from different urns weakly independent. This could be seen as an argument against this representation X. Finally, it is hoped that the axiomatic and empirical investigation of "stochastic independence" would give some new momentum to the literature on games with uncertainty averse agents.

APPENDIX

Proof of Lemma 1

The first claim directly follows from the definition of $C|_{\sigma_1} \times C|_{\sigma_2}$. To see the second claim assume that $\pi \in Ext(C|_{\sigma_1\sigma_2})$ implies that $\pi = \pi|_{\sigma_1} \times \pi|_{\sigma_2}$ for some $\pi|_{\sigma_1} \in C|_{\sigma_1}$ and $\pi|_{\sigma_2} \in C|_{\sigma_2}$. Observe that there exist finite subsets $\{\pi^i\}_{i\in I} \in Ext(C|_{\sigma_1})$, $\{\pi^j\}_{j\in J} \in Ext(C|_{\sigma_2})$ and sets of positive parameters $\{\lambda^i\}_{i\in I}, \{\kappa^j\}_{j\in J}$ such that $\sum_{i\in I} \lambda^i = 1$, $\sum_{j\in J} \kappa^j = 1$ $\pi|_{\sigma_1} = \sum_{i\in I} \lambda^i\pi^i$ and $\pi|_{\sigma_2} = \sum_{j\in J} \kappa^j\pi^j$. Consequently π can be expressed as

$$\pi = (\sum_{i\in I} \lambda^i\pi^i) \times (\sum_{j\in J} \kappa^j\pi^j) = \sum_{(i,j)\in I\times J} \lambda^i\kappa^j(\pi^i \times \pi^j)$$

where $\sum_{(i,j)\in I\times J} \lambda^i\kappa^j = 1$, $\lambda^i\kappa^j > 0$ for all $(i,j) \in I \times J$ and $\pi^i \times \pi^j \in Ext(C|_{\sigma_1} \times C|_{\sigma_2})$. This in turn implies that any $\pi' \in C|_{\sigma_1\sigma_2}$ can be expressed as a convex combination of the set of $Ext(C|_{\sigma_1} \times C|_{\sigma_2})$. Therefore $C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1} \times C|_{\sigma_2}$ holds as claimed. The subset relation $C|_{\sigma_1} \times C|_{\sigma_2} \subset C|_{\sigma_1}\Box C|_{\sigma_2}$ always holds as $(C|_{\sigma_1} \times C|_{\sigma_2})(\cdot|E)|_{\sigma_j} = C|_{\sigma_j}$ for all $E \in \sigma_i$ and $i \neq j$.

To see that $C|_{\sigma_1}\Box C|_{\sigma_2}$ is convex consider two priors $\pi^1, \pi^2 \in C|_{\sigma_1}\Box C|_{\sigma_2}$ and fix $\lambda \in (0,1)$. Let $E \in \sigma_i$ and $i \neq j$ and observe that

$$\lambda\pi^1(\cdot|E)|_{\sigma_j} + (1-\lambda)\pi^2(\cdot|E)|_{\sigma_j} \in C|_{\sigma_j}$$

since $\pi^1(\cdot|E)|_{\sigma_j}, \pi^2(\cdot|E)|_{\sigma_j} \in C|_{\sigma_j}$ which is itself a convex set. Consequently the set $C|_{\sigma_1}\Box C|_{\sigma_2}$ is convex. Counterexamples for the inverse inclusions can be found in the text right after the statement of the Lemma.

For the remaining proof let me define $U_\pi(f)$ as $\sum_{\omega\in\Omega} u(f(\omega))\pi(\omega)$, in words $U_\pi(f)$ represents the expected utility of f when using prior π to evaluate the probability on Ω.

Proof of Theorem 3

Assume that $co(C(\cdot|E)|_{\sigma_2}) = C|_{\sigma_2}$.[2] Calculate $U(f_E x_f)$ as follows:

$$U(f_E x_f) = \min_{\pi\in C}\left[\pi(E)\sum_{\omega\in\Omega} u(f(\omega))\pi(\omega|E) + (1-\pi(E))U(f)\right]$$

The equality $co(C(\cdot|E)|_{\sigma_2}) = C|_{\sigma_2}$ implies that $\sum_{\omega\in\Omega} u(f(\omega))\pi(\omega|E)$ is bounded from below by $U(f)$ and that this bound is attained for some $\pi \in C$. This implies the equality $U(f_E x_f) = U(f)$.

Suppose there existed a $\rho \in co(C(\cdot|E)|_{\sigma_2})/C|_{\sigma_2}$. Since $C|_{\sigma_2}$ is convex and closed there exists a hyperplane that separates ρ from $C|_{\sigma_2}$, so there exists a vector $\xi \in u(\Delta(X))^\Omega$ such that $\min_{\pi\in C|_{\sigma_2}} \pi\cdot\xi > \rho\cdot\xi$ and such that $\xi_\omega \neq \xi_{\omega'}$ only occurs when there exist $G, G' \in \sigma_2$ such that $G\cap G' = \emptyset$ and $\omega \in G, \omega' \in G'$. Let f be a σ_2-measurable act such that $u(f(\omega)) = \xi_\omega$ for all $\omega\in\Omega$, which implies that $\min_{\pi\in C|_{\sigma_2}} U_\pi(f) > U_\rho(f)$. Let π' be such that $\rho(G) = \pi'(G|E)$ for all $G\in\sigma_2$. Observe that

$$\begin{aligned} U(f_E x_f) = \min_{\pi\in C} U_\pi(f_E x_f) \leq \\ \sum_{\omega\in E} u(f(\omega))\pi'(\omega) + \pi'(\overline{E})U(f) = \pi'(E)\sum_{\omega\in\Omega} u(f(\omega))\rho(\omega) + \pi'(\overline{E})U(f) < \\ \pi'(E)U(f) + \pi'(\overline{E})U(f) = U(f) \end{aligned}$$

So there exists an f for which $f_E x_f \prec f$. We conclude that the weak independence of E from σ_2 implies that $co(C(\cdot|E)|_{\sigma_2}) \subset C|_{\sigma_2}$.

To prove the inverse inclusion suppose we had a $\pi^* \in C|_{\sigma_2}/co(C(\cdot|E)|_{\sigma_2})$. Since $co(C(\cdot|E)|_{\sigma_2})$ is convex and closed there exists a hyperplane that separates π^* from

[2] An example that $co(C(\cdot|E)|_{\sigma_2}) = C(\cdot|E)|_{\sigma_2}$ need not hold is available from the author upon request.

$co(C(\cdot|E)|_{\sigma_2})$, mathematically there exists a vector $\xi \in u(\Delta(X))^{\Omega}$ such that $\min_{\pi \in co(C(\cdot|E)|_{\sigma_2})} \pi \cdot \xi > \pi^* \cdot \xi$ and such that $\xi_\omega \neq \xi_{\omega'}$ only occurs when there exist $G, G' \in \sigma_2$ such that $G \cap G' = \emptyset$ and $\omega \in G, \omega' \in G'$. Let f be a σ_2-measurable act such that $u(f(\omega)) = \xi_\omega$ for all $\omega \in \Omega$, which implies that $\min_{\pi \in co(C(\cdot|E)|_{\sigma_2})} U_\pi(f) > U_{\pi^*}(f)$. The latter implies that $\min_{\pi \in C} \pi(E) U_{\pi(\cdot|E)}(f) + (1 - \pi(E))U(f) > U(f)$ for any event E with $\pi(E) > 0$ for some $\pi \in C$. This implies that $f_E x_f \succ f$ as $U(f_E x_f) = \min_{\pi \in C} \pi(E) U_{\pi(\cdot|E)}(f) + (1-\pi(E))U(f)$. In sum $co(C(\cdot|E)|_{\sigma_2}) = C|_{\sigma_2}$ holds if and only if E is independent of σ_2.

Proof of Theorem 4

To see that weak independence implies $C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$ observe that σ_1, σ_2 are weakly independent if and only if $C|_{\sigma_1\sigma_2}(\cdot|E)|_{\sigma_i} = C|_{\sigma_i}$ for $i \neq j$ and all $E \in \sigma_j$. Consequently $C|_{\sigma_1\sigma_2}(\cdot|E)|_{\sigma_i} \subset C|_{\sigma_i}$ holds for $i \neq j$ an and all $E \in \sigma_j$.

Now assume $C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$ was true. This implies that $C|_{\sigma_1\sigma_2}(\cdot|E)|_{\sigma_i} \subset C|_{\sigma_i}$ for $i \neq j$ and $E \in \sigma_j$. To show weak independence it has to be shown that this relation holds with equality. Observe that for any $E \in \sigma_1$ and any f that is σ_2-measurable

$$U(f_E x_f) = \min_{\pi \in C|_{\sigma_1\sigma_2}} U_\pi(f_E x_f) \geq \min_{\pi \in C|_{\sigma_1} \Box C|_{\sigma_2}} U_\pi(f_E x_f) = \\ \min_{\pi \in C|_{\sigma_1}, \pi' \in C|_{\sigma_2}} [\pi(E) U_{\pi'}(f) + (1 - \pi(E))U(f)] = U(f)$$

So for $f_E x_f \sim f$ to be violated we would have to have a σ_2-measurable act f and an event $E \in \sigma_1$ such that $f_E x_f \succ f$. This in turn would have to imply that $\sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|E) > U(f)$ holds for all $\pi \in C$. It would also have to hold that $f_{\overline{E}} x_f \succsim f$, which would in turn imply that $\sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|\overline{E}) \geq U(f)$ for all $\pi \in C$. Summing up the last two expressions we obtain for all $\pi \in C$ that $U_\pi(f) > U(f)$ as $U_\pi(f)$ can be written as

$$\pi(E) \sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|E) + (1 - \pi(E)) \sum_{\omega \in \Omega} u(f(\omega))\pi(\omega|\overline{E})$$

This contradiction the definition of $U(f)$ as $\min_{\pi \in C} U_\pi(f)$.

Proof of Theorem 5

I will show first that $\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} u(f_E g) = \min_{\pi\in C|_{\sigma_1}\Box C|_{\sigma_2}} u(f_E g)$ holds for any acts f, g that are σ_1- and σ_2-measurable respectively and any $E \in \sigma_1$. To see this observe that,

$$\min_{\pi\in C|_{\sigma_1}\Box C|_{\sigma_2}} U_\pi(f_E g) = \min_{\pi\in C|_{\sigma_1}\Box C|_{\sigma_2}} \left[\pi(E)U_{\pi(\cdot|E)}(f) + \sum_{\omega\notin E} u(g(\omega))\pi(\omega)\right] \geq$$
$$\min_{\pi\in(C|_{\sigma_1}\Box C|_{\sigma_2})|_{\sigma_1}, \pi'\in(C|_{\sigma_1}\Box C|_{\sigma_2})(\cdot|E)|_{\sigma_2}} \left[\pi(E)U_{\pi'}(f) + \sum_{\omega\notin E} u(g(\omega))\pi(\omega)\right] \geq$$
$$\min_{\pi\in C|_{\sigma_1} \pi'\in C|_{\sigma_2}} \left[\pi(E)U_{\pi'}(f) + \sum_{\omega\notin E} u(g(\omega))\pi(\omega)\right] = \min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} u(f_E g)$$

Where $(C|_{\sigma_1}\Box C|_{\sigma_2})|_{\sigma_1} \subset C|_{\sigma_1}$ and $(C|_{\sigma_1}\Box C|_{\sigma_2})(\cdot|E)|_{\sigma_2} \subset C|_{\sigma_2}$ follow from the definition of the square set. The opposite inequality $\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} u(f_E g) \geq \min_{\pi\in C|_{\sigma_1}\Box C|_{\sigma_2}}$ follows from the observation that $C|_{\sigma_1}\times C|_{\sigma_2} \subset C|_{\sigma_1}\Box C|_{\sigma_2}$.

Next assume that $C|_{\sigma_1}\times C|_{\sigma_2} \subset C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1}\Box C|_{\sigma_2}$ holds and show that $f_E g \sim x_{f_E} g$ holds for all f, g σ_1 and σ_2 measurable respectively and all $E \in \sigma_2$. The subset relation together with the preliminary observation made above imply the following:

$$\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} U_\pi(f_E g) \geq \min_{\pi\in C|_{\sigma_1\sigma_2}} U_\pi(f_E g) \geq \min_{\pi\in C|_{\sigma_1}\Box C|_{\sigma_2}} U_\pi(f_E g)$$
$$\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} U_\pi(f_E g) = \min_{\pi\in C|_{\sigma_1}\Box C|_{\sigma_2}} U_\pi(f_E g) \Rightarrow$$
$$\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} U_\pi(f_E g) = \min_{\pi\in C|_{\sigma_1\sigma_2}} U_\pi(f_E g).$$

By the same logic $\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} U_\pi(x_{f_E} g) = \min_{\pi\in C|_{\sigma_1\sigma_2}} U_\pi(x_{f_E} g)$ holds. It is therefore sufficient to show that

$$\min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} U_\pi(f_E g) = \min_{\pi\in C|_{\sigma_1}\times C|_{\sigma_2}} U_\pi(x_{f_E} g).$$

To see this equality observe that

$$\min_{\pi \in C|_{\sigma_1} \times C|_{\sigma_2}} U_\pi(f_E g) =$$

$$\min_{\pi \in C|_{\sigma_1} \times C|_{\sigma_2}} \left[\sum_{\omega \in E} u(f(\omega))\pi(\omega) + \sum_{\omega \notin E} u(g(\omega))\pi(\omega) \right] =$$

$$\min_{\pi' \in C|_{\sigma_1}} \left[\pi'(E) \min_{\pi \in C|_{\sigma_2}} \left[\sum_{\omega \in \Omega} u(f(\omega))\pi(\omega) \right] + \sum_{\omega \notin E} u(g(\omega))\pi'(\omega) \right] =$$

$$\min_{\pi' \in C|_{\sigma_1}} \left[\pi'(E)U(f) + \sum_{\omega \notin E} u(g(\omega))\pi'(\omega) \right] = U(x_{f_E} g).$$

I show next that strong independence of σ_1 and σ_2 implies that $C|_{\sigma_1} \times C|_{\sigma_2} \subset C|_{\sigma_1\sigma_2} \subset C|_{\sigma_1} \Box C|_{\sigma_2}$ holds. Since strong implies weak independence $C|_{\sigma_1\sigma_2}$ has to be a subset of $C|_{\sigma_1} \Box C|_{\sigma_2}$. All that remains to be shown is that $C|_{\sigma_1} \times C|_{\sigma_2}$ has to be a subset of C. Assume there existed a $\pi^* \in C|_{\sigma_1} \times C|_{\sigma_2} / C|_{\sigma_1\sigma_2}$. W.l.o.g. assume that $\pi^* \in Ext(C|_{\sigma_1} \times C|_{\sigma_2})$, so that $\pi^*|_{\sigma_1} \in Ext(C|_{\sigma_1})$ and $\pi^*|_{\sigma_2} \in Ext(C|_{\sigma_2})$.

Next define two acts f, g as follows. Let f be σ_2-measurable and let $U(f) = U_{\pi^*}(f) < U_\pi(f)$ for all $\pi \in C$ such that $\pi|_{\sigma_2} \neq \pi^*|_{\sigma_2}$. Let g be σ_1-measurable and let $g(E) = x_f$, $U(g) = U_{\pi^*}(g) < U_\pi(g)$ for all $\pi \in C$ such that $\pi|_{\sigma_1} \neq \pi^*|_{\sigma_1}$. Next calculate $U(x_{f_E} g)$ and compare its value to $U(f_E g)$.

$$U(x_{f_E} g) = \min_{\pi \in C|_{\sigma_1\sigma_2}} \left[\pi(E)U(f) + \sum_{\omega \notin E} \pi(\omega)u(g(\omega)) \right] =$$

$$\min_{\pi \in C|_{\sigma_1}} \left[\pi(E) \min_{\pi' \in C|_{\sigma_2}} \left[\sum_{\omega \in \Omega} u(f(\omega))\pi'(\omega) \right] + \sum_{\omega \notin E} \pi(\omega)u(g(\omega)) \right] =$$

$$\min_{\pi \in C|_{\sigma_1} \times C|_{\sigma_2}} \left[\pi(E) \left(\sum_{\omega \in \Omega} u(f(\omega))\pi'(\omega) \right) + \sum_{\omega \notin E} \pi(\omega)u(g(\omega)) \right] <$$

$$\min_{\pi \in C|_{\sigma_1\sigma_2}} \left[\pi(E) \left(\sum_{\omega \in \Omega} u(f(\omega))\pi(\omega) \right) + \sum_{\omega \notin E} \pi(\omega)u(g(\omega)) \right] = U(f_E g)$$

The inequality holds as for all $\pi \in C|_{\sigma_1\sigma_2}$ we have that either $\pi|_{\sigma_1} \neq \pi^*|_{\sigma_1}$ or $\pi|_{\sigma_2} \neq \pi^*|_{\sigma_2}$ or both. So we either have that $U(f_E g) = U(x_E g) > U(x_{f_E} g)$ with $x \succ x_f$ or

$U(f_E g) = U_\pi(x_{f_E} g) > U(x_{f_E} g)$ with $\pi|_{\sigma_1} \neq \pi^*|_{\sigma_1}$, or a combination of both which also implies a strict inequality.

Proof of Lemma 4

To prove this Lemma a last definition is needed. Say that the algebra σ^* of all subsets of Ω contains all events E_i for an index set I. For any algebra on Ω the set $I(\sigma) \subset I$ is called the *partitional base* of σ if $\{E_i\}_{i \in I(\sigma)}$ is a partition of Ω and $E \in \sigma$ if and only if $E = \bigcup_{i \in I'} E_i$ for some $I' \subset I$. Observe that for any $E \in \sigma$ there exists a unique subset $I' \subset I(\sigma)$ such that E can be represented as a union of the sets E_i for all $i \in I'$.

The following example shows that σ_1 need not be weakly independent of σ_2 even though all events in the two algebras are weakly independent of each other. Let $\{G_1, G_2, G_3\}$ be a partitional base of σ_2 and let $C|_{\sigma_2} = co(\pi_1, \pi_2)$ with

$$\pi_1(G_1) = 1/4, \pi_1(G_2) = 1/2, \pi_1(G_3) = 1/4$$
$$\pi_2(G_1) = 3/4, \pi_2(G_2) = 1/8, \pi_2(G_3) = 1/8$$

Let there be an event E such that $C|_{\sigma_2 \sigma_E} = co(\{\pi_1, \pi_2\} \times \{1/2\}, \pi')$ where $\pi'|_{\sigma_2, \sigma_E}$ is given as follows:

$$\pi'(G_1 \cap E) = 17/64, \quad \pi'(G_2 \cap E) = 8/64, \quad \pi_1(G_3 \cap E) = 7/64$$
$$\pi'(G_1 \cap \overline{E}) = 15/64, \quad \pi'(G_2 \cap \overline{E}) = 12/64, \quad \pi_2(G_3 \cap \overline{E}) = 5/64.$$

Observe that $\pi'|_{\sigma_2} = \frac{1}{2}\pi_1|_{\sigma_1} + \frac{1}{2}\pi_2|_{\sigma_2}$, and $\pi'(E) = 1/2$ so $co(\{\pi_1, \pi_2\} \times \{1/2\}, \pi')|_{\sigma_2} = C|_{\sigma_2}$ as already claimed above and $co(\{\pi_1, \pi_2\} \times \{1/2\}, \pi')(E) = C(E) = \{1/2\}$. Next observe that

$$\pi'(G_1|E) = \frac{17}{32} \in C(G_1) = [\frac{1}{4}, \frac{3}{4}], \quad \pi'(G_2|E) = \frac{8}{32} \in C(G_2) = [\frac{1}{8}, \frac{1}{2}]$$
$$\text{and } \pi'(G_3|E) = \frac{7}{32} \in C(G_3) = [\frac{1}{8}, \frac{1}{2}].$$

So $C(G_i|E) = C(G_i)$ for $i = 1, 2, 3$ which in turn implies that E is weakly independent of G for the set of beliefs C. Next suppose that E was weakly independent of σ_2, we would then have to have that $\pi'(\cdot|E) \in co(\pi_1, \pi_2)$. Suppose this held true. A contradiction is obtained as the equation $\frac{1}{4} = \pi'(G_2|E) = \alpha\pi_1(G_2)+(1-\alpha)\pi_2(G_2) = \alpha\frac{1}{2}+(1-\alpha)\frac{1}{8}$ implies that $\alpha = \frac{1}{3}$. But $\pi'(G_3|E) \neq \frac{1}{3}\pi_1(G_3) + \frac{2}{3}\pi_2(G_3)$. Therefore E is not weakly independent of σ_2.

The next example show that the strong independence of all events in two algebras does not imply that the two algebras are strongly independent.

Let there be a algebra σ_1 on Ω with $I(\sigma_1) = \{1, 2, 3\}$. Define a set of priors C_1 on σ_1, such that $Ext(C_1) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$, $\min_{\pi\in C_1}(\pi(E_1)) \neq \pi_4(E_1)$ and $\max_{\pi\in C_1}(\pi(E_1)) \neq \pi_4(E_1)$. Define C_2 on σ_2 with $I(\sigma_2) = \{4, 5, 6\}$ such that $Ext(C_2) = \{\pi'_1, \pi'_2, \pi'_3, \pi'_4\}$ where $\pi_j(E_i) = \pi'_j(E_{i+3})$, in short C_1 and C_2 are identical with the one difference that C_1 is defined with $\{E_1, E_2, E_3\}$ as a partitional base and C_2 is defined with $\{E_4, E_5, E_6\}$ as the partitional base. Also assume that $\sigma_1 \cap \sigma_2 = \{\emptyset, \Omega\}$.

Then define $C|_{\sigma_1\sigma_2} = co(\{\pi_1, \pi_2, \pi_3\} \times \{\pi'_4\} \cup \{\pi_4\} \times \{\pi'_1, \pi'_2, \pi'_3\} \cup \{\pi_1, \pi_2, \pi_3\} \times \{\pi'_1, \pi'_2, \pi'_3\})$. Observe that $C(E) \times C(G) = C|_{\sigma_E\sigma_G}$ for all $E \in \sigma_1, G \in \sigma_2$, so any such pair of events is strongly independent. On the other hand $C|_{\sigma_1} \times C|_{\sigma_2} = C_1 \times C_2$ is a strict superset of $C|_{\sigma_1\sigma_2}$ as $\pi_4 \times \pi'_4$ belongs to the former but not to the latter, so σ_1 and σ_2 are not strongly independent.

$\square$

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