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Pareto-optimal assignments by hierarchical exchange

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Abstract Papai's 2000 hierarchical exchange mechanisms for house allocation problems determine matchings as the outcome of multiple rounds of trading cycles. Any hierarchical exchange mechanism can be defined through a structure of ownership, which determines the ownership of houses after any round of trading cycles. Given a permutation of agents, a "permuted" hierarchical exchange mechanism can be constructed by consistently permuting agents over the entire structure of ownership. The paper shows that for any Pareto-efficient matching and any hierarchical exchange mechanism, there is a permutation of agents in the ownership structure such that the induced permuted hierarchical exchange mechanism leads to this matching.

1 Introduction

The literature on matching mechanisms studies optimal mechanisms for the one-to-one allocation of indivisible goods, typically called houses, to agents. The class of hierarchical exchange mechanisms plays a prominent role in matching theory. Papai (2000) defined and characterized this class. Hierarchical exchange mechanisms determine matchings via multiple rounds of trade in cycles. Each round starts with some initial ownership allocation of houses. Agents then point to their most preferred houses, while houses point to their owner. At least one cycle forms. Any agent in such a cycle is assigned the house he is pointing to. The agents and the houses that are not matched in the present round (or any of the preceding rounds) go on to the next round.

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In this article, I show that for any fixed Pareto optimum and hierarchical exchange mechanism one can permute the roles played by the agents such that the desired Pareto optimum is the result of the permuted mechanism. This result can be viewed as the converse of the result that any outcome of a hierarchical exchange mechanism is Pareto-optimal, which was shown by Papai (2000) as part of her characterization of the class of hierarchical exchange mechanisms.

Any hierarchical exchange mechanism is fully defined through an "ownership structure" o which determines the ownership of all houses at the beginning of any round of the mechanism. Serial dictatorship and Gale's top trading cycles mechanism are special cases of hierarchical exchange mechanisms, for housing problems with an identical number of houses and agents. In the case of a serial dictatorship, some agent starts out owning all houses. His appropriation of one of the houses can be interpreted as the formation of a trading cycle of length one. Next, another agent, the second dictator, inherits all remaining houses. In Gale's top trading cycles mechanism, each agent starts out owning exactly one house. The ownership at any of the ensuing rounds is fully determined through the so-called persistence requirement, which demands that ownership of houses is terminated only if an agent leaves the mechanism.

For any ownership structure o, let $[o]$ be the set of ownership structures that is identical to o up to a renaming of agents. So if $\widetilde{o}$ is the ownership structure that describes the serial dictatorship in which agent 1 is the first dictator, agent 2 the second dictator and so forth, then $[\widetilde{o}]$ is the class of all serial dictatorships, that can be obtained by re-ordering the agents into all possible sequences of dictators. If $\widehat{o}$ denotes one particular top trading cycles mechanism then $[\widehat{o}]$ describes the set of all top trading cycles mechanism, obtained through all permutations of the initial endowments. It turns out that $[\widehat{o}]$ and $[\widetilde{o}]$ describe the same set; this was shown by Abdulkadiroglu and Sonmez (1998). It is the goal of the present article to extend this statement to all hierarchical exchange mechanisms. More formally, take any hierarchical exchange mechanism o and define the set of mechanisms $[o]$ that obtains via permuting the agents in the mechanism. The set of Pareto optima coincides with the set of outcomes of the mechanisms in $[o]$. This statement can be split into two parts: first, any outcome of a hierarchical exchange mechanism is Pareto-optimal. Second, for any hierarchical exchange mechanism o and any Pareto optimum there exists a permutation of the agents such that the Pareto optimum is the outcome of the mechanism. The first result has been shown by Papai (2000). The present paper proves the second half.

2 Definitions

A *housing problem* (N, H, R) consists of a finite set of houses H, a finite set of agents $N := \{1, \cdots, n\}$, and a profile $R := (R_1, \cdots, R_n)$. of the agents' Preferences on H. Preferences are assumed to be strict throughout.[1] The set of all such preference profiles is denoted by $\mathcal{R}$.

[1] The expression hR_ih' means that agent i weakly prefers house h to house h'. Since preferences are assumed to be strict hR_ih' and $h'R_ih$ both obtain only if $h = h'$.

The goal of a mechanism is to obtain matchings between houses and agents. Any bijective function $\sigma : H_\sigma \to N_\sigma$ for subsets $H_\sigma \subset H$ and $N_\sigma \subset N$ is called a **submatching**. Agent $\sigma(h)$ is **assigned** house h under the submatching σ. A submatching with $H_\sigma = H$ or $N_\sigma = N$ (or both) is called a **matching**. The set of all matchings is denoted by $\mathcal{M}$. The set of all submatchings that are not themselves matchings is denoted by $\overline{\mathcal{M}}$. For any particular submatching $\sigma \in \overline{\mathcal{M}}$, the sets of agents and houses not matched by σ are denoted as $\overline{N}_\sigma$ and $\overline{H}_\sigma$, respectively. In addition to their interpretation as functions, I also use σ to denote the set of house-agent pairs $\{(h, i) : \sigma(h) = i\}$. So house h is matched to agent i according to σ if and only if $(h, i) \in \sigma$.

Formally a mechanism is a function $M : \mathcal{R} \to \mathcal{M}$. A matching σ is considered Pareto optimal if there exists no other matching $\sigma' \neq \sigma$ such that every agent is at least as well off under σ' as under σ.[2] Since N and H are fixed, I denote the set of all Pareto-optimal matchings for (N, H, R) by $PO(R)$.

In a hierarchical exchange mechanism $M : \mathcal{R} \to \mathcal{M}$ matchings are obtained from preferences through a system of ownership rights together with a trading process. Hierarchical exchange mechanisms were first defined by Papai (2000). Here I use the definition introduced by Pycia and Unver (2011). A hierarchical exchange mechanism is generated by the following two components: an allocation of ownership rights to agents and a procedural description of trade for the given ownership rights.

An **ownership structure** $o = (o_\sigma)_{\sigma \in \overline{\mathcal{M}}}$ consists of a function $o_\sigma : \overline{H}_\sigma \to \overline{N}_\sigma$ for each submatching $\sigma \in \overline{\mathcal{M}}$. Any function o_σ assigns the ownership of all houses that are not matched by σ to the unmatched agents. So $o_\sigma(h)$ is the owner of house $h \in \overline{H}_\sigma$ at the submatching σ. The functions o_σ need not be submatchings, as some agent might own more than one house. To serve as the base of a hierarchical exchange mechanism, the full ownership structure o has to satisfy the following requirement, which says, informally, that an agent who owns some house at some point in the mechanism must continue to own this house for as long as he is unmatched.

Persistence: Let $\sigma' = \sigma \cup \{(i, h)\}$ for some $i \in \overline{N}_\sigma$ and $h \in \overline{H}_\sigma$. Let $i' \in \overline{N}_{\sigma'}$. Then $o_\sigma(h') = i'$ implies $o_{\sigma'}(h') = i'$.[3]

The following iterative procedure describes the mechanism:

Start: Let $\sigma_1 = \emptyset$.
Round k: let $\sigma = \sigma_k$.
Ownership: Determine the ownership of the houses remaining in the mechanism by $o_\sigma : \overline{H}_\sigma \to \overline{N}_\sigma$.
Pointing: Each house in $h \in \overline{H}_\sigma$ points to its owner $o_\sigma(h)$. Each agent $i \in \overline{N}_\sigma$ points to his most preferred house among the houses remaining in the mechanism (agent i points to house h_i if $h_i R_i h$ for all $h \in \overline{H}_\sigma$).

[2] Note that the assumption that preferences are strict implies that at least some agent must strictly prefer σ' to σ, since these two matchings differ and since all agents weakly prefer σ' to σ.

[3] In Pycia and Unver (2011) this requirement is called "consistency" in their discussion of Papai's 2000 hierarchical exchange mechanisms on page 12. The same requirement appears as (R4) "persistence of ownership" on page 18 in the general definition of trading cycles mechanisms.

Cycles: At least one cycle forms. Let σ' be the set of pairs (i, h) of houses and agents such that agent i and house h take part in a cycle and agent i points to house h.

Continuation: Define $\sigma_{k+1} : = \sigma \cup \sigma'$. If σ_{k+1} is a matching, the mechanism terminates. If not, continue with round $k + 1$ of the mechanism.

Since hierarchical exchange mechanisms are defined through ownership structures, I denote any such mechanisms by the ownership structure o that defines it. I write $o(R)$ for the outcome of the hierarchical exchange mechanism o at the housing problem (N, H, R). I generally denote the serial dictatorship in which agent i is the ith dictator for all $i \in N$ by $\widetilde{o}$ (so $\widetilde{o}_\sigma(h) = i$ for all $h \in \overline{H}_\sigma$ if $\{1, \cdots, i-1\} = N_\sigma$). The main result of the paper pertains to permuted mechanisms, which I define next.

Definition 1 Let $M : \mathcal{R} \to \mathcal{M}$ be a matching mechanism. Let $p : N \to N$ be a permutation. Then the **permuted mechanism** $M^p : \mathcal{R} \to \mathcal{M}$ is defined through $M^p(R)(h) = p(M(R_{p(1)}, \cdots, R_{p(n)})(h))$ for all $h \in H$. The set of all permutations of some mechanism M is defined as $[M] : = \{M^p \mid p \text{ is a permutation}\}$. Finally $[M](R) : = \{\mu : \exists p \text{ such that } \mu = M^p(R)\}$.

In a permuted exchange mechanism, agent $p^{-1}(i)$ assumes the role of agent i. This is best illustrated using serial dictatorships. Consider the case with just 3 agents and the permutation p with $p(1) = 2$, $p(2) = 3$ and $p(3) = 1$. To calculate the outcome of the permuted mechanism we first need to substitute agent $p(i)$'s preference for agent i's preference. Next we need to calculate the outcome of the serial dictatorship $\widetilde{o}(R_{p(1)}, R_{p(2)}, R_{p(3)}) = \widetilde{o}(R_2, R_3, R_1)$. According to this serial dictatorship, agent 1 gets the best possible house according to agent 2's preferences, say this house is h^*, so $\widetilde{o}(R_2, R_3, R_1)(h^*) = 1$. In addition to permuting the preferences, we need to permute the roles of agents as recipients. In the permuted serial dictatorship agent 2 is to play the role of the first dictator. To make sure that he obtains h^* the matching $\widetilde{o}(R_2, R_3, R_1)$ needs to be permuted with p. The recipient of house h^* under the permuted serial dictatorship is thus $p(\widetilde{o}(R_2, R_3, R_1)(h^*)) = p(1) = 2$.

Any permutation of a hierarchical exchange mechanism o can be obtained via a permutation of the agents in the ownership structure. Let a permuted submatching $p \circ \sigma$ be defined by $(p \circ \sigma)(h) = p(h(\sigma))$ for all $h \in H_\sigma$. Note that $p \circ \sigma$ bijectively maps H_σ to $N_{p \circ \sigma} : = \{i : i = p(j) \text{ for some } j \in N_\sigma\}$. Given this definition, $o^p : = (o^p_\sigma)_{\sigma \in \overline{\mathcal{M}}}$ can be defined by $o^p_\sigma = p(o_{p \circ \sigma}(h))$ for all $h \in \overline{H}_\sigma$ and all $\sigma \in \overline{\mathcal{M}}$. In the above example $o^p_\emptyset(h) = 2$ holds for all h, since $p(o_{p \circ \emptyset}(h)) = p(o_\emptyset(h)) = p(1) = 2$. So in the permuted serial dictatorship, agent 2 acts as the first dictator. More generally, if, according to the ownership structure o, agent i^* is the owner of house h^* at σ ($i^* = o_\sigma(h^*)$), then agent $p(i^*)$ is the owner of h^* according to the mechanism o^p at the submatching $p \circ \sigma$. The only difference between a hierarchical exchange mechanism o and a permutation of the mechanism o^p lies in the permutation of the roles played by the agents. The main result of the present paper concerns $[o](R)$, the set of all matchings that can be obtained by using some permutation of the hierarchical exchange mechanism o in the housing problem (N, H, R).

Any ownership structure o defines a hierarchical exchange mechanism. However, multiple ownership structures might define the same hierarchical exchange mechanism. In this sense, hierarchical exchange mechanisms are not uniquely represented by ownership structures. Since we are interested in the class of permuted mechanisms and not in their particular representations, I note the following Lemma.

Lemma 1 *Let the ownership structures* $\underline{o}$ *and* $\overline{o}$ *define the same hierarchical exchange mechanism M. Then* $\overline{o}^p$ *and* $\underline{o}^p$ *both define the hierarchical exchange mechanism* M^p.

The proof of this lemma relies on a unique canonical representation of hierarchical exchange mechanisms. Papai (2000) defines and discusses such representations using inheritance trees. I cannot directly apply Papai's 2000 result here, since I define hierarchical exchange mechanisms through ownership structures o. Since my contribution on this point does not substantially add to Papai's 2000 treatment, I relegate the proof to the appendix.

3 The result

Theorem 1 *Fix a housing problem and a hierarchical exchange mechanism. Then a matching is Pareto-optimal if and only if it is the outcome of some permutation of the given mechanism; formally* $[o](R) = PO(R)$ *holds for all hierarchical exchange mechanisms o and all housing problems (N, H, R).*

Let me give an informal preview of the proof. To that end, suppose that μ, with $\mu(h_i) = i$ for all $i \in N$ is the outcome of the serial dictatorship $\widetilde{o}$. Fix some hierarchical exchange mechanism o. The proof inductively defines a permutation p for consecutive groups of agents such that μ is the outcome of the o^p. The definition of p starts out by making agent 1 the initial owner of house h_1. If there is any other agent i who most prefers h_i out of the grand set of houses, then the permutation p is defined to make the largest set of such agents i initial owners of "their" respective houses h_i.

Now consider the submatching σ^* that matches all the initial owners i just mentioned to "their" houses h_i. The definition of p is continued assigning ownership rights at the submatching σ^*: o_{σ^*} by the same procedure as described above for the assignment of roles at the initial submatching $\emptyset$. That is: define p such that the agent i^* with the lowest index among the unmatched is assigned ownership of h_{i^*}. Then proceed with all agents i who most prefer "their" house h_i out of the set of houses which are not part of the matching σ^*. This process inductively defines the permutation p for all agents. Moreover the outcome of the permuted mechanism must be μ: an agent is matched in the first round if and only if he is assigned ownership through p in the first round of the definition of p. Moreover, all of these matches are in accordance with μ. The same holds for all ensuing rounds.

Proof The subset relation $[o](R) \subset PO(R)$ holds since hierarchical exchange mechanisms are Pareto-optimal, as shown by Papai (2000).

To see the inverse subset relation $PO(R) \subset [o](R)$, consider some $\mu \in PO(R)$. We know from Svensson (1994) that any Pareto-optimal allocation is the outcome of some serial dictatorship. Assume without loss of generality that μ matches house h_i to agent i for all $i \in N$ and that μ is the outcome of the serial dictatorship $\widetilde{o}$.

Inductively define a permutation p as follows.

To start let $k = 1$: $\sigma_1 = \emptyset$.
Round k: Let $\sigma := \sigma_k$. Define $i_k := \min\{i \in \overline{N}_\sigma\}$.
Step α: Define p such that $p(o_{p\circ\sigma}(h_{i_k})) = i_k$, meaning that under p agent i_k is the owner of h_{i_k}. Let $C = \{i_k\}$.
Step β: If there is some $i \in \overline{N}_\sigma \setminus C$ such that $h_i R_i h$ for all $h \in \overline{H}_\sigma$ and $p(o_{p\circ\sigma}(h_i)) \notin C$ then let $p(o_{p\circ\sigma}(h_i)) = i$ and set C to be the union of the preceding C with i and return to Step β. If there is no other $i \in \overline{N}_\sigma \setminus C$ with $h_i R_i h$ for all $h \in \overline{H}_\sigma$ and $p(o_{p\circ\sigma}(h_i)) \notin C$, then set $\sigma_{k+1} := \sigma_k \cup \{(i, h_i) : i \in C\}$. Start round $k + 1$ of the process with σ_{k+1}.

This inductive procedure uniquely assigns a role to every agent in the mechanism. It remains to be checked whether μ is the outcome of $o^p(R)$. This proof is done by induction. Define C_k as the set of agents who are assigned ownership in round k of the procedure described above. Suppose that up to some round $k - 1$ any agent $i \in C_1 \cup \cdots \cup C_{k-1}$ is matched with house h_i and no other agent is matched. I show that any agent $i \in C_k$ is matched to h_i at round k—which, according to the inductive hypothesis, starts with submatching σ_k.

By construction, any agent $i \in C_k$ owns house h_i under the permutation p at the submatching σ_k. Agent i_k prefers house h_{i_k} to all other houses that are still available since agent i_k would choose h_{i_k} under $\widetilde{o}$ and $i_k = \min\{i \in \overline{N}_{\sigma_k}\}$. If there are any other agents $i \in C_k$ these prefer h_i to all other available houses by construction. So any $i \in C_k$ owns and most prefers h_i and therefore appropriates it.

Now suppose some other house was assigned in round k. For this to happen we would have to have an additional cycle $i_1 \to h_{i_2} \to i_3 \to h_{i_4} \to \cdots \to i_s = i_1$ at round k. First observe that we cannot have that $i_j = i_{j+1}$ for any j odd (if we had $i_j = i_{j+1}$ agent i_j would most prefer house h_{i_j} out of the set of all remaining houses; the permutation p would therefore have to be such that agent i_j is assigned ownership of house h_{i_j} in round k of the procedure used to define p). Define k_j such that agent i_j is assigned ownership of a house in round k_j of the definition procedure for p, formally $i_j \in C_{k_j}$. Observe that any agent $j \in C_l$ with $l > k$ must point to a house h_i with $i \in C_{l'}$ and $l' < l$ at round k. The rationale is that $h_i R_i h$ holds for all houses $h \in C_m$ with $m \geq l$ when $i \in C_l$. To point to a house which he prefers to h_i (which is still available at round k) the agent must point to a house in $C_1 \cup \cdots C_{l-1}$. So we must have $k_j > k_{j+1}$ for all odd j. Next by persistence any house h_i with $i \in C_l$ can only be owned by agents in $C_1 \cup \cdots \cup C_{l-1} \cup \{i\}$ in the trading rounds preceding round l. This latter observation implies that $k_j \geq k_{j+1}$ holds for all even j. In sum there cannot be such a cycle, since $k_j \geq k_{j+1}$ would have to hold for all $j = 1 \cdots s - 1$ with $k_j > k_{j+1}$ for all odd j and $k_1 = k_s$. So it cannot be that any other agents are matched at round k. It can be concluded that $\mu = o^p(R)$. □

4 Discussion

There is an analogy between Papai's 2000 and the present result on the Pareto optimality of trade on the one hand and the First and Second Fundamental Theorems of Welfare Economics on the other hand. Papai (2000) shows, parallel to the First Theorem, that any outcome of trade is Pareto optimal. Conversely the present paper shows— analogously to the Second Theorem—that any Pareto optimum can be achieved through trade when a given set of property rights is correctly allocated to the agents. Of course the analogy is not perfect: trade in the present mechanism does not correspond exactly to the market trades of general equilibrium theory.

Hierarchical exchange mechanisms constitute a large and very general set; various subclasses have been characterized in the literature [see Ma (1994); Svensson (1999); Ergin (2000); Miyagawa (2002); Ehlers et al. (2002); Ehlers and Klaus (2004); Kesten (2009); Sonmez and Unver (2006); Ehlers and Klaus (2007), and Velez (2008)]. The result presented here extends to any of these subclasses.

Pycia and Unver (2011) characterize the class of *all* strategy proof, non-bossy and Pareto-optimal matching mechanisms. This class which includes Papai's 2000 hierarchical exchange mechanisms, also determines matchings as the outcome of trade in cycles. Pycia and Unver's 2011 class of "Trading Cycles Mechanisms" generalize hierarchical exchange mechanisms insofar as there are two forms of control over houses in trading cycles mechanisms: ownership and brokerage. Ownership is defined just as in hierarchical exchange mechanisms. Brokers have strictly less rights than owners. While a broker may freely exchange any house he controls with any other house, he is barred from appropriating the house he brokers. Theorem 1 also holds for the still larger class of trading cycles mechanisms. The proof of this more general result adapts the arguments given in the proof of Theorem 1 to the setting of Pycia and Unver's 2011 trading cycles mechanisms. The more general result (Theorem 2) and its proof can be found in the Appendix of the online version of this paper.

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5 Appendix

An ownership structure o uniquely defines a hierarchical exchange mechanism. To see that the converse does not hold, consider the example of Gale's top trading cycles mechanism: specifying any ownership rights beyond the first owner of each house is unnecessary for this case. More formally, take any two top trading cycles mechanisms $\overline{o}$ and $\underline{o}$ which coincide in their ownership assignment at the first round of the mechanism. Say each prescribes that agent i owns house h_i at the beginning of the mechanism, so $\overline{o}_{\emptyset}(h_i) = \underline{o}_{\emptyset}(h_i) = i$ for all $i \in N$. Then $\overline{o}$ and $\underline{o}$ both describe the same mechanism, no matter what assumption we make on the ownership after submatchings which can never occur. They can for example prescribe $\underline{o}_{\sigma^*}(h_2) = 4$ and $\overline{o}_{\sigma^*}(h_2) = 3$ where $\sigma^* = \{(2, h_1)\}$, as this submatching cannot occur given the initial assignment of ownership.

To obtain a unique representation of hierarchical exchange mechanisms, I define a submatching σ as **reachable** under o if there exists some preference profile R such that some round of the hierarchical exchange mechanism starts with the submatching σ. In the above example, σ^* is not reachable for either $\overline{o}$ or $\underline{o}$. Define $\mathbf{o} = (\mathbf{o}_\sigma)_{\overline{\mathcal{M}}(\mathbf{o})}$ to be a **reachable ownership structure** if there exists an ownership structure o such that all submatchings $\overline{\mathcal{M}}(\mathbf{o})$ are reachable under o and $\mathbf{o}_\sigma = o_\sigma$ for all $\sigma \in \overline{\mathcal{M}}(\mathbf{o})$. The two ownership structures $\overline{o}$ and $\underline{o}$ are identical on all reachable submatchings. So $\overline{\mathbf{o}} = \underline{\mathbf{o}}$. The following proposition generalizes the observation that two ownership structures that are identical on the set of reachable submatchings describe the same hierarchical exchange mechanism. The proposition parallels Papai's 2000 observations on unique representations through inheritance trees for the environment of ownership structures.

Proposition 1 *There exists a bijection between the set of hierarchical exchange mechanisms and the set of reachable ownership structures.*

Proof Any hierarchical exchange mechanism $M : \mathcal{R} \rightarrow \mathcal{M}$ can be represented using an ownership structure o. The reachable ownership structure $\mathbf{o}$ defines the same mechanism, since only submatchings σ that are reached for some R matter to the determination of the outcome of the mechanism. Now suppose two ownership structures o and o' describe two different hierarchical exchange mechanisms. Then the two must lead to a different outcome for some housing problem (N, H, R). Let k be the first round in which the assignments differ according to the two mechanisms. Let that round start with submatching σ. This implies that σ is reachable under either mechanism, so also the reachable ownership structures must differ, and hence $\mathbf{o} \neq \mathbf{o}'$. □

I next use this observation on the unique representation of hierarchical exchange mechanisms to prove Lemma 1.

Proof Let the ownership structures $\underline{o}$ and $\overline{o}$ represent the same hierarchical exchange mechanism M. By the preceding proposition we know that the two ownership structures have the same reachable set $\overline{\mathcal{M}}(\overline{\mathbf{o}}) = \overline{\mathcal{M}}(\underline{\mathbf{o}}) : = \mathcal{N}$ and that the two ownership structures coincide on the reachable set: $\underline{o}_\sigma = \overline{o}_\sigma$ for all $\sigma \in \mathcal{N}$. This implies that $p(\underline{o}_{p\circ\sigma}) = p(\overline{o}_{p\circ\sigma})$ for all $\sigma \in \mathcal{N}$ or equivalently $\underline{o}^p_\sigma = \overline{o}^p_\sigma$ for all σ for which there exists a $\sigma' \in \mathcal{N}$ such that $p \circ \sigma' = \sigma$, which describes the reachable set of submatchings under $\underline{o}^p$ and $\overline{o}^p$. This in turn implies that $\underline{o}^p$ and $\overline{o}^p$ indeed describe the same hierarchical exchange mechanism. □

Consequently the proof of Theorem 1 does not depend on a particular representation of a hierarchical exchange mechanism by an ownership structure o. If some permutation p can be used on one ownership structure that represents a hierarchical exchange mechanism to obtain some Pareto optimum, then the same permutation can be used on any other ownership structure that represents the same hierarchical exchange mechanism to obtain this Pareto optimum.

References

Abdulkadiroglu A, Sonmez T (1998) Random serial dictatorship and the core from random endowments in house allocation problems. Econometrica 66(3):689–701

Ehlers L, Klaus B (2004) Resource monotonicity for house allocation problems. Int J Game Theory 32: 545–560

Ehlers L, Klaus B (2007) Consistent house allocation. Econ Theory 30:260–274

Ehlers L, Klaus B, Papai S (2002) Strategy-proofness and population monotonicity in house allocation problems. J Math Econ 38:329–339

Ergin HI (2000) Consistency in house allocation problems. J Math Econ 34:77–97

Kesten O (2009) Coalitional strategy-proofness and resource monotonicity for house allocation problems. Int J Game Theory 38:17–21
Ma J (1994) Strategy-proofness and strict core in a market with indivisibilities. Int J Game Theory 23:75–83
Miyagawa E (2002) Strategy-proofness and the core in house allocation problems. Games Econ Behav 38:347–361
Papai S (2000) Strategyproof assignment by hierachical exchange. Econometrica 68(6):1403–1433
Pycia M, Ünver MU (2011) Incentive compatible allocation and exchange of discrete resources. Available at SSRN: http://ssrn.com/abstract=1079505 or http://dx.doi.org/10.2139/ssrn.1079505
Tayfun S, Utku Unver M (2006) Kidney exchange with good Samaritan donors: A characterization. Boston College and University of Pittsburgh, (Working paper)
Svensson LG (1994) Queue allocation of indivisible goods. Soc Choice Welf 11:323–330
Svensson LG (1999) Strategy-proof allocation of indivisible goods. Soc Choice Welf 16:557–567
Velez RA (2008) Revisiting consistency in house allocation problems. (Working paper)