Online Appendix for Random Serial Dictatorship: The One and Only

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1 Introduction

This online appendix contains the proofs of Theorem 2, Proposition 1, and Proposition 2. I prove Theorem 2 and Proposition 1 by induction over the number of agents n.

Start: If $n = 1$ there is exactly one good mechanism: match the agent with his highest ranked house. This mechanism has a unique representation as a trading and braiding mechanism with one control rights function that assigns ownership of every house to the single agent. All statements on lax trading and braiding mechanisms (including Proposition 1) are trivially satisfied with only one agent.

Hypothesis: Theorem 1 and Proposition 1 hold with fewer than n agents.

Step: Theorem 1 and Proposition 1 hold with n agents given that the hypothesis holds.

The proof of the Step for Theorem 1 has many chapters. Braids are good (Section 3). Lax mechanisms are well-defined and good (Sections 4 and 5). Representations as trading and braiding mechanisms are unique (Section 6). The proof of the converse direction of Theorem 2, that any good mechanism M can be represented as a trading and braiding mechanism, starts with a list of arguments that are used repeatedly in the sequel (Section 7).

For M to be representable as a trading and braiding mechanism a function $c_\emptyset : H \to N \times \{o, b\}$ needs to be defined (Section 8). This function $c_\emptyset$ is shown to have a set of properties in line with the properties of control rights functions in trading and braiding mechanisms (Section 9). These properties are used to show that $c_\emptyset$ satisfies (C1), (C2), and (C3) implying in particular that braids are the only alternative to trading rounds (Section 10). If M is not a braid its outcome at R is consistent with any submatching achieved in a first trading round under $c_\emptyset$ at R and any such submatching is followed by a well-defined trading and braiding (sub-)mechanism $c[\nu]$ (Section 11). These submechanisms $c[\nu]$ together with $c_\emptyset$ define a traiding and braiding mechanism c that satisfies all conditions (C1)-(C6) (Section 12).

Pycia and Unver [4] also sets out to characterize the set of all good mechanisms. While their result turned out to be wrong (their trading cycles mechanisms do not admit braids) their approach and in particular the insight that houses might be brokered are indispensable for the present characterization. Many of the following chapters lean on the Pycia and Unver [4] proof. To disentangle the present contributions from Pycia and Unver's [4], let me give an overview of the differences between the two proofs.

First, my proof is by induction over the number of all agents n. This makes the present proof that all trading and braiding mechanisms are good more concise than the Pycia and Unver [4] proof of the equivalent claim. Next, Pycia and Unver [4] does not consider braids. Neither the statement that braids are good nor its proof appear in Pycia and Unver [4]. The first steps of the current proof that any good mechanism has a representation as a trading and braiding mechanism exactly follow Pycia and Unver [4]. I state the first result in this context, Lemma 2 [Pycia and Unver [4]: Lemma 9], without proof. This result formalizes the pivotal idea in Pycia and Unver [4]: if all agents' rankings agree on the two top ranked houses then the recipient of the top ranked house does not depend on the agents' rankings of the other houses. For all remaining lemmas I do state proofs even though some of these lemmas appear in Pycia and Unver [4]. In some cases the new proofs are shorter. Moreover, the complete proof can be understood without going back and forth between two different approaches and sets of notation. I will revisit the comparison between the two proofs in Sections 8 through 12.

Some ancillary statements in the current paper do not correspond to results in Pycia and Unver [4]. Given that the definition of a lax mechanism only requires the matching of at least one cycle at every trading round, it needs to be shown that lax mechanisms are well-defined in the sense that outcomes do not depend on the order in which trading cycles are removed (Section 4). While Section 6 shows that any good mechanism has a unique representation as a trading and braiding mechanism, Pycia and Unver [4] makes no such uniqueness claim. The fact braids cannot be represented as Pycia and Unver [4] trading cycles mechanisms follows from the uniqueness result in Section 6.

The online appendix contains proofs of two further results that are used in the main body of the paper. In Section 13, I prove Proposition 1 on how to derive equivalent representations of good mechanisms. This Proposition plays an important role in the proof of Theorem 1, when β is used to to define consecutive mechanisms c^k, c^{k+1}. In Section 15 I prove Proposition 2, which demonstrates that with 3 agents and 3 houses there is exactly one ordinally strategy proof and ex post Pareto optimal mechanism that treats equals equally: random serial dictatorship. This result is used to show that the symmetrization of any braid equals random serial dictatorship.

2 Further Notation and Concepts

Any profile of preferences in which all agents agree that h is the best house is denoted R^h, so hR_i^hH for all $i \in N$. Similarly the profiles R^{gh} and R^{egh} are such that all agents agree on the ranking of the top two and respectively top three houses: $gR_i^{gh}hR_i^{gh}H \setminus \{g\}$ and $eR_i^{egh}gR_i^{egh}hR_i^{egh}H \setminus \{e, g\}$ for all $i \in N$. Different profiles where all agents rank the same house at the top, say e, are denoted $\hat{R}^e$ and $\tilde{R}^e$. Alternatively I write $R_i : e\ g$ if R_i ranks e and g at the top, the notation $R_i : e = \nu(i)$ means that agent i ranks e, which equals $\nu(i)$, at the top. Two preferences R_i and R_i' **coincide** on $H' \subset H$ if $eR_ig \Leftrightarrow eR_i'g$ holds for all $e, g \in H'$. Two profiles of preferences R and R' coincide on $H' \subset H$ if R_i coincides with R_i' on H' for all $i \in N$. If R_i and R_i' coincide on H' then their restrictions to H', $\overline{R}_i$ and $\overline{R}_i'$, are identical. Let $\mathcal{N}(c_\emptyset)$ be the the set of direct c-successors to $\emptyset$ and let $\mathcal{N}(c_\emptyset)(R)$ be the subset of direct

c-successors to $\emptyset$ that are reachable under c at R. Since these sets coincide for two lax trading and braiding mechanisms c and c' with $c_\emptyset = c'_\emptyset$ only the control rights function $c_\emptyset$ is used to define $\mathcal{N}(c_\emptyset)$ and $\mathcal{N}(c_\emptyset)(R)$.

For the proof, let me be precise about requirements (C1), (C2), (C2)', and (C3) on control right functions c_ν.

(C1) If $c_\nu(e) = (\cdot, b)$, $c_\nu(g) = (\cdot, b)$, and $e \neq g$ hold for some $e, g \in H$ then $H = \{e, f, g\}$ and $c_\nu(e) = (i, b)$, $c_\nu(f) = (j, b)$, $c_\nu(g) = (j', b)$, and $i \neq j \neq j' \neq i$ hold for some $i, j, j' \in N$.

(C2) If $c_\nu(e) = (i_b, b)$ holds for exactly one $e \in H$ and $i_b \in N$ then there exist $f, g \in H \setminus \{e\}$ and $i, j \in N \setminus \{i_b\}$ such that $c_\nu(f) = (i, o)$, $c_\nu(g) = (j, o)$, $f \neq g$, and $i \neq j$.

(C2)' If $c_\nu(e) = (i_b, b)$ holds for exactly one $e \in H$ and $i_b \in N$ then there exist $g \in H \setminus \{e\}$ and $i \in N \setminus \{i_b\}$ such that $c_\nu(e) = (i, o)$.

(C3) If $c_\nu(e) = (i_b, b)$ for some $e \in H$ and $i_b \in N$ then $c_\nu(h) \neq (i_b, o)$ holds for all $h \in H \setminus \{e\}$.

3 Braids are good

The following claim was first made in Bade [1]. Neither the claim nor its proof appear in Pycia and Unver [4]. The recent revision, Pycia and Unver [5], proves that braids (now called 3 broker* mechanisms) are good.

Lemma 1 *The braid B^ω is good.*

Proof Since $B^\omega(R)(i) = \emptyset$ and $B^\omega(R'_i, R_{-i}) = B^\omega(R)$ holds for all $i \notin N_\omega$, R'_i and R it is w.l.o.g to assume $N = \{1, 2, 3\}$. To fix ideas also let $\omega = (e, f, g)$. B^ω is Pareto optimal as $B^\omega(R) \in Mini(R) \subset PO(R)$ holds for any $R \in \mathcal{R}$. The alternative representation B' is useful in the upcoming arguments.

I. If $\{\omega\} = PO(R)$ then $B'(R) :\, = \omega$.

for II and III let $\{i, j, j'\} = \{1, 2, 3\}$ and denote the two maximally avoidant matchings ω' and ω''.

II. If $\omega'(i) R_i \omega''(i)$ and $R_j : \omega(i) = \omega'(j)$, then $B'(R) :\ = \omega'$.

III. If $\omega'(i) R_i \omega''(i)$, $R_j : \omega(j)$ and $R_{j'} : \omega(i) = \omega'(j)$, then $B'(R) :\ = (\omega'(i), \omega(j), \omega(i))$.

While I applies if and only if $PO(R) = \{\omega\}$, III applies if and only if $PO(R) \neq \{\omega\}$ and $\{\omega', \omega''\} \cap PO(R) = \emptyset$. So I, II, and III partition $\mathcal{R}$, if $\{\omega', \omega''\} \cap PO(R) \neq \emptyset$ implies that R is covered by II. To see that this holds suppose not. Letting $\omega' = (g, e, f)$ and $\omega'' = (f, g, e)$, suppose w.l.o.g that $\omega' \in PO(R)$ and $R_2 : e = \omega'(2) = \omega(1)$. For R not to be covered by II 1 must prefer $f = \omega''(1)$ to $\omega'(1) = g$ and 3 may not rank e highest. Given $f R_1 g$, ω' can only be Pareto optimal if $f R_3 g$, so $R_3 : f$ must hold. But then R is covered by II as $e = \omega'(2) R_2 \omega''(2) = g$ and $R_3 : \omega'(3) = \omega(2) = f$, a contradiction.

To see that $B^\omega(R) = B'(R)$ holds for all $R \in \mathcal{R}$, first assume $PO(R) = \{\mu\}$, implying $B^\omega(R) = \mu$. If $\mu = \omega$, then $B'(R) = \omega$ holds by I. If $\mu = \omega' = (g, e, f)$, then $B'(R) = \omega'$ holds by II as agent 2 ranks $\omega'(2) = e = \omega(1)$ highest and as $R_1 : g$ implies $g = \omega'(1) R_1 \omega''(1)$. If $\mu \notin \{\omega, \omega', \omega''\}$, then some agent, say 2, must rank his avoidance match highest: $R_2 : f = \omega(2) = \mu(2)$. For $\mu \neq \omega$ to be the unique Pareto optimum, we must have $R_1 : g = \omega(3) = \mu(1)$ and $R_3 : e = \omega(1) = \mu(3)$. By III we obtain $B'(R) = \mu$.

Now assume that $PO(R)$ is not a singleton, say two agents rank $\omega(1) = e$ highest. Assume w.l.o.g that $g = \omega'(1) R_1 \omega''(1) = f$. First let R be covered by II, so $R_2 : e = \omega'(2)$, $B'(R) = \omega'$, and $\omega' \in Mini(R) \subset \{\omega', \omega''\}$. If in addition $R_3 : e$, then $B^\omega(R) = \omega'$ holds since 1 prefers ω' to ω''. If $R_3 : h \neq e$ then $B^\omega(R) = \omega'$ holds since ω' matches 2 (the only agent other than 1 who ranks e at the top) with e. Finally consider the case that R is covered by III, so $R_2 : f$, $R_3 : e = \omega'(2)$, and $B'(R) = (g, f, e)$. Since $P(R)$ is not a singleton $R_1 : e\ g$ must hold and $B^\omega(R)$ equals (g, f, e), the only Pareto optimum that matches 3 to e. In sum, B^ω and B' define the same mechanism called B in the remainder of the proof of Lemma 1.

Only in two cases is an agent matched with his avoidance math: Either R is covered by I, or R is covered by III: $\omega'(i) R_i \omega''(i)$, $R_j : \omega(j)$, $R_{j'} : \omega(i) = \omega'(i)$ and j is matched with $\omega(j)$ under $B(R)$. So we have

(O) : $B(R)(i) = \omega(i)$ implies $R_i : \omega(i)$ and no other agent ranks $\omega(i)$ highest.

To see that B is strategy proof and non-bossy fix a profile R. If R is covered by I all agents obtain their best house. Moreover $B(R) \neq B(R'_i, R_{-i})$ holds only if $R'_i : h \neq \omega(i)$. But in that case (O) implies $B(R)(i) = \omega(i) \neq B(R'_i, R_{-i})(i)$.

As an example of a profile that is covered by II, consider R with $g = \omega'(1) R_1 \omega''(1) = f$ and $R_2 : e = \omega(1) = \omega'(2)$, so $B(R) = \omega' = (g, e, f)$. Given $R_2 : e = \omega(1)$ and (O) 1 cannot obtain e, the only house he might possible prefer to $g = B(R)(1)$, 2 obtains his best house, and 3 has no impact on the outcome. By the last statement $B(R)(3) = B(R'_3, R_{-3})(3) \Rightarrow B(R) = B(R'_3, R_{-3})$ trivially holds. For $B(R)(2) = e = B(R'_2, R_{-2})(2)$ and $B(R) \neq B(R'_2, R_{-2})$ to hold $B(R'_2, R_{-2})$ must equal (f, e, g). So (R'_2, R_{-2}) is covered by III with $f R_1 g$ a contradiction to the assumption that $g R_1 f$. $B(R) = (g, e, f) \neq B(R'_1, R_{-1})$ and $B(R)(1) = B(R'_1, R_{-1})(1)$ imply $B(R'_1, R_{-1})(2) \neq e$. The latter only holds if $R_3 : e$ and $f R'_1 g$. But then $B(R'_1, R_{-1}) = \omega''$ holds by II and we obtain the contradiction $B(R'_1, R_{-1})(1) = \omega''(1) \neq \omega'(1) = B(R)(1)$.

As an example of a profile that is covered by III, consider R with $g = \omega'(1) R_1 \omega''(1) = f$, $R_2 : f = \omega(2)$, and $R_3 : e = \omega'(2)$, so $B(R) = (g, f, e)$. By $R_3 : e$ and (O), 1 cannot obtain $e = \omega(1)$, the only house he might possibly rank above $M(R)(1) = g$; 2 and 3 obtain their best houses. Fixing the submatching $\{(1, g)\}$, $\{(2, f), (3, e)\}$ is the only Pareto optimal submatching for $2, 3$, and fixing the submatching $\{(3, e)\}$, $\{(1, g), (2, f)\}$ is the only Pareto optimal submatching. So $B(R)(i) = B(R'_i, R_i)(i) \Rightarrow B(R) = B(R'_i, R_{-i})$ holds for $i = 1, 3$ and any R'_i. Finally (O) implies $M(R'_2, R_{-2})(1) \neq e$ for any R'_2, so the implication also holds for $i = 2$.

Mutatis mutandum the above arguments apply to generic profiles R and B is strategy proof and non-bossy. □

4 Lax mechanisms are well-defined

Fix a lax mechanism c for n agents and a profile of preferences R. To see that c is well-defined we need to first check that the algorithm always specifies some step to be taken and terminates with a matching. Secondly, if the algorithm allows for multiple choices at some step, we need to check that all these choices lead to the same matching.

To see the first note that c_ν is specified for any possible round of the algorithm.[1] Now consider a c-relevant ν with at most one broker. By the definition of c_ν, every unmatched house is controlled by an unmatched agent. So every $h \in \overline{H}_\nu$ points to an $i \in \overline{N}_\nu$. Every owner in $\overline{N}_\nu$ points to his most preferred house in $\overline{H}_\nu$. Every broker in $\overline{N}_\nu$ points to his most preferred owned house in $\overline{H}_\nu$. Assumption (C2)' ensures that there is at least one owned house at ν. Since there are finitely many houses and agents at least one cycle forms. Since at least one cycle must be matched in every round the algorithm terminates with a matching.

To see the second we need to show that the outcome of the algorithm does not depend on the order in which trading cycles are removed. Assume that c is not a braid and let $\emptyset \neq \nu \subset \nu'$ arise out of matching sets of cycles that form at $\emptyset$ under c and R. By the hypothesis of the induction $c[\nu'](\overline{R})$ (with $\overline{R}$ the restriction of R to $\overline{H}_\nu$ and $\overline{N}_\nu$) does not depend on the order in which trading cycles are matched. It therefore suffices to show ν' is reachable under c at R. If $\nu = \nu'$ the result trivially holds. So suppose $\nu \subsetneq \nu'$. Since no broker may point to the house he brokers at least one agent in $N_{\nu'} \setminus N_\nu$ is an owner at $\emptyset$. Given (C4) this agent must also be an owner at ν and $c[\nu]$ cannot be a braid.

Case I: $c_\emptyset(h) = c_\nu(h)$ holds for all $h \in H_{\nu'} \setminus H_\nu$. So any $h \in H_{\nu'} \setminus H_\nu$ points to the same agent at $\emptyset$ and at ν. Any agent $i \in N_{\nu'} \setminus N_\nu$ points to the same house at $\emptyset$ and at ν (since hR_iH implies $hR_i\overline{H}_\nu$ and since $hR_iH \setminus \{h_b\}$ implies $hR_i\overline{H}_\nu \setminus \{h_b\}$, which matters if i brokers h_b). So any cycle at $\emptyset$ that is not immediately removed remains a cycle at ν. As the inductive hypothesis applies to $c[\nu]$ we may next match the cycles that result in $\nu' \setminus \nu$ and ν' reachable under c at R.

Case II: $c_\emptyset(h_b) \neq c_\nu(h_b)$ holds for some $h_b \in H_{\nu'} \setminus H_\nu$. By (C4) $c_\emptyset(h_b) = (i_b, b)$ holds for some $i_b \in N$. As a broker i_b must point to some owned house h at $\emptyset$. Let $c_\emptyset(h) = (i, o)$. By (C4) agent i continues to own h at ν. By (C5) i is the only agent who owns houses at $\emptyset$ and ν. So $i_b \to h \to i \to h_b$

[1]A trading and braiding mechanism c specifies c_ν for any c-relevant ν. A submatching ν is relevant if it occurs for some path of cycle removal at some R in the trading algorithm.

is the only cycle that forms at $\emptyset$ but remains unmatched under ν and $\nu' = \nu \cup \{(i, h_b), (i_b, h)\}$. By (C6) i owns h_b at ν. Since i points to h_b at $\emptyset$ we have $h_b R_i H$ and consequently $h_b R_i \overline{H}_\nu$. So at ν, h_b points to i and i to h_b. By the inductive hypothesis, we may remove this one cycle and $\nu \cup \{(i, h_b)\}$ is reachable under c and R. (C6) and $c_\emptyset(h) = (i, o)$ imply that i_b owns h at $\nu \cup \{(i, h_b)\}$. Since $h R_{i_b} H \setminus \{h_b\}$ implies $h R_{i_b} \overline{H}_\nu \setminus \{h_b\}$ a cycle just involving i_b and h forms. By the inductive hypothesis we may eliminate this one cycle and $\nu' = \nu \cup \{(i, h_b), (i_b, h)\}$ is reachable under c at R.

5 Lax mechanisms are good

Fix a lax mechanism c for n agents, a profile of preferences R, an agent i, and a preference R'_i. If $\nu \in \mathcal{N}(c_\emptyset)(R)$ and $i \notin N_\nu$ then $c(R) = \nu \cup c[\nu](\overline{R})$ and $c(R'_i, R_{-i}) = \nu \cup c[\nu](\overline{R}'_i, \overline{R}_{-i})$ hold for $\overline{R}$ and $\overline{R}'_i$ the restrictions of R and R'_i to $\overline{N}_\nu, \overline{H}_\nu$. Since there are fewer than n agents in $c[\nu]$, it is by the inductive hypothesis good. The proof that c is strategyproof and non-bossy is split into two cases.

Case I: i stays unmatched for some $\nu \in \mathcal{N}(c_\emptyset)(R)$. Fix such a ν with $i \notin N_\nu$ and recall that $c[\nu](\overline{R})(i) = c(R)(i)$ and $c[\nu](\overline{R}'_i, \overline{R}_{-i})(i) = c(R'_i, R_{-i})(i)$. Since $c[\nu]$ is strategyproof, we have $c[\nu](\overline{R})(i) \overline{R}_i c[\nu](\overline{R}'_i, \overline{R}_{-i})(i)$ and consequently $c(R)(i) R_i c(R'_i, R_{-i})(i)$. Moreover $c(R)(i) = c(R'_i, R_{-i})(i)$ holds if and only if $c[\nu](\overline{R})(i) = c[\nu](\overline{R}'_i, \overline{R}_{-i})(i)$. Since $c[\nu]$ is non-bossy the latter implies $c[\nu](\overline{R}) = c[\nu](\overline{R}'_i, \overline{R}_{-i})$ and we have $c(R) = \nu \cup c[\nu](\overline{R}) = \nu \cup c[\nu](\overline{R}'_i, \overline{R}_{-i}) = c(R'_i, R_{-i})$.

Case II: The only $\nu \in \mathcal{N}(c_\emptyset)(R)$ is such that $i \in N_\nu$. Suppose there was a cycle not involving i at $\emptyset$ under (R'_i, R_{-i}). As all agents other than i have the same preferences in R and (R'_i, R_{-i}) this cycle also forms at $\emptyset$ under c and R contradicting the assumption that $\{\nu\} = \mathcal{N}(c_\emptyset)(R)$. In sum, i is part of the unique cycle at $\emptyset$ under (R'_i, R_{-i}) and $c(R'_i, R_{-i})(i)$ is the house that i points to at $\emptyset$ given R'_i. If $c(R'_i, R_{-i})(i) = c(R)(i)$ then the cycle that yields ν also forms at $\emptyset$ under (R'_i, R_{-i}) and $c(R'_i, R_{-i})$ equals $\nu \cup c[\nu](\overline{R}) = c(R)$. So c is non-bossy. Since $c(R)(i)$ is the R_i-best house among all houses that i may point to at $\emptyset$ under c, $c(R)(i) R_i c(R'_i, R_{-i})(i)$ holds and c is strategyproof.

To see that $c(R)$ is Pareto optimal at R, suppose there existed a matching

$\mu \neq c(R)$ with $\mu(i) R_i c(R)(i)$ for all $i \in N$ and $\mu(i_b) \neq c(R)(i_b)$ for some $i_b \in N_\nu$ with $\nu \in \mathcal{N}(c_\emptyset)(R)$. Since $\nu(i) R_i H$ holds for any owner $i \in N_\nu$, i_b is a broker: $c_\emptyset(h_b) = (i_b, b)$ for some h_b. Since h_b is the only house that i_b may not point to at $\emptyset$, $\mu(i_b)$ must equal h_b. Since $i_b \in N_\nu$, $\nu(i^*) = h_b$ must hold for some i^*, who is by (C1) an owner and we have $h_b R_{i^*} H$. We obtain the contradiction that i^* strictly prefers $c(R)(i^*) = \nu(i^*) = h_b$ to $\mu(i^*) \neq h_b$ and $\nu(i) = \mu(i) = c(R)(i)$ holds for all $i \in N_\nu$. Since $c(R) \setminus \nu = c[\nu](\overline{R})$ and since $c[\nu]$ is by the inductive hypothesis Pareto optimal, $c(R)$ is Pareto optimal.

6 Uniqueness

Fix two trading and braiding mechanisms c and c^* with $c(R) = c^*(R)$ for all $R \in \mathcal{R}$.

First suppose that c is a braid B^ω with $H = \{e, g, f\}$ and $N_\omega = \{1, 2, 3\}$. Since B^ω maps R^{eg} and R^{ef} to the two (different) matchings ω' and ω'' with $N_{\omega'} = N_{\omega''} = \{1, 2, 3\}$ that maximally avoid ω, B^ω matches e to two different agents at R^{eg} and R^{ef}, implying $c^*_\emptyset(e) = (\cdot, b)$. By the same token $c^*_\emptyset(g) = (\cdot, b)$ also holds and c^* must be a braid.[2] For B^ω to equal another braid B^{ω^*}, N_{ω^*} must equal $\{1, 2, 3\}$. If $\omega(i) \neq \omega^*(i)$ for all $i \in \{1, 2, 3\}$ then ω^* maximally avoids ω and $B^\omega(R) = \omega^*$ holds for some R where all agents announce the same preference. Since any matching is Pareto optimal at R, $B^{\omega^*}(R)$ cannot equal ω^*. If $\omega \neq \omega^*$ but $\omega(i) = \omega^*(i)$ for some $i \in \{1, 2, 3\}$ say $\omega(3) = \omega^*(3)$, then $\omega(1) \neq \omega^*(1)$ and $\omega(2) \neq \omega^*(2)$ must hold. Define R such that $R_1 : \omega(1)\ \omega^*(1)$, $R_2 : \omega^*(2)\ \omega(2)$, $R_3 : \omega(3)$. Given $PO(R) = \{\omega, \omega^*\}$ we have $B^\omega(R) = \omega^* \neq \omega = B^{\omega^*}(R)$. So if c is a braid then c^* must be the same braid ($\omega = \omega^*$).

So suppose that according to $c_\emptyset$ and $c^*_\emptyset$ there is at most one brokered house. Fix any $e \in H$. If $c_\emptyset(e) = (i, o)$ and $c^*_\emptyset(e) = (j, o)$ then $c(R^e)(i) = e = c^*(R^e)(j)$ implies $i = j$. If $c_\emptyset(e) = (i, b)$ and $c^*_\emptyset(e) = (j, b)$ then $c(R^{eg})(i) = g = c^*(R^{eg})(j)$ implies $i = j$. If $c_\emptyset(e) = (i, b)$ and $c^*_\emptyset(e) = (j, o)$, then $c_\emptyset(g) = (j', o)$ holds by (C2) for some $g \neq e$ and $j' \notin \{i, j\}$ yielding the contradiction $c(R^{eg})(j') = e = c^*(R^{eg})(j)$. In sum, $c_\emptyset = c^*_\emptyset$ must hold for

[2] This argument shows that braids cannot be represented as Pycia and Unver [4] trading cycles mechanisms.

$c(R) = c^*(R)$ for all $R \in \mathcal{R}$. Since $c_\emptyset = c^*_\emptyset$, ν is a direct c-successor of $\emptyset$ if and only if it is a direct c^*-successor of $\emptyset$. The hypothesis of the induction implies that $c[\nu]$ is identical to $c^*[\nu]$ for any such $\nu \in \mathcal{N}(c_\emptyset)$.

7 A collection of arguments

Fix an arbitrary good mechanism M, a profile of preferences R, and a deviation R'_i. Let $M(R)(i) = e$. The following arguments are used throughout the next sections. Strategy proofness implies that nothing changes for agent i when he ranks $e = M(R)(i)$ at least as high under R'_i as under R_i:

SP-I If $eR_ih \Rightarrow eR'_ih$ for all $h \in H$, then $M(R'_i, R_{-i})(i) = e$.

Since M is non-bossy we additionally obtain:

SP-NB If $eR_ih \Rightarrow eR'_ih$ for all $h \in H$, then $M(R'_i, R_{-i}) = M(R)$.

Let g rank directly below e according to R_i, let e rank directly below g according to R'_i and let this be the only difference between R_i and R'_i (so eR_ig, gR'_ie, and $hR_ih' \Leftrightarrow hR'_ih'$ if $\{h, h'\} \neq \{e, g\}$), then

SP-II $M(R'_i, R_{-i})(i) \in \{e, g\}$.

In combination with Pareto optimality the preceding observation yields

SP-PO If $M(R)$ is not Pareto optimal at (R'_i, R_{-i}), then $M(R'_i, R_{-i})(i) = g$.

8 The Definition of $c_\emptyset$

Lemma 2 below shows that if $M(\hat{R}^{eg})(i) = e$ holds for a particular $\hat{R}^{eg}$ then $M(R^{eg})(i) = e$ holds for any R^{eg}. This lemma is identical to Pycia and Unver [4] Lemma 9 and I do not provide a proof. The lemma crucially simplifies the problem of characterizing all good mechanisms. Thanks to Lemma 2 only a few top ranked houses matter in the upcoming arguments.

Lemma 2 *[Pycia and Unver [4], Lemma 9] Fix any $\hat{R}^{eg}$, $\tilde{R}^{eg}$, $\hat{R}^{feg}$, and $\tilde{R}^{feg}$. Then $M(\hat{R}^{eg})(i^*) = e$ implies $M(\tilde{R}^{eg})(i^*) = e$, and $M(\hat{R}^{feg})(i^*) = e$ implies $M(\tilde{R}^{feg})(i^*) = e$.*

Following Pycia and Unver [4], define a function $c_\emptyset : H \to N \times \{o, b\}$. If $M(R^{eg})(i) = e$ holds for all R^{eg} with $g \neq e$ let $c_\emptyset(e) = (i, o)$ if not let $c_\emptyset(e) = (i_b, b)$ where i_b is such that $M(R^{eg})(i_b) = g$ for some R^{eg}. To see that $c_\emptyset$ is well-defined we need to check that there exists a unique agent i_b who obtains the second best house in any profile R^{eg} when $M(R^{eg})(i) = e = M(R^{eh})(j)$ holds for some g, h and $i \neq j$. Lemma 3, which is equivalent to Pycia and Unver [4] Lemma 10, does this.

Lemma 3 *[Pycia and Unver [4], Lemma 10] Let $M(\tilde{R}^{eg})(1) = e = M(\hat{R}^{ef})(2)$ for some $\tilde{R}^{eg}, \hat{R}^{ef}$. Then there exists an agent i_b such that $M(R^{eh})(i_b) = h$ for any R^{eh} with $h \neq e$.*

Proof Fix some R^{efg} with $M(R^{efg})(i_b) = f$. Let $\tilde{R}^{efg}$ and $\tilde{R}^{egf}$ coincide with $\tilde{R}^{eg}$ on $H \setminus \{f\}$. By Lemma 2 $M(\tilde{R}^{efg})(i_b)$ equals f. Switching $\tilde{R}_i^{efg}$ to $\tilde{R}_i^{eg}$ for all $i \neq i_b$ SP-NB yields $M(\tilde{R}_{i_b}^{efg}, \tilde{R}_{-i_b}^{eg}) = M(\tilde{R}^{efg})$. If $M(\tilde{R}_{i_b}^{egf}, \tilde{R}_{-i_b}^{eg})(i_b) = f = M(\tilde{R}_{i_b}^{efg}, \tilde{R}_{-i_b}^{eg})(i_b)$ then $M(\tilde{R}_{i_b}^{egf}, \tilde{R}_{-i_b}^{eg}) = M(\tilde{R}^{efg})$ as M is non-bossy. Lemma 2 and $M(\tilde{R}^{eg})(1) = e = M(\hat{R}^{ef})(2)$ imply the contradiction $M(\tilde{R}_{i_b}^{egf}, \tilde{R}_{-i_b}^{eg})(1) = e = M(\tilde{R}^{egf})(2)$. SP-II ($M(R_{i_b}^{egf}, R_{-i_b}^{eg})(i_b) \in \{g, f\}$) and SP-I then imply $g = M(\tilde{R}_{i_b}^{egf}, \tilde{R}_{-i_b}^{eg})(i_b) = M(\tilde{R}^{eg})(i_b)$. By Lemma 2 $M(R^{eg})(1)$ equals e for any R^{eg} and the above arguments imply $M(R^{eg})(i_b) = g$. Switching the roles of g and f in the above arguments and using $M(R^{egf})(i_b) = g$ we obtain $M(R^{ef})(i_b) = f$ for any R^{ef}. To prove $M(R^{eh})(i_b) = h$ for all R^{eh} with $h \notin \{e, g, f\}$ apply the above arguments to R^{eh} and R^{egh} if $M(R^{eh})(e) = 1$ and to R^{eh} and R^{efh} otherwise. $\square$

9 Properties of $c_\emptyset$

Lemmas 4, 5, and 6 show that $c_\emptyset$ satisfies a range of properties. Say i owns g and i_b brokers e. If i ranks e above all other houses and if i_b ranks g above all other houses (except possibly e), then i obtains e and i_b g. If i ranks g at the top he gets it. As a broker i_b does not control any house other than e. Lemmas 4, 5, and 6 condense the Lemmas 2 and 3 which are not (directly) used after the current section. The following proof of Lemma 5 significantly simplifies the Pycia and Unver [4] proof by induction over the set of unmatched agents. Lemmas 4 and part c) of Lemma 6 are small preliminary results whose content also appears interspersed in the arguments in Pycia and Unver [4]. Part a) of Lemma 6 corresponds to Pycia and Unver [4] Lemma 15. Part b) of Lemma 6) is not shown in Pycia Unver [4].

Lemma 4 *Let $c_\emptyset(e) = (1, b)$ and $M(\hat{R}^{eg})(2) = e$ for some $\hat{R}^{eg}$. If R such that $R_2 : e$ and either $R_1 : g$ or $R_1 : e\ g$ then $M(R)(2) = e$ and $M(R)(1) = g$.*

Proof Let R^{eg} coincide with R on $H \setminus \{e, g\}$. Lemma 2 and $M(\hat{R}^{eg})(2) = e$ imply $M(R^{eg})(2) = e$. Lemma 3 and $c_\emptyset(e) = (1, b)$ imply $M(R^{eg})(1) = g$. Dropping e and g in all rankings SP-NB yields $M(R^{eg}) = M(R)$, in particular $M(R)(1) = g$ and $M(R)(2) = e$. □

Lemma 5 *[Pycia and Unver [4], Lemma 11] If $c_\emptyset(e) = (1, o)$ and $R_1 : e$, then $M(R)(1) = e$.*

Proof Let R^e, R^{fe}, R^{eg}, R^{efg} and R^{ef} coincide with R on all statements that are not explicitly mentioned. Define j and f via $M(R^{eg})(i) = g$ and $M(R^e)(i) = f$. Case 1: fR_ig and $f \neq g$. SP-PO, SP-NB and $e = M(R_i^{efg}, R_{-i}^{ef})(1)$ yield $f = M(R_1^{fe}, R_i^{efg}, R_{-\{1,i\}}^{ef})(1) = M(R_1^{fe}, R_i^{efg}, R_{-\{1,i\}}^{e})(1)$. If $f = M(R_1^{ef}, R_i^{efg}, R_{-\{1,i\}}^{e})(1)$ then $f = M(R_1^{ef}, R_i^{eg}, R_{-\{1,i\}}^{e})(1)$ holds by SP-NB. A contradiction arises since $M(R^{eg})(1) = e$, $M(R^{eg})(i) = g$ and SP-NB imply $M(R_1^{ef}, R_i^{eg}, R_{-\{1,i\}}^{e})(1) = e$. So f cannot equal $M(R_1^{ef}, R_i^{efg}, R_{-\{1,i\}}^{e})(1)$ which, by SP-PO, must equal e. This, $M(R^e)(i) = f$ and SP-NB imply $M(R_1^{ef}, R_i^{efg}, R_{-\{1,i\}}^{e}) = M(R_i^{efg}, R_{-i}^{e}) = M(R^e)$. Case 2. gR_if: The inductive application of SP-NB yields $M(R^{eg}) = M(R_i^{eg}, R_{-i}^{e}) = M(R^e)$. So $M(R^e)(1) = e$ holds in either case; $R_1 : e$ and SP-NB imply $M(R)(1) = e$. □

Lemma 6 *Let $c_\emptyset(e) = (1, b)$. Then*

a) $c_\emptyset(h) \neq (1, o)$ holds for all $h \in H$.

b) $c_\emptyset(h) \neq (1, b)$ holds for all $h \in H \setminus \{e\}$.

c) If $c_\emptyset(g) = (i, o)$ then $M(R^{eg})(i) = e$.

Proof a) Suppose $c_\emptyset(g) = (1, o)$ for some $g \neq e$. Since $c_\emptyset(e) = (1, b)$ there exist $1 \neq i \neq j \neq 1$ and $f \notin \{e, g\}$ with $M(R^{eg})(i) = e = M(R^{ef})(j)$.

Lemma 5, $c_\emptyset(g) = (1, o)$ and SP-II imply $M(R_i^{ge}, R_{-i}^{eg})(1) \in \{e, g\}$. SP-PO and $M(R^{eg})(i) = e$ yield $M(R_i^{ge}, R_{-i}^{eg})(i) = g$. This and SP-I imply $M(R_i^{ge}, R_{-i}^{eg})(1) = e = M(R_1^{ef}, R_i^{ge}, R_{-\{1,i\}}^{eg})(1)$. Lemma 4 and $M(R^{ef})(j) = e$, yield the contradiction $M(R_1^{ef}, R_i^{ge}, R_{-\{1,i\}}^{eg})(j) = e$.

b) Suppose $c_\emptyset(g) = (1, b)$ for some $g \neq e$ and $M(R^{eg})(2) = e$. Since 1 brokers g two different agents must obtain g under R^{gh} for different h. So there exist an $i \notin \{1, 2\}$ and $f \notin \{e, g\}$ such that $M(R^{gf})(i) = g$. Lemma 4 together with $c_\emptyset(e) = (1, b)$, $M(R^{eg})(2) = e$, $c_\emptyset(g) = (1, b)$ and $M(R^{gf})(i) = g$ yields the contradiction $M(R_{\{1,i\}}^{gf}, R_{-\{1,i\}}^{eg})(1) = g = M(R_{\{1,i\}}^{gf}, R_{-\{1,i\}}^{eg})(i)$.

c) Lemma 5, $c_\emptyset(g) = (i, o)$, and SP-II imply $M(R^{eg})(i) \in \{e, g\}$. But $c_\emptyset(e) = (1, b)$ implies $M(R^{eg})(1) = g$ and $M(R^{eg})(i) = e$ must hold. □

10 Braids, (C1), (C2), and (C3)

If there is at most one brokered house according to $c_\emptyset$, then part a) of Lemma 6 implies that the broker of this house does not own any house as required by (C3). The definition of $c_\emptyset$ implies that there must be at least two owners under $c_\emptyset$ if there is exactly one broker under $c_\emptyset$ as required by (C2). Finally, Lemma 7 shows that M is a braid if at least two houses are brokered according to $c_\emptyset$ which therefore satisfies (C1).

The following Lemma 7 has no counterpart in Pycia and Unver [4] which claims in Lemmas 12 and 13, that a stronger version of (C1), according to which there is at most one broker at any ν, holds for any good mechanism. Pycia and Unver [5] corrects this error with the new Lemma 11 which is identical to the following Lemma 7.

Lemma 7 *Let $c_\emptyset(e) = (1, b)$ and $c_\emptyset(g) = (k, b)$ for $e \neq g$ and some $k \in N$. Then $| H |= 3$ and M is a braid.*

Proof W.l.o.g. assume that $M(R^{eg})(2) = e = M(R^{ef})(3)$ for some R^{eg}, R^{ef}.

Claim 1: House g is brokered by agent 3, so $k = 3$.

Since 1 brokers e, 1 cannot broker g by part b) of Lemma 6. Suppose $k = 2$ and therefore $M(R^{ge})(2) = e$. Then SP-NB yields $M(R^{eg}) =$

$M(R^{ge}_{\{1,2\}}, R^{eg}_{-\{1,2\}}) = M(R^{ge})$, in particular $M(R^{ge}_{\{1,2\}}, R^{eg}_{-\{1,2\}})(1) = g$ and $M(R^{ge}_{\{1,2\}}, R^{eg}_{-\{1,2\}})(2) = e$. SP-PO and SP-I applied to 1's choice yield $M(R^{ge}_2, R^{eg}_{-2})(1) = e = M(R^{ef}_1, R^{ge}_2, R^{eg}_{-\{1,2\}})(1)$. Lemma 4, $M(R^{ef})(3) = e$, and $c_\emptyset(e) = (1, b)$ imply the contradiction $M(R^{ef}_1, R^{ge}_2, R^{eg}_{-\{1,2\}})(3) = e$ and neither $k = 2$ nor $M(R^{ge})(2) = e$ hold. $M(R^{eg})(2) = M(R^{eg}_2, R^{ge}_{-2})(2) = e$ (by SP-NB), and $M(R^{ge})(2) \in \{e, g\}$ (by SP-PO) then imply $M(R^{ge})(2) = g$. Now suppose $k > 3$. Lemma 4 together with $M(R^{ef})(3) = e$, $c_\emptyset(e) = (1, b)$, $M(R^{ge})(2) = g$ and $c_\emptyset(g) = (k, b)$ yields the contradiction

$$M(R^{ge}_{\{2,k\}}, R^{ef}_{-\{2,k\}})(k) = e = M(R^{ge}_{\{2,k\}}, R^{ef}_{-\{2,k\}})(3).$$

Claim 2: There is no $h \in H$ and $i > 3$ such that $c_\emptyset(h) = (i, \cdot)$.

Suppose $c_\emptyset(h) = (i, o)$ held for some $i > 3$ and $h \in H$. Lemma 4, $M(R^{eh})(i) = e$ (which holds by Lemma 6), $c_\emptyset(e) = (1, b)$, $M(R^{ge})(2) = g$, and $c_\emptyset(g) = (3, b)$ (as established in Claim 1) yields the contradiction

$$M(R^{eh}_{\{1,i\}}, R^{ge}_{-\{1,i\}})(i) = e = M(R^{eh}_{\{1,i\}}, R^{ge}_{-\{1,i\}})(3).$$

Next suppose $c_\emptyset(h) = (i, b)$ held for some $i > 3$ and $h \in H$. Let j be such that $M(R^{he})(j) = h$. Lemma 4 implies $M(R^{he}_{\{i,j\}}, R_{-\{i,j\}})(i) = e$ for any R. If $j \notin \{1, 2\}$, Lemma 4, $c_\emptyset(e) = (1, b)$, and $M(R^{eg})(2) = e$ imply the contradiction $M(R^{he}_{\{i,j\}}, R^{eg}_{-\{i,j\}})(2) = e$. Lemma 4 also leads to a contradiction in the remaining two cases: If $j \notin \{1, 3\}$ $c_\emptyset(e) = (1, b)$ and $M(R^{ef})(3) = e$ imply $M(R^{he}_{\{i,j\}}, R^{ef}_{-\{i,j\}})(3) = e$. If $j \notin \{2, 3\}$ $c_\emptyset(g) = (3, b)$ and $M(R^{ge})(2) = g$ (both established in Claim 1) imply $M(R^{he}_{\{i,j\}}, R^{ge}_{-\{i,j\}})(3) = e$.

Claim 3: $c_\emptyset(f) = (2, b)$ and $H = \{e, f, g\}$.

Let $c_\emptyset(h) = (i, \cdot)$ for some $h \notin \{e, g\}$. Claim 2 implies $i \leq 3$. By Lemma 6 no broker controls more than one house. So $c_\emptyset(e) = (1, b)$, and $c_\emptyset(g) = (3, b)$ implies $i \neq 1, 3$, in particular $c_\emptyset(f) = (2, \cdot)$. Part c) of Lemma 6 and $M(R^{ef})(2) \neq e$ then imply $c_\emptyset(f) = (2, b)$. By Lemma 6 2 does not control any other house and H equals $\{e, f, g\}$.

Claim 4: Fix $\omega = (e, f, g)$. If $R \in \{R^{eg}, R^{gf}, R^{fe}\}$ then $M(R) = B^\omega(R) = (g, e, f)$, if $R \in \{R^{ef}, R^{ge}, R^{fg}\}$ then $M(R) = B^\omega(R) = (f, g, e)$.

Since $c_\emptyset(g) = (3, b)$ and $M(R^{ge})(2) = g$ there exist a $h \notin \{e, g\}$ and a $j \notin \{2, 3\}$ such that $M(R^{gh})(j) = g$. Since there are only three houses $h = f$. If j is not equal 1 Lemma 4 implies the contradiction

$$M(R^{eg}_{\{1,2\}}, R^{gf}_{-\{1,2\}})(1) = g = M(R^{eg}_{\{1,2\}}, R^{gf}_{-\{1,2\}})(j).$$

Claim 2 implies $M(R^{fg})(2) = g$ and $M(R^{fe})(2) = e$. By SP-NB $e = M(R^{ef})(3) = M(R^{ef}_3, R^{fe}_{-3})(3)$. SP-II then implies $M(R^{fe})(3) \in \{e, f\}$. Given $M(R^{fe})(2) = e$, $M(R^{fe})(3)$ must equal f. Applying the same arguments mutatis mutandis we obtain $M(R^{fg})(1) = f$. In sum we know which agents in $\{1, 2, 3\}$ are matched with the two top ranked houses for any $R^{hh'}$ with $\{h, h'\} \subset \{e, f, g\}$. Lemma 4, $c_\emptyset(g) = (3, b)$, and $M(R^{gf})(1) = g$ imply $M(R^{ge}_1, R^{eg}_2, R^{gf}_3)(3) = f$. SP-NB implies $M(R^{eg}) = M(R^{ge}_1, R^{eg}_2, R^{gf}_3)$ and in sum we obtain $M(R^{eg}) = (g, e, f)$. Mutatis mutandis the same arguments prove the claim for R^{ef}, R^{ge}, R^{gf}, R^{fe}, and R^{fg}.

Claim 5: $M(R) = B^\omega(R)$ holds for all $R \in \mathcal{R}$.

To show that M equals B^ω with $\omega = (e, f, g)$, $\omega' = (g, e, f)$, and $\omega'' = (f, g, e)$ I separately consider some profiles R that are covered by case I, II, and III as defined in Lemma 1. The arguments for each one of these examples apply mutatis mutandis to all other profiles that are covered by the same case. As a profile that is covered by II consider R with $g = \omega'(1) R_1 \omega''(1) = f$ and $R_2 : \omega(1) = e$. Claim 4 and SP-NB imply $\omega' = M(R^{eg}) = M(R) = B^\omega(R)$.

As an example that is covered by III consider R with $g R_1 f$, $R_2 : f$, and $R_3 : e = \omega(1) = \omega'(2)$. The preceding paragraph yields $M(R^{eg}_1, R^{ef}_{-1}) = (g, e, f) = M(R^{eg}_1, R^{fe}_{-1})$. SP-PO then implies $M(R^{eg}_1, R^{fe}_2, R^{ef}_3)(2) = f$ and $M(R^{eg}_1, R^{fe}_2, R^{ef}_3)(3) = e$. That paragraph also yields $M(R^{fe}_2, R^{ef}_{-2})(1) = f$, and by SP-I $M(R^{eg}_1, R^{fe}_2, R^{ef}_3)(1) \neq \emptyset$. In sum we obtain $M(R^{eg}_1, R^{fe}_2, R^{ef}_3) = (g, f, e)$ and by SP-NB $M(R^{eg}_1, R^{he}_2, R^{eh}_3) = M(R) = B^\omega(R)$.

Finally let $R_1 : e$, $R_2 : f$ and $R_3 : g$, so R is covered by I. Since (R'_i, R_{-i}) for $R'_i : \omega(j)$ is covered by II or III (as analysed above), i is matched under $M(R'_i, R_{-i})$. Since M is strategyproof $M(R)(i) \neq \emptyset$. Since ω is the only Pareto optimum matching $\{1, 2, 3\}$, $M(R)$ equals $\omega = B^\omega(R)$. $\square$

11 Pointing

Assume that M is not a braid and fix some R°. Lemma 8 shows that the outcome $M(R^\circ)$ contains the submatching achieved at R° in the first round of any trading and braiding mechanism with $c_\emptyset$ the control rights function at $\emptyset$, so $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$ implies $\nu \subset M(R^\circ)$. Lemma 9 shows that the calculation of $M(R^\circ)$ can be split into the calculation of a such submatching and the outcome of a well-defined submechanism. This submechanism inherits the property of being good from M and can by the inductive hypothesis be represented as a trading and braiding mechanism $c[\nu]$. In sum we obtain that $c(R) = M(R)$ for any $R \in \mathcal{R}$. Lemmas 8 and 9 correspond to Pycia and Unver [4] Section F3 which shows that at any R the outcome of the trading cycles mechanism constructed in the preceding sections equals the outcome of the underlying good mechanism. For the next two Lemmas fix a submatching $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$. W.l.o.g. assume $N_\nu = \{1, \cdots, m\}$ and $c_\emptyset(h_i) = (i, \cdot)$ for all $i \leq m$.

Lemma 8 *Any submatching that arises out of matching one cycle under $c_\emptyset$ at R° is part of the outcome $M(R^\circ)$: $\nu \in \mathcal{N}(c_\emptyset)(R^\circ)$ implies $\nu \subset M(R^\circ)$.*

Proof **Case 1**: $m = 1$. So $\nu = \{(1, h_1)\}$. Since only an owner may point to a house he controls and since an owner may point to any house, we have $c_\emptyset(h_1) = (1, o)$ and $R_1^\circ : h_1$. Lemma 5 then implies $M(R^\circ)(1) = h_1$ and $\nu \subset M(R^\circ)$.

For the remaining cases 2, 3, and 4 let $m > 1$. W.l.o.g assume that $\nu(i) = h_{i+1}$ for all $i < m$ and $\nu(m) = h_1$.

Case 2: $c_\emptyset(h_i) = (i, o)$ for all $i \leq m$. For all $i \leq m$ let $R_i^* : \nu(i)\ h_i$. For all preference statements that have not been explicitly mentioned let R^* and R° coincide. Suppose we had $M(R^*)(i^*) \neq \nu(i^*)$ for at least one $i^* \leq m$, say $i^* = m$. Lemma 5 and $c_\emptyset(h_m) = (m, o)$ imply $M(R'_m, R^*_{-m})(m) = h_m$ for $R'_m : h_m$. SP-II implies that $M(R^*)(m) \in \{h_1, h_m\}$. The assumption $M(R^*)(m) \neq \nu(m) = h_1$ then implies $M(R^*)(m) = h_m$. So $M(R^*)(m-1)$ differs from $\nu(m-1) = h_m$. Inductively applying these arguments to all agents in the cycle we obtain that $M(R^*)(i) = h_i$ for all $i \leq m$. This contradicts the Pareto optimality of M since each $i \leq m$ strictly prefers $\nu(i)$

to h_i. So $\nu \subset M(R^*)$ must hold. Dropping h_i in the rankings of all agents $i \leq m$ SP-NB yields $M(R^*) = M(R^\circ)$.

For the remaining cases 3 and 4 assume that $c_\emptyset(h_m) = (m, b)$. By (C2) $c_\emptyset(h_i) = (i, o)$ holds for all $i < m$.

Case 3: $m = 2$: $R_1^\circ : h_2$ and either $R_2^\circ : h_1$ or $R_2^\circ : h_2\ h_1$ must hold for ν to arise out of matching a single cycle and $\nu \subset M(R^\circ)$ holds by Lemma 4 and part c) of Lemma 6.

Case 4: $m > 2$: Define $R_{m-2}^* : h_{m-1}\ h_1$, $R_m^* : h_m\ h_1\ h_{m-1}$ and $R_i^* : \nu(i)\ h_i$ for all other $i \leq m$. For all preference statements that have not been explicitly mentioned let R^* and R° coincide. Under $M(R'_{m-2}, R^*_{-(m-2)})$ with $R'_{m-2} : h_1$ the owners $\{1, \cdots, m-2\}$ form a pointing cycle and $M(R'_{m-2}, R^*_{-(m-2)})(m-2) = h_1$ holds by Case 2. By Case 3 $M(R'_m, R^*_{-m})(m) = h_{m-1}$ holds for $R'_m : h_m\ h_{m-1}$. Strategyproofness implies $M(R^*)(m-2) R^*_{m-2} h_1$, $M(R^*)(m) R^*_m h_{m-1}$, and $M(R^*)(m) \neq h_m$. So $m-2$ and m must be matched to the houses $\{h_1, h_{m-1}\}$ at R^*. Pareto optimality requires that $M(R^*)(m-2) = h_{m-1}$ and $M(R^*)(m) = h_1$. Lemma 5, $c_\emptyset(h_1) = (1, o)$, and SP-I imply $M(R^*)(1) R_1^* h_1$. Since $M(R^*)(m) = h_1$, $M(R^*)(1)$ must equal h_2. Inductively applying these arguments to all other agents in N_ν we obtain $\nu \subset M(R^*)$. Applying SP-NB to drop h_1 in R^*_{m-2}, h_{m-1} and h_m in R^*_m, and h_i in all other rankings R_i^* with $i \leq m$ we obtain $M(R^*) = M(R^\circ)$. $\square$

For any R let $\overline{R}$ be its restriction to $\overline{N}_\nu\ \overline{H}_\nu$. Let $\overline{\mathcal{R}}$ be the set of all such restrictions. Let $\overline{\mathcal{M}}$ be the set of submatchings ν' such that $\nu \cup \nu'$ is a matching. Define a trading and braiding mechanism $c[\nu] : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ by $c[\nu](\overline{R}) := M(R) \setminus \nu$ whenever $\nu \in \mathcal{N}(c_\emptyset)(R)$.

Lemma 9 *The mechanism $c[\nu]$ is well-defined.*

Proof Fix three profiles R_{-N_ν}, R'_{-N_ν}, and R''_{-N_ν} with $\overline{R}$ as their restriction to $\overline{H}_\nu$ and such that $\overline{H}_\nu R''_i H_\nu$ holds for all $i \in \overline{N}_\nu$. By Lemma 8, $\nu \subset M(R^\circ_{N_\nu}, R_{-N_\nu})$ and $\nu \subset M(R^\circ_{N_\nu}, R'_{-N_\nu})$. By SP-NB $M(R^\circ_{N_\nu}, R_{-N_\nu}) = M(R^\circ_{N_\nu}, R''_{-N_\nu}) = M(R^\circ_{N_\nu}, R'_{-N_\nu})$ and consequently $M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu = M(R^\circ_{N_\nu}, R'_{-N_\nu}) \setminus \nu$. So $M^\circ : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ with $M^\circ(\overline{R}) = M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu$ where R_{-N_ν} has $\overline{R}$ as its restriction to $\overline{H}_\nu$ is a well-defined mechanism.

Now let $M^* : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ be defined via $M^*(\overline{R}) = M(R^*_{N_\nu}, R_{-N_\nu})$ where the restriction of R_{-N_ν} to $\overline{H}_\nu$ equals $\overline{R}$ and $R^*_{N_\nu}$ is fixed such that ν forms as a cycle at $\emptyset$ under $c_\emptyset$ at any $(R^*_{N_\nu}, R_{-N_\nu})$. To see that $M^* = M^\circ$, fix some some $\overline{R} \in \overline{\mathcal{R}}$, let R_{-N_ν} have $\overline{R}$ as its restriction of $\overline{H}_\nu$ and note

$$\begin{aligned} \nu \cup M^*(\overline{R}) = M(R^*_{N_\nu}, R_{-N_\nu}) = \\ M(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) = \cdots = M(R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu}) = \\ M(R^\circ_{N_\nu}, R_{-N_\nu}) = \nu \cup M^\circ(\overline{R}). \end{aligned}$$

For ν to arise at $\emptyset$ under $(R^\circ_{N_\nu}, R_{N_\nu})$ as well as under $(R^*_{N_\nu}, R_{N_\nu})$ we have $R^*_i : \nu(i)$, $R^\circ_i : \nu(i)$ for all owners $i \leq m$ and $\nu_i(i) R^*_i H \setminus \{h_i\}$, $\nu_i(i) R^\circ_i H \setminus \{h_i\}$ for a broker $i \leq m$ (if there is one) where $c_\emptyset(h_i) = (i, b)$ and h_i is the house he brokers. So the cycle yielding ν also arises under $c_\emptyset$ at any of the intermediate profiles $(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}) \cdots (R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu})$. Lemma 8 implies that $\nu \subset M(R^\circ_1, R^*_{\{2,\cdots,m\}}, R_{-N_\nu}), \cdots, \nu \subset M(R^\circ_{\{1,\cdots,m-1\}}, R^*_m, R_{-N_\nu})$. Due to the non-bossiness of M the matchings of all agents in $\overline{N}_\nu$ stay constant as well.

To see that M° is strategy proof fix an agent $i \in \overline{N}_\nu$, a profile $\overline{R}$ and a deviation $\overline{R}'_i$ for agent i. Let the restrictions of R'_i and R_{-N_ν} to $\overline{H}_\nu$ be $\overline{R}'_i$ and $\overline{R}$. The definition of M° and the strategyproofness of M then imply

$$M^\circ(\overline{R})(i) = M(R^\circ_{N_\nu}, R_{-N_\nu})(i) R_i M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})(i) = M^\circ(\overline{R}'_i, \overline{R}_{-i})(i).$$

To see that M° is non-bossy let $M^\circ(\overline{R})(i) = M^\circ(\overline{R}'_i, \overline{R}_{-i})(i)$. Since $i \notin N_\nu$ we obtain $M(R^\circ_{N_\nu}, R_{-N_\nu})(i) = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})(i)$. The non-bossiness of M then implies $M(R^\circ_{N_\nu}, R_{-N_\nu}) = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})})$ and therefore

$$M^\circ(\overline{R}) = M(R^\circ_{N_\nu}, R_{-N_\nu}) \setminus \nu = M(R^\circ_{N_\nu}, R'_i, R_{-(N_\nu \setminus \{i\})}) \setminus \nu = M^\circ(\overline{R}'_i, \overline{R}_{-i}).$$

Since $M(R^\circ_{N_\nu}, R_{-N_\nu}) = \nu \cup M^\circ(\overline{R})$ is Pareto optimal at $(R^\circ_{N_\nu}, R_{-N_\nu})$, M° is Pareto optimal. So M° is good. By the hypothesis of the induction it can be represented as a trading and braiding mechanism $c[\nu]$. □

12 Defining a trading and braiding mechanism c

Combine the definition of $c_\emptyset$ in Section 8 with Lemma 9 on submechanisms $c[\nu^*]$ to define a control rights structure c via $c_\nu := c[\nu^*]_{\nu'}$ if $\nu = \nu^* \cup \nu'$,

$\nu^* \in \mathcal{N}(c_\emptyset)$, and ν' $c[\nu^*]$-relevant. Since a submatching $\nu \neq \emptyset$ is c-relevant if and only if it can be split into a submatching $\nu^* \in \mathcal{N}(c_\emptyset)$ and a $c[\nu^*]$-relevant ν', c is defined on all c-relevant submatchings.

We know from Section 10 that $c_\emptyset$ satisfies (C1), (C2) and (C3). Since $c[\nu^*]$ is a trading and braiding mechanism for any $\nu^* \in \mathcal{N}(c_\emptyset)$, (C1), (C2), and (C3) are satisfied for any c-relevant $\nu \neq \emptyset$. Moreover (C4), (C5), and (C6) are satisfied by any pair of a c-relevant $\nu^\circ \neq \emptyset$ with a direct c-successor ν of ν°. In the following Lemma 10 I show that (C4), (C5), and (C6) are also satisfied for $\emptyset$ and any $\nu \in \mathcal{N}(c_\emptyset)$. Lemma 10 corresponds to Pycia and Unver [4] Lemmas 16, 17, 18, and 19 on the requirements that link control rights functions.

Lemma 10 *(C4), (C5) and (C6) hold for $\emptyset$ and $\nu \in \mathcal{N}(c_\emptyset)$.*

Proof **(C4)** Fix $i^* \notin N_\nu$ with $c_\emptyset(e) = (i^*, o)$. Lemma 8 implies $\{(i^*, e)\} \subset M(R^\circ_{N_\nu}, R^e_{-N_\nu})$ for any $R^e_{-N_\nu}$. Since $M(R^\circ_{N_\nu}, R^e_{-N_\nu}) \backslash \nu = c[\nu](\overline{R}^e)$, $c[\nu](\overline{R}^e)(i^*) = e$ holds for any $\overline{R}^e$ and i^* owns e under $c[\nu]_\emptyset = c_\nu$.

(C5) Let $i_b, i, j \notin N_\nu$ be such that $c_\emptyset(e) = (i_b, b)$, $c_\emptyset(g) = (i, o)$, and $c_\emptyset(f) = (j, o)$. Lemma 8 implies $M(R^\circ_{N_\nu} R^{eg}_{-N_\nu})(i_b) = g$, $M(R^\circ_{N_\nu} R^{eg}_{-N_\nu})(i) = e = M(R^\circ_{N_\nu} R^{ef}_{-N_\nu})(j)$. Since $M(R^\circ_{N_\nu} R^{eg}_{-N_\nu}) \backslash \nu = c[\nu](\overline{R}^{eg})$ and $M(R^\circ_{N_\nu} R^{ef}_{-N_\nu}) \backslash \nu = c[\nu](\overline{R}^{ef})$, we have $c[\nu](\overline{R}^{eg})(i) = e = c[\nu](\overline{R}^{ef})(j)$ and e is not owned at $c[\nu]_\emptyset$. Since $[\nu](\overline{R}^{eg})(i_b) = g$, i_b brokers e under $c[\nu]_\emptyset = c_\nu$.

(C6) Let $c_\emptyset(g) = c_\nu(g) = (i, o)$, $c_\emptyset(e) = (i_b, b) \neq c_\nu(e)$ and $i_b \notin N_\nu$. Fix any $h \in H_\nu$. Lemma 8 implies $\{(i, e), (i_b, g)\} \subset M(R^\circ_{N_\nu}, R^{eg}_{-N_\nu}) = \nu \cup c[\nu](\overline{R}^{eg})$ for any R^{eg}. So $c[\nu](\overline{R}^{eg})(i) = e$ and $c[\nu](\overline{R}^{eg})(i_b) = g$ holds for any $\overline{R}^{eg}$ and e can neither be brokered by i nor be owned by some $j \neq i$ at ν. If $c_\nu(e) = (j, b)$ with $j \in \overline{N}_\nu \setminus \{i_b, i\}$ the contradiction $c[\nu](\overline{R}^{eg})(j) = g$ results. So $c_\nu(e) = (i, o)$ must hold and $M(R^\circ_{N_\nu}, R^{eg}_{-N_\nu})$ can be calculated as $\nu \cup \{(i, e)\} \cup c[\nu \cup \{(i, e)\}](\overline{\overline{R}}^g)$ with $\overline{\overline{R}}^g$ the restriction of R^{eg} to $\overline{H}_\nu \setminus \{e\}$ and $\overline{N}_\nu \setminus \{i\}$. In sum i_b is matched with g under $c[\nu \cup \{(i, e)\}]$ whenever he ranks g above all remaining houses and therefore owns g under $c[\nu \cup \{(i, e)\}]_\emptyset = c_{\nu \cup \{(i,e)\}}$.

13 Proof of Proposition 1

Fix a lax trading and braiding mechanism $\bar{c}$ with n agents and use the rules given in Proposition 1 to define a control rights structure c.

Case 1. $\bar{c}_\emptyset$ satisfies (C2). The construction of c uniquely defines $c_\emptyset$ as $\bar{c}_\emptyset$. For any $\nu^* \in \mathcal{N}(c_\emptyset) = \mathcal{N}(\bar{c}_\emptyset)$, $c[\nu^*]$ is, by the inductive hypothesis, the trading and braiding mechanism that represents $\bar{c}[\nu^*]$. For any c-relevant $\nu \neq \emptyset$ we have $c_\nu = c[\nu^*]_{\nu'}$ for some $\nu^* \in \mathcal{N}(c_\emptyset)$ and the $c[\nu^*]$-relevant ν' with $\nu = \nu^* \cup \nu'$. If ν can alternatively be represented as $\overline{\nu}^* \cup \overline{\nu}'$ with $\overline{\nu}^* \in \mathcal{N}(c_\emptyset)$ and $\overline{\nu}'$ $c[\overline{\nu}^*]$-relevant, then $c[\nu^*]_{\nu'} = c[\overline{\nu}^*]_{\overline{\nu}'}$ holds since $c[\nu^*]$ and $c[\overline{\nu}^*]$ are both derived from $\bar{c}$ via Proposition 1.

Case 2. $\bar{c}_\emptyset$ does not satisfy (C2), so $\bar{c}_\emptyset(h_b) = (i_b, b)$ and $\bar{c}_\emptyset(h) = (i^*, o)$ holds for two different $i_b, i^* \in N$, some $h_b \in H$ and all $h \in H \setminus \{h_b\}$. The construction of c requires $c_\emptyset(h) \colon = (i^*, o)$ for all $h \in H$ and $c_{\{(i^*,h_b)\}}(h) \colon = (i_b, o)$ for all $h \in H \setminus \{h_b\}$. To see that c_ν is defined for all c-relevant ν first note that exactly one submatching is c- but not $\bar{c}$-relevant: $\{(i^*, h_b)\}$. The control rights function $c_{\{(i^*,h_b)\}}$ is defined such that any direct c-successor of $\{(i^*, h_b)\}$ can be represented as $\{(i^*, h_b), (i_b, h)\}$ for some $h \neq h_b$. Since $\{(i^*, h)\}$ and $\{(i^*, h_b), (i_b, h)\}$ are $\bar{c}$-relevant for any $h \neq h_b$, $c[\{(i^*, h)\}]$ and $c[\{(i^*, h_b), (i_b, h)\}]$ are by the inductive hypothesis trading and braiding mechanisms. For any c-relevant $\nu \notin \{\{(i^*, h)\}, \{(i^*, h_b), (i_b, h)\}$ for some $h \neq h_b\}$ there exists a unique $\nu^* \in \{\{(i^*, h)\}, \{(i^*, h_b), (i_b, h)\}$ for some $h \neq h_b\}$ and a unique $c[\nu^*]$-relevant ν' such that $\nu = \nu^* \cup \nu'$ and c_ν is uniquely defined as $c_\nu = c[\nu^*]_{\nu'}$.

To see that c is a trading and braiding mechanism in either case, first note that (C1)-(C6) are by the inductive hypothesis satisfied in the sub-mechanisms $c[\nu^*]$ as defined above. If Case 1 applies then $c_\emptyset$ satisfies (C1), (C2), and (C3), since $\bar{c}_\emptyset$ does. Now consider ν° and a direct c-successor $\nu \in \mathcal{N}(c_\emptyset)$. If $c_\nu = \bar{c}_\nu$ (C4), (C5), and (C6) are automatically satisfied at $\emptyset$ and ν. If $c_\nu \neq \bar{c}_\nu$, then $\bar{c}_\nu(h_b) = (i_b, b)$ and $\bar{c}_\nu(h) = (i^*, o)$ hold for two different $i_b, i^* \in \overline{N}_\nu$, some $h_b \in \overline{H}_\nu$ and all $h \in \overline{H}_\nu \setminus \{h_b\}$. By construction $c_\nu(h) = (i^*, o)$ holds for all $h \in \overline{H}_\nu$. Since $\bar{c}$ satisfies (C4) and since $\bar{c}_\nu(h) = (i, o) \Rightarrow c_\nu(h) = (i, o)$ holds for all $(h, i) \in \overline{H}_\nu \times \overline{N}_\nu$, (C4) is satisfied by c at $\emptyset$ and ν. (C5) trivially holds since only one agent owns houses under

$c_\emptyset$ and c_ν. For (C6) to have any grip on c at $\emptyset$ and ν $c_\emptyset(h_b) = (i_b, b)$ must hold. In that case the construction of $\overline{c}$ implies $c_{\nu\cup\{(i^*,h_b)\}}(h) = (i_b, o)$ for all $h \in \overline{H}_\nu \setminus \{h_b\}$ exactly as required by (C6). If Case 2 applies, then $\emptyset$ and any $\nu^* \in \{\{(i^*, h)\}, \{(i^*, h_b), (i_b, h)\}$ for $h \in H \setminus \{h_b\}\}$ are c-dictatorial and (C1)-(C6) trivially hold in the remaining cases.

To see that $c(R) = \overline{c}(R)$ for all $R \in \mathcal{R}$ fix an arbitrary R together with a $\nu \in \mathcal{N}(\overline{c}_\emptyset)(R)$. If $\nu \notin \mathcal{N}(c_\emptyset)(R)$ then ν equals $\{(i^*, h_b), (i_b, h)\}$ with $\overline{c}_\emptyset(h_b) = (i_b, b)$ and $\overline{c}_\emptyset(h) = (i^*, o)$. Moreover, agent i^* ranks h_b at the top and $\{(i^*, h_b)\}$ forms under c at $\emptyset$. By the construction of c we have $c_{\{(i^*,h_b)\}}(h) = (i_b, o)$. Since $\nu = \{(i^*, h_b), (i_b, h)\} \in \mathcal{N}(\overline{c}_\emptyset)(R)$ agent i_b prefers h to all other houses in $H \setminus \{h_b\}$. So $\nu = \{(i^*, h_b), (i_b, h)\}$ is reachable under c at R. Let $\overline{R}$ be the restriction of R to $\overline{N}_\nu$ and $\overline{H}_\nu$. By the inductive hypothesis Proposition 1 holds for $\overline{c}[\nu]$ and $c[\nu]$ and $\overline{c}(R) = \nu \cup \overline{c}[\nu](\overline{R})$ equals $\nu \cup c[\nu](\overline{R}) = c(R)$.

14 Relations with other sets of good mechanisms

A trading and braiding mechanism qualifies as a Pycia and Uver [4] trading cycles mechanism if (C1) is replaced by the requirement that there is at most one broker at any given round of the mechanism. A trading and braiding mechanism without brokerage or braids belongs to the class of hierarchical exchange mechanisms, characterized by Papai [3]. In this subclass (C1), (C2), (C3), (C5), and (C6) are trivially satisfied. A trading and braiding mechanism is GTTC if $\mid N \mid = \mid H \mid$ and if there exists a matching μ such that $c_\emptyset(\mu(i)) = (i, o)$ for all $i \in N$.

Just like Shapley and Scarf's [6] original definition of Gale's top trading cycles, the definition of trading and braiding cycles requires that at least one cycle is matched at any trading round. Papai's [3] hierarchical exchange mechanisms and Pycia and Unver's [4] trading cycles mechanisms in contrast require that all cycles at any given round have to be matched at that round.[3]

[3] Carroll [2] shows that the order of the elimination of trading cycles does not matter in top trading cycles mechanisms.

The freedom to remove trading cycles in any order considerably simplifies the proof that any lax mechanism is good and the proof of Theorem 1.[4] Theorem 2 contains a *unique* representation of good mechanisms: any good mechanism can be represented as exactly one trading and braiding mechanism. In contrast Pycia and Unver's [4] trading cycles mechanisms and Papai's [3] hierarchical exchange mechanism have multiple representations.

Strategyproofness and Pareto optimality are better founded than nonbossiness as principles of mechanism design and one might wonder about a characterization of the set of all strategy proof and Pareto optimal mechanisms. Given that all three axioms are repeatedly used in the proof that any good mechanism can be represented as a trading and braiding mechanism a simple extension of the same proof to the grand set of all strategy proof and Pareto optimal mechanisms is out of the question. However, the representation of good mechanisms as trading and braiding mechanisms can be used to construct a class of Pareto optimal and strategy proof mechanisms. Simply modify control rights structures insofar as that the inheritance of houses not only depends on submatchings ν but also on the preferences of the matched agents. Such a modified control rights structure c maps any combination of a c-relevant ν and a profile of preferences R to a control rights function. Keeping (C1)-(C6) intact we would have to additionally impose that a c-relevant ν and two profiles of preferences R and R' can only be mapped to two different control rights functions if $R_i \neq R'_i$ holds for some $i \in N_\nu$. A serial dictatorship in which the second dictator depends on the first dictator's preferences over houses he did not choose is the simplest example of such a bossy mechanism. The inductive proof that any trading and braiding mechanism is good can be extended to show that any mechanism, defined through such a modified control rights structure, is strategyproof and Pareto optimal. Similarly, only a few extra steps are required to extend the proof of Theorem 1 to this wider class to strategy proof and Pareto optimal mechanisms. The

[4]Successive mechanisms c^k and c^{k+1} in the proof of Theorem 1 typically differ only with respect to two agents. To show that such successive mechanisms are s-equivalent I need to zoom to a trading round where at least one of these two agents has to be matched. I do so by eliminating as many cycles as possible while keeping the two crucial agents unmatched.

question whether any Pareto optimal and strategyproof mechanisms can be represented by such a modified control rights structure awaits some new ideas and techniques of proof.

15 Proof of Proposition 2

Fix an arbitrary profile of preferences R. Derive 9 independent linear equations from the Pareto optimality, strategy proofness and equal treatment of equals of $\mathfrak{M}$ to uniquely determine the 9 values $\mathfrak{M}(R)[i,h]$. Since random serial dictatorship satisfies the named properties it equals $\mathfrak{M}$.

Case (I) there is a unique Pareto optimum at R. Ex post Pareto optimality requires that $\mathfrak{M}(R)$ assigns probability 1 to this matching. Case (II) $R_1 = R_2 = R_3$. Equal treatment of equals requires $\mathfrak{M}(R)[i,h] = \frac{1}{3}$ for all i, h. Case (III) (R_1^c, R_{-1}^{ab}). Ex post Pareto optimality implies $\mathfrak{M}(R_1^c, R_{-1}^{ab})[1,c] = 1$. Equal treatment of equals implies $\mathfrak{M}(R_1^c, R_{-1}^{ab})[2,\cdot] = \mathfrak{M}(R_1^c, R_{-1}^{ab})[3,\cdot]$ and therefore $\mathfrak{M}(R_1^c, R_{-1}^{ab})[i,h] = \frac{1}{2}$ for $i = 2,3$ and $h = a, b$. Case (IV) (R_2^{ac}, R_{-2}^{ab}). Ex post Pareto optimality implies $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[2,b] = 0$; ordinal strategyproofness and (II) imply that $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[2,a] = \frac{1}{3}$. Equal treatment of equals implies $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[1,\cdot] = \mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[3,\cdot]$. The unique solution to this system of linear equations is $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[2,c] = \frac{2}{3}$, $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[i,a] = \frac{1}{3}$ for $i = 1,2,3$, $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[i,b] = \frac{1}{2}$ and $\mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[i,c] = \frac{1}{6}$ for $i = 1,3$.

Case (V) (R_1^b, R_{-1}^{ab}). Ex post Pareto optimality implies $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,a] = 0$. Equal treatment of equals implies $\mathfrak{M}(R_1^b, R_{-1}^{ab})[2,\cdot] = \mathfrak{M}(R_1^b, R_{-1}^{ab})[3,\cdot]$. Ordinal strategy-proofness and (II) imply $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,a] + \mathfrak{M}(R_1^b, R_{-1}^{ab})[1,b] = \mathfrak{M}(R^{ab})[1,a] + \mathfrak{M}(R^{ab})[1,b] = \frac{2}{3}$. The unique solution of this system of linear equations is $\mathfrak{M}(R_1^b, R_{-1}^{ab})[i,a] = \frac{1}{2}$, $\mathfrak{M}(R_1^b, R_{-1}^{ab})[i,b] = \frac{1}{6}$, $\mathfrak{M}(R_1^b, R_{-1}^{ab})[i,c] = \frac{1}{3}$ for $i = 2,3$, $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,b] = \frac{2}{3}$ and $\mathfrak{M}(R_1^b, R_{-1}^{ab})[1,c] = \frac{1}{3}$.

Case (VI) $(R_1^b, R_2^{ac}, R_3^{ab})$. By ex post Pareto optimality $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,a]$ and $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2,b]$ equal 0. By ordinal strategyproofness and (IV)

$$\frac{5}{6} = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,a] + \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1,b] = \mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[1,a] + \mathfrak{M}(R_2^{ac}, R_{-2}^{ab})[1,b].$$

By ordinal strategyproofness and (V) $\mathfrak{M}(R_1^b, R_{-1}^{ab})[2,a] = \frac{1}{2} = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2,a]$.

The unique solution of this system of linear equations is $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1, b] = \frac{5}{6}$, $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[1, c] = \frac{1}{6}$, $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2, c] = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[2, a] = \mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[3, a] = \frac{1}{2}$, $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[3, b] = \frac{1}{6}$ and $\mathfrak{M}(R_1^b, R_2^{ac}, R_3^{ab})[3, c] = \frac{1}{3}$. Mutatis mutandis all profiles of preferences are covered by Cases (I) through (VI).

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