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# Nash equilibrium in games with incomplete preferences★

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**Summary.** This paper investigates Nash equilibrium under the possibility that preferences may be incomplete. I characterize the Nash-equilibrium-set of such a game as the union of the Nash-equilibrium-sets of certain derived games with complete preferences. These games with complete preferences can be derived from the original game by a simple linear procedure, provided that preferences admit a concave vector-representation. These theorems extend some results on finite games by Shapley and Aumann. The applicability of the theoretical results is illustrated with examples from oligopolistic theory, where firms are modelled to aim at maximizing *both* profits and sales (and thus have multiple objectives). Mixed strategy and trembling hand perfect equilibria are also discussed.

**Keywords and Phrases:** Incomplete preferences, Nash equilibrium, multi-objective programming, Cournot Equilibrium.

**JEL Classification Numbers:** D11, C72, D43.

## 1 Introduction

The theory of incomplete preferences is an important subfield of decision theory, which is designed to include in its realm statements such as "I don't know if I prefer alternative $a$ or $b$" *in addition* to the statements "I prefer $a$ to $b$," and "I am indifferent between $a$ and $b$". The fundamentals of this theory have been laid out in the seminal contributions of Aumann (1962) and Bewley (1986), and it has been

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pursued further in the recent literature.[1] The importance of this theory becomes even more apparent when one considers the behavior of economic agents made up of collections of individuals (such as coalitions). However, a very large fraction of the work on incomplete preferences concerns only individual choice problems; only few authors have studied strategic interaction between agents with incomplete preferences. The present paper focuses on precisely this issue, and proposes a way to study strategic interactions in which agents sometimes remain indecisive.

An immediate observation is that, since incomplete preferences can leave many options unranked, equilibrium sets in games that allow for incomplete preferences can be considerably larger than in games with complete preferences. While this may at first seem undesirable, we should note that if one picks some plausible, but incorrect, complete preferences to represent the ranking of an indecisive agent, a good range of equilibria may be overlooked. The relevance of this issue becomes even more apparent when considering games that do not have any equilibria. Maybe some of the players' preferences are, in reality, not as complete as the model would have us believe. Incorporating this indecision to the model might, in fact, enlarge the original (empty) equilibrium set, thereby solving the nonexistence problem.[2]

The potential lack of information on the part of the modeler is another reason for modeling the preferences of some players as incomplete. An outside observer might only be able to establish some baseline about an agent's preferences, such as that the agent would prefer a lottery that stochastically dominates another, or that her preferences are single peaked with a certain bliss point. However, it may be difficult to go beyond such a baseline assumption, and ascertain the precise trade-offs an agent would be willing to make. It is a consequence of one of our main results (Theorem 1) that games with incomplete preferences can be used as a tool for *robust* modelling of such strategic situations. It may be useful to assume that agents effectively possess incomplete preferences, in a way to encompass all plausible formulations of their actual preference profiles. With this conservative modelling strategy one would be able to identify those action profiles that arise as an equilibrium for the actual complete preferences of the agents.

After we develop the basic Nash equilibrium theory with incomplete preferences (Sects. 3 and 4), we illustrate the workings of this theory by means of examples from oligopoly theory. Motivated by the long-standing debate on the modelling of the objectives of the firm, we investigate a scenario in which firms have incomplete preferences: we assume that they might not be able to rank all combinations of profits, revenues, sales and possibly other variables. We show that this model allows us to substantially mitigate a well known nonexistence problem of oligopoly theory, the Edgeworth paradox of capacity-constrained Bertrand competition. Using the theory developed in this paper, we are also able to give an upper bound on the set

[1] Among the recent papers that develop a utility theory for incomplete preferences are Dubra, Maccheroni, and Ok (2004), Mandler (2001), Ok (2002), and Sagi (2003). The choice theoretic foundations of incomplete preferences are, on the other hand, examined in Danan (2003), Eliaz and Ok (2004), and Mandler (2004).

[2] Bade (2003) and Roemer (1999, 2001) all tackle the nonexistence problem of the models of multidimensional political competition between two parties by assuming that parties' preferences are incomplete.

of all reasonable Cournot equilibria when firms care not only about their profits but also about revenues and sales. Finally, we apply the assumption of incomplete preferences to the celebrated Kreps-Scheinkman model of oligopolistic competition, and show that this modified model has a pure strategy equilibrium, whereas the equilibrium of the original model involves complicated off-the-equilibrium path mixing.

In passing, we note that most work on games with incomplete preferences focuses only on the problem of the existence of equilibrium (cf. Ding, 2000; Shafer and Sonnenschein, 1975; Yu and Yuan, 1998; and the references cited therein). By contrast, our objective here is to obtain operational characterizations of Nash equilibrium sets of such games. In this sense, our paper is closer in spirit to that of Shapley (1959), who characterizes the set of all mixed strategy Nash equilibria in vector-valued two-player zero-sum games. This characterization has been extended by Aumann (1962) to a larger class of matrix games. In particular, we show here that the set of Nash equilibria of *any* game with incomplete preferences can be characterized in terms of certain derived games with complete preferences. Provided that all players' preferences can be represented by concave functions, we can sharpen this result further; in this case it suffices for the characterization of the equilibrium set to look at games with complete preferences that are derived from the original game by a simple *linear* procedure. We conclude with a discussion of trembling hand perfect equilibria in games with incomplete preferences.

## 2 Preliminaries

Throughout this paper $G = \{(A_i, \succsim_i)_{i\in I}\}$ will denote an arbitrary (normal-form) game. Where $I$ is a (finite) set of players, player $i$'s nonempty action space is denoted by $A_i$ and $\succsim_i$ is player $i'$s preference relation on the outcome space $A := \underset{i\in I}{\times} A_i$. Each preference relation $\succsim_i$ is assumed to be transitive and reflexive but, in contrast to the standard theory, need not be complete. Some player $i$ is *indifferent* between $a$ and $b$, denoted by $a \sim_i b$, if and only if $a \succsim_i b$ and $b \succsim_i a$. Player $i$ *strictly prefers* an outcome $a$ to $b$, denoted by $a \succ_i b$, if and only if $a \succsim_i b$ but not $b \succsim_i a$.

We say that a preference relation $\succsim'$ on $A$ is a *completion* of another preference relation $\succsim$ on $A$, if $\succsim'$ is complete, and if $a \succsim b$ implies $a \succsim' b$ and $a \succ b$ implies $a \succ' b$. We say that a game $G' = \{(A_i, \succsim_i')_{i\in I}\}$ is a *completion* of a game $G = \{(A_i, \succsim_i)_{i\in I}\}$ if $\succsim_i'$ is a completion of $\succsim_i$ for each $i$. A transitive and reflexive relation $\succsim'$ is called a *transitive closure* of a reflexive relation $\succsim$ if $\succsim'$ is the smallest transitive and reflexive relation such that $a \succsim b$ implies $a \succsim' b$, we write $\succsim' = tc(\succsim)$. It is easy to show that $tc(\succsim) = \bigcup_{i=0}^{\infty} \succsim^i$ where $a \succsim^0 b$ if, and only if, $a \succsim b$ and $a \succsim^i b$ (for $i > 0$) if, and only if, there exist $a_1, a_2, ..., a_i$ such that $a \succsim a^1 \succsim ...a^i \succsim b$.

There is a natural way of extending the standard notion of Nash equilibrium to the present framework. An action profile $a = (a_1, ..., a_{|I|})$ is a Nash equilibrium if and only if no agent has an incentive to deviate from her own action given every one else's action. More formally, the profile $a$ is a *Nash equilibrium* if for no player

$i$ there exists an action $a_i' \in A_i$ such that $(a_i', a_{-i}) \succ_i (a_i, a_{-i})$. If each player's preference relation is complete, this definition reduces to the common definition of the Nash equilibrium. In what follows, we denote the set of all Nash equilibria of a game $G$ by $N(G)$.

## 3 A general characterization result

In this section we shall characterize the set of all Nash equilibria of a game $G$ with incomplete preferences as the union of all Nash-equilibrium sets of all completions of $G$. This characterization reduces the problem of identifying Nash equilibria of a game with incomplete preferences to the familiar problem of obtaining Nash equilibria of a collection of games with complete preferences. The proof of the theorem is based on the following Lemma 1 which amounts to a generalization of the classical theorem by Szpilrajn (1930).

*Lemma 1. Let $\succsim$ be a preference relation on some nonempty set $A$, let $B$ be a nonempty subset of $A$, and let $a^*$ be a maximal point of $\succsim$ in $B$. Then there exists a completion $\succsim'$ of $\succsim$ such that $a^*$ is a maximal point of $\succsim'$ in $B$.*

*Proof.* Define the following two partial orders on the quotient set $A/_\sim$: Define $\succsim^q$ by $[a] \succsim^q [b]$ iff $a \succsim b$. Define $\succsim^*$ by $[a] \succsim^* [b]$ iff $[a] = [a^*]$ and there exists a $b'$ in $B$ such that $[b] = [b']$. Define $\succsim^u$ as the union of $\succsim^q$ and $\succsim^*$ and $\succsim^t$ as the transitive closure of $\succsim^u$, $\succsim^t := tc(\succsim^u)$. We need to show that $[a] \succ^q [b]$ implies $[a] \succ^t [b]$. Suppose not, that is, suppose that for some $[a], [b]$ in $A/_\sim$ and some $n \in \mathbb{N}$ we have $[a] \succ^q [b]$ and there exist some distinct $[a^1], [a^2], ..., [a^n] \in A/_\sim$ with $[b] := [a^0] \succ^u [a^1] \succ^u [a^2] \succ^u ... \succ^u [a^n] \succ^u [a] := [a^{n+1}]$, where all inequalities $[a^{i-1}] \succ [a^i]$ are strict, since we have $[a^{i-1}] \neq [a^i]$ for all $i$. Since $\succsim^q$ is transitive, there must be (exactly) one $1 \leq i \leq n+1$ such that $[a^{i-1}] \succ^* [a^i]$. So $[a^{i-1}] = [a^*]$ and $[a^i] = [b']$ for some $b'$ in $B$. Let us rearrange the above chain as $[b'] \succ^q [a^{i+1}] \succ^q .... \succ^q [a] \succ^q [b] \succ^q [a^1] \succ^q ..... \succ^q [a^*]$. The transitivity of $\succsim^q$ implies in turn that $[b'] \succ^q [a^*]$, a contradiction. It follows from Szpilrajn's theorem that there exists a completion of $\succsim^t$, call it $\succsim''$ . By construction $\succsim''$ is also a completion of $\succsim^q$ and $\succsim^*$which implies that also with respect to $\succsim''$, $[a^*]$ is a maximum in the set of all $[b]$ for $b \in B$. Finally define $\succsim'$ on $A$ by $a \succsim' b$ if and only if $[a] \succsim'' [b]$. It is easily checked that $\succsim'$ fulfills our requirements. □

Observe that Lemma 1 applies to sets $A$ with infinitely many elements. If $A$ were finite we would not need Szpilrajn's theorem for the proof. The following fact is now easy to obtain.

**Theorem 1.** *Let $G = \{(A_i, \succsim_i)_{i \in I}\}$ be any game. Then*

$$N(G) = \bigcup \{N(G') : G' \textit{is a completion of } G\}.$$

*Proof.* The "$\supseteq$" part is obvious. To see the "$\subseteq$" part of the claim, let $a^*$ be a Nash equilibrium of $G$, and define $B_i := \{(a_i, a_{-i}^*) : a_i \in A_i\}$ for all players $i$. So for any player $i$, $a^*$ is a maximal point of $\succsim_i$ in $B_i$. By Lemma 1, there exists a completion $\succsim_i'$ of $\succsim_i$ for each player $i$ such that $a^*$ is maximal point of $\succsim_i'$ in $B_i$. Consequently $a^*$ is a Nash equilibrium of the completion $G' = \{(A_i, \succsim_i')_{i \in I}\}$. □

Theorem 1 establishes a strong relation between games with complete preferences and games with incomplete preferences. In particular, it allows us to commute back and forth between games with complete and incomplete preferences when calculating equilibrium sets. The problem of equilibria computation in a game with incomplete preferences is thus reduced to a known problem: the computation of equilibria in games with complete preferences.

But at times it can also be useful to model a situation as a game with incomplete preferences, even though we suspect that the preferences of all agents are complete. This case arises when one does not know the preferences of players precisely. We can then specify the preferences of the players as incomplete preorders, consisting only of the preference statements we feel safe to posit. Theorem 1 then says that the equilibria of any completion of a game must lie within the equilibrium set of the game with incomplete preferences. In other words, non-equilibria are robust under improvements in our knowledge about the preferences of the players. So Theorem 1 on the one hand allows us to simplify the solution of games with incomplete preferences, on the other hand it justifies the use of models with incomplete preferences as tools of robust modelling, when the preferences of the players are not known in detail to the modeler.[3]

## 4 The case of vector-valued utility functions

The preceding characterization theorem relies on the concept of a completion. In general, however, the set of all completions of a game is not easy to determine. To develop a more operational theory we shall now restrict our attention to games in which all players have representable preference relations. Following the recent literature on the representation of incomplete preferences, we shall consider preference relations $\succsim$ that are representable in the sense that there exists a function $u : A \to \mathbb{R}^n$ such that $a \succsim b$ iff $u(a) \geq u(b)$.[4,5] Such vector-valued utility functions are convenient since the problem of maximizing a utility is formally equivalent to the well studied problem of Pareto-optimization. Any $n$-person Pareto-optimization problem can simply be mapped to a problem of maximizing a preference relation that is representable in the $\mathbb{R}^n$ by identifying the utility-vector representing the incomplete preference relation with the vector of all the $n$ persons' utilities. The utility possibility frontier then corresponds to the set of all maximal points of the incomplete preference relation.

[3] This approach presupposes that only the modeller does not know the preferences of the players. If we assume that the players are equally ignorant about the preferences of the other players the robustness result of Theorem 1 breaks down: Given certain priors about the other players, some player might chose an action in equilibrium, that she would never choose when preferences where common knowledge.

[4] See Ok (2002) for an axiomatic treatment of such a vector-valued utility representation.

[5] Notation: For any $n \in \mathbb{N}$ and $a, b \in \mathbb{R}^n$ $a \geq b$ signifies $a_i \geq b_i$ for all $i$; $a > b$ signifies $a \geq b$ but not $b \geq a$. Finally $a \gg b$ iff $a_i > b_i$ for all $i$.

In what follows, by $G = \left\{(A_i, u^i)_{i\in I}\right\}$ we mean the game $G = \{(A_i, \succsim_i)_{i\in I}\}$ where $u^i : A \to \mathbb{R}^{m_i}$ represents $\succsim_i$ in the sense as defined above. For any set of vectors $\beta = \{\beta^1, ..., \beta^{|I|}\}$ with $\beta^i \in \mathbb{R}^{m_i}$ we define the game

$$G_\beta := \left\{(A_i, \beta^i u^i)_{i\in I}\right\}$$

where $\beta^i u^i : A \to \mathbb{R}$ is defined as the dot product of $\beta^i$ and $u^i$,that is $\beta^i u^i := \sum_{j=1}^{m_i} \beta^i_j u^i_j$. To simplify our notation we let

$$\Delta := \left\{\{\beta^1, ..., \beta^{|I|}\} : \beta^i \in \Delta^{m_i} \text{ for all } i\right\} \text{ and } \Delta_+ := \Delta \cap \mathbb{R}^{\Sigma m_i}_{++}$$

where $\Delta^{m_i}$ denotes the $m_i - 1$ dimensional simplex.

Since in this section we are switching back and forth between the game $G = \left\{(A_i, u^i)_{i\in I}\right\}$ and the derived games $G_\beta$, it makes sense to indicate the utility function in the definition of the best response correspondence. So we denote player $i$'s *best response correspondence with respect to his utility function* $u^i$ as $BR_{u^i}$, that is, $BR_{u^i} : A_{-i} \rightrightarrows A_i$ is defined by

$$BR_{u^i}(a_{-i}) := \arg\max_{b_i\in A_i} u^i(b_i, a_{-i}).$$

For any $\beta^i \gg 0$, the function $\beta^i u^i$ represents a completion of the preferences represented by $u^i$. So, for any $\beta \in \Delta_+$, the game $G_\beta$ is a *(linear)* completion of the game $G$. Therefore, by applying Theorem 1 we know that $N(G_\beta)$ is a subset of $N(G)$ for all $\beta \in \Delta_+$. In some cases we can also use such collections of vectors to describe an upper bound on $N(G)$, or even the full set $N(G)$. If we restrict the utility functions of all players to be concave, and all action spaces to be convex, we can derive such an upper bound. The arguments that are commonly being used to defend concavity (such as decreasing marginal utility) also apply to multidimensional utilities. Alternatively, when considering some coalition whose utility is simply the vector of the utilities of its members, the concavity of the coalition's utility can be a consequence of the concavity of the utility functions of its constituents. Following its proof, we will show that Theorem 2 does not extend to quasiconcave utility functions.

**Theorem 2.** *Let $G = \left\{(A_i, u^i)_{i\in I}\right\}$ be a game such that each $A_i$ is a nonempty convex subset of some finite dimensional Euclidean space and each $u^i$ is concave in $a_i$. Then*

$$\bigcup \left\{N(G_\beta) : \beta \in \Delta_+\right\} \subseteq N(G) \subseteq \bigcup \left\{N(G_\beta) : \beta \in \Delta\right\}.$$

*Proof.* The first inclusion is clear from the discussion above. To see the second inclusion, pick any $a \in N(G)$, and fix any $i \in I$. Then we have $a_i \in BR_{u^i}(a_{-i}) = \arg\max_{b_i\in A_i} u^i(b_i, a_{-i})$. Define

$$Y(a_{-i}) := \left\{x \in \mathbb{R}^{m_i} : u^i(b_i, a_{-i}) \geq x \text{ for some } b_i\right\}.$$

and

$$X(u^i(a)) := \{x \in \mathbb{R}^{m_i} : u^i(a) < x\}$$

Observe that both of these sets are convex. To see that $Y(a_{-i})$ is convex pick any $x^i, x^{i\prime} \in Y(a_{-i})$ and any $\lambda \in (0,1)$. Then there exist some $b_i, b_i' \in A_i$ with $x^i \leq u_i(b_i, a_{-i})$ and $x^{i\prime} \leq u_i(b_i', a_{-i})$. While the convexity of all $A_i$ implies $\lambda b_i + (1-\lambda)b_i' \in A_i$, the concavity of all $u^i$ implies $u^i(\lambda b_i + (1-\lambda)b_i', a_{-i}) \geq \lambda x^i + (1-\lambda)x^{i\prime}$. Observe furthermore that by the maximality of $u^i(a)$ in $Y(a_{-i})$, we have $X(u^i(a)) \cap Y(a_{-i}) = \emptyset$. So by Minkowski's separating hyperplane theorem, there exists some vector $p^i \in \mathbb{R}^{m_i}$ and some constant $c$ such that $p^i x \leq c$ for all $x \in Y(a_{-i})$ and $p^i x \geq c$ for all $x \in X(u^i(a))$. Since $x < u^i(a)$ implies $x \in Y(a_{-i})$ and $x > u^i(a)$ implies $x \in X(u^i(a))$, we have $p^i \geq 0$ and $p^i u^i(a) = c$. So there exists some $\beta^i \in \Delta^{m_i}$ such that $\beta^i u^i(a) \geq \beta^i x$ for all $x \in Y(a_{-i})$. Since $u^i(A_i, a_{-i})$ a subset of $Y(a_{-i})$, we also have that $\beta^i u^i(a) \geq \beta^i u^i(b_i, a_{-i})$ which implies that $a_i \in BR_{\beta^i u^i}(a_{-i})$. Since $i \in I$ is arbitrary, this yields a $\beta$ in $\Delta$ such that $a \in N(G_\beta)$. □

The proof of Theorem 2 does not extend to quasiconcave functions, as the sum of two quasiconcave functions is not necessarily itself quasiconcave. Take the following trivial one-player game with a convex action space $A = \{(x,y) \in [0,1]^2 : x + y = 1\}$ and a quasiconcave utility defined by $u(x,y) = (x^2, y^2)$. While $(\frac{1}{2}, \frac{1}{2})$ is an equilibrium in this game, there does not exist any $\beta \in \Delta$ such that $(\frac{1}{2}, \frac{1}{2}) \in \arg\max_{x,y \in A} \beta u$.

Theorem 2 gives an upper and a lower bound on the set of all Nash equilibria. In applications we would not expect that there would be many elements in $\bigcup \{N(G_\beta) : \beta \in \Delta\}$ that are not contained in $\bigcup \{N(G_\beta) : \beta \in \Delta_+\}$, that is, the bulk of $N(G)$ is likely to be contained in $\bigcup \{N(G_\beta) : \beta \in \Delta_+\}$. In particular, if we assume componentwise strict concavity - as would be for example reasonable when we investigate players that are made up of individuals that each have strictly concave utility functions - the characterization at hand provides one with a full description of the set of all Nash equilibria of a game. This claim is proved next.

**Lemma 2.** *Let $G = \{(A_i, u^i)_{i \in I}\}$ be a game such that, for some $i$, $A_i$ is a convex subset of some finite dimensional Euclidean space and $u^i$ is concave in $a_i$. For some component $j \in \{1, ..., m_i\}$ let $u_j^i : A \to R$ be strictly concave in $a_i$. Then for any $\beta^i \in \Delta$ such that $\beta_j^i > 0$ and any $a_{-i} \in A_{-i}$ we have $BR_{\beta^i u^i}(a_{-i}) \subseteq BR_{u^i}(a_{-i},)$.*

*Proof.* Fix any $a_{-i}$ and any $\beta^i \in \Delta$ such that $\beta_j^i > 0$. Assume that there exist an $a_i'$ and some $\beta^i \geq 0$ with $\beta_j^i > 0$ such that $a_i' \in BR_{\beta^i u^i}(a_{-i})$ but $a_i' \notin BR_{u^i}(a_{-i})$. This implies that there exists an $a_i'' \in A_i$ such that $u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$. It follows that $\beta^i u_i(a_i'', a_{-i}) \geq \beta^i u_i(a_i', a_{-i})$, and since $a_i' \in BR_{\beta^i u^i}(a_{-i}, A_i)$, we have $\beta^i u_i(a_i'', a_{-i}) = \beta^i u_i(a_i', a_{-i})$. The strict concavity of $u_j^i(., a_{-i})$ and positivity of $\beta_j^i$ together with the concavity of $u^i(., a_{-i})$ imply that $\beta^i u^i(., a_{-i})$ is a strictly concave function. Since a strictly concave function is maximized at a unique point, we conclude that $a_i'' = a_i'$ contradicting our assumption that $u^i(a_i'', a_{-i}) > u^i(a_i', a_{-i})$. □

**Theorem 3.** *Let $G = \{(A_i, u^i)_{i \in I}\}$ be a game such that each $A_i$ is a nonempty convex subset of a finite dimensional Euclidean space and every component of each $u^i$ is strictly concave in $a_i$. Then $N(G) = \bigcup \{N(G_\beta) : \beta \in \Delta\}$.*

*Proof.* Since every component of each player's utility functions is strictly concave in her own action, we know that

$$\Delta{=}\{\beta{\in}\Delta\text{: for all } i, \beta^i_k{>}0 \text{ for at least one } k \text{ for which } u^i_k \text{ is strictly concave}\}.$$

So combining Lemma 2 and Theorem 2, we have

$$\bigcup\left\{BR_{\beta^i u^i}(a_{-i}) : \beta^i \in \Delta\right\} \subseteq BR_{u^i}(a_{-i}) \subseteq \bigcup\left\{BR_{\beta^i u^i}(a_{-i}) : \beta^i \in \Delta\right\}.$$

It follows that $N(G) = \bigcup\{N(G_\beta) : \beta \in \Delta\}$. □

## 5 Applications to oligopoly theory

We now illustrate the theory developed so far by studying how to incorporate multiple objectives in some standard models of oligopolistic competition. The following alternatives to profits have been suggested as objectives for the oligopolistic firm. Firms might concentrate on maximizing revenues or sales, possibly as imperfect proxies for long run profits. Due to the difficulty in evaluating managerial efforts, the executives of a firm might be judged according to the relative performance of the firm, and this might compel managers to focus on the market share in terms of profits, sales and revenues which suggests another set of possible objectives of the firm.[6]

At the very least, it seems worthwhile to explore the implications of the hypothesis that objectives other than profits play a role in the firms decision making, when profits are above a certain threshold (e.g. nonnegative). We model the preferences of firms such that they depend only on profits and sales[7]. More precisely, we investigate the following preference structure on the part of the firms: When making profits, a firm prefers a situation $a$ to a situation $b$ if in $a$ it has at least as much profit and sales as in $b$. If, however, situation $a$ is better according to either one of the criteria, while $b$ is better according to the other criterion, then the firm is undecided between these two options. If the firm is making losses in situation $a$ or $b$ it prefers the one with the lower losses (or equivalently higher profits) no matter how these two situations compare according to the sales of the firm. For ease of presentation, we focus in what follows on duopolies. We assume that both firms produce a homogenous good at constant marginal cost $c > 0$, and that at a price $p$ the market demand is $1 + c - p$ as long as this expression is positive, otherwise market demand is 0. By convention, $i$ and $j$ denote two *different* firms in what follows.

[6] See, for instance, Baumol (1959), Fershtman and Judd (1987), Galbraith (1967), Holmstrom (1982), Marris (1964), Simon (1964), and Sklivas (1987) for arguments in favor of modelling firms as pursuing objectives that deviate from profit maximization.

[7] Revenues and market shares can w.l.o.g. be dropped from consideration as they are monotone transformations of profits and sales (at least as long as profits are positive).

*5.1 Cournot competition*

Consider the Cournot model in which firms choose their production levels. Here $A_i = \mathbb{R}_+$, and the utility function of firm $i$ is defined on $\mathbb{R}^2_+$ by

$$u^i(q) := (\pi^i(q), v^i(q)),$$

where $\pi^i$ denotes the common profit function, and $v^i(q) := q_i$ if $q_i \leq 1 - q_j$ and $v^i(q) := 1 - q_j$ otherwise. Here $v^i(q)$ represents the sales of firm $i$ as long as profits are nonnegative. If firm $i$ incurs losses at the output profile $q$, then $v^i(q)$ takes a constant value; the particular value of this constant, $1 - q_j$, is chosen to obtain a utility function that is concave and continuous in the firm's own action $q_i$. We denote the resulting game $\left\{(A_i, u^i)_{i=1,2}\right\}$ by $G^C$.[8]

We now compute $N(G^C)$. Since each firm's objective function $u^i(q)$ is concave in the firm's own quantity, and since the first component $\pi^i(q)$ is even strictly concave Theorem 2 and Lemma 2 readily yield

$$\bigcup \left\{N(G^C_\beta) : \beta \in \Delta, \beta^i_1 > 0\right\} \subseteq N(G^C) \subseteq \bigcup \left\{N(G^C_\beta) : \beta \in \Delta\right\}. \quad (*)$$

In the linearly completed game $G^C_\beta$, firm $i$'s objective function is $\beta^i u^i$, where

$$\beta^i u^i(q) = \begin{cases} \beta^i_1 \pi^i(q) + \beta^i_2 q_i, & \text{if } q_i \leq 1 - q_j \\ \beta^i_1 \pi^i(q) - \beta^i_2 (1 - q_j), & \text{if } q_i > 1 - q_j \end{cases}.$$

So, in the completed game $G^C_\beta$, firm $i$'s best response to $q_j$ is: $\frac{1-q_j}{2} + \frac{\beta^i_2}{2\beta^i_1}$ if this expression is in the interval $[0, 1 - q_j]$, if this expression is smaller than any value in this interval then the best response is not to sell anything, otherwise firm $i$'s best response is $1 - q_j$. This implies that

$$\bigcup \left\{N(G^C_\beta) : \beta \in \Delta, \beta^i_1 > 0\right\} = \left\{q \in \mathbb{R}^2_+ : \frac{1}{2}(1 - q_j) \leq q_i \leq 1 - q_j \text{ for } i = 1, 2\right\}$$
$$= \bigcup \left\{N(G^C_\beta) : \beta \in \Delta\right\}.$$

Combining this with (*), we conclude that

$$N(G^C) = \left\{q \in \mathbb{R}^2_+ : \frac{1}{2}(1 - q_j) \leq q_i \leq 1 - q_j \text{ for } i = 1, 2\right\}.$$

Figure 1 illustrates this analysis: the lines $AB$ and $CD$ represent the reaction curves when firms maximize profits. The line $AD$ represents the reaction curves of the two firms when they maximize sales subject to nonnegativity of profits. All points in the triangle $ADE$ represent Nash equilibria of $G^C$. For any point $q$ in this triangle, there exists a $\beta$ such that $q \in N(G^C_\beta)$. The line $AGH$ represents reaction

[8] A range of alternative formulations of this game $\left\{(A_i, v^i)_{i\in I}\right\}$, with $v^i$ being a monotone transformation of $u^i$ for $i = 1, 2$, yield the same set of Nash equilibria. The present formulation in convenient, for it allows Theorem 2 and Lemma 2 to be applied. Finally, observe that profits play an important role in the preferences of the firm, as they enter $u^i$ both directly and also indirectly (via sales).

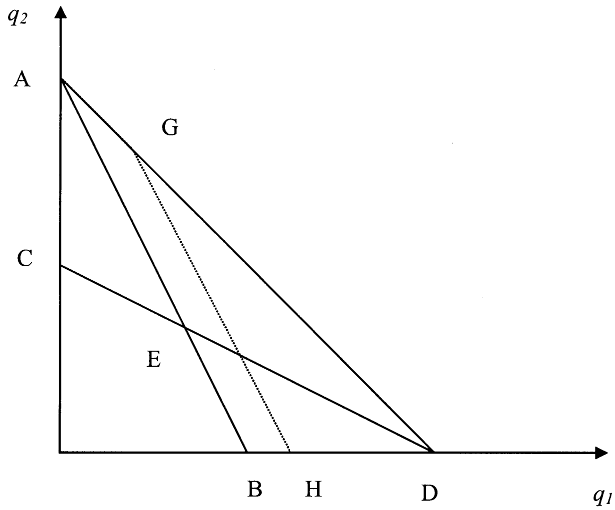


**Figure 1**

curve of firm 1 for its completed utility $\beta^1 u^1$. For small $q_2$, firm 1 best-responds with $\frac{1-q_2}{2} + \frac{\beta_2^i}{2\beta_1^i}$; observe that this line (the portion $GH$ of the line $AGH$) has the same slope as the firm's reaction curve according to the traditional model (the line $AB$). For large $q_2$, the response with a quantity $q_1 = \frac{1-q_2}{2} + \frac{\beta_2^i}{2\beta_1^i}$ would yield negative profits and firm 1 is best off by setting its quantity to $1 - q_2$; this explains the portion $AG$ of its reaction curve. Observe that if more weight is placed on profits in the completion, the line $AGH$ would shift closer to the line $AB$.[9]

If we assume that the objective function of a firm should be some positive combination of their revenues, sales, and profits (as long as the firm does not make any losses, and if it does, it only tries try to reduce them), we know by Theorem 1 that the equilibrium outcomes of the Cournot model must lie within this triangle. This application of Theorem 1 helps us to exclude a wide range of action profiles that cannot be equilibria for any reasonable complete formulation of the game.

### *5.2 Bertrand competition*

To show that Nash equilibrium sets in games with incomplete preferences need not be large (as opposed to what the previous example might suggest), we now study the above multi-objective model within Bertrand competition. We again assume that firms have incomplete preferences as long as profits are nonnegative, preferring higher sales and profits. The unique Nash equilibrium in this case accords perfectly with the standard case: the firms set their prices equal to marginal cost. This is not

[9] Notice that in this case $N(G^C)$ coincides with the convex hull of the Nash equilibria in the "extreme" games $G^C_{\pi^1,\pi^2}$, $G^C_{\pi^1,v^2}$, $G^C_{v^1,\pi^2}$ and $G^C_{v^1,v^2}$ (where $G^C_{f^1,f^2}$ denotes the game in which firm $i$'s utility is the function $f^i$). This is a peculiar consequence of the fact that best response correspondences in the "extreme" games are linear. The observation does not generalize.

surprising since no sales or revenue motive could give a firm an incentive to deviate from the unique equilibrium of the price competition game played amongst profit maximizing duopolists. The uniqueness of this equilibrium is also not unexpected since in all relevant cases profits and sales move in the same direction as the own price of a firm changes. This can be interpreted as a confirmation of the robustness of the Bertrand equilibrium: By Theorem 1, any completion of the present Bertrand model either has no equilibrium or its only equilibrium is the classical one. To deviate from the classical equilibrium firms must have preferences that cannot be represented as increasing combinations of their profits, revenues and/or sales.

There is, however, more to this story. Edgeworth (1925) showed in his famous critique of Bertrand-competition that games played amongst capacity constrained price setters can fail to have any Nash equilibria. In the literature on oligopolistic competition, this observation is called the *Edgeworth paradox.*[10] Interestingly, this paradox does not arise in the present multi-objective Bertrand-model. To demonstrate, let both firms face some identical capacity constraint $K \leq \frac{1}{2}$. As usual, we assume that the firm with the lower price serves the customers that are willing to pay most for the good, the firm with the higher price serves the rest if there is any. More precisely, given prices $p_i$ and $p_j$, the demand is shared in the following way. If $p_i < p_j$, then firm $i$ faces the full market demand, that is, $D^i(p_i, p_j) = 1+c-p_i$, where $D^i(p_i, p_j)$ denotes the demand for firm $i$ given the prices $p_i$ and $p_j$. If $p_i > p_j$ and $K \geq 1+c-p_j$, then no residual demand remains for firm $i$: $D^i(p_i, p_j) = 0$. If, on the other hand, $1+c-p_j > K$ (and still $p_i > p_j$) firm $i$ faces a residual demand of $D^i(p_i, p_j) = 1 + c - p_i - K$. If both firms set an equal price $p_i = p_j = p$ they share the market-demand equally: $D^i(p_i, p_j) = \frac{1+c-p}{2}$. The duopolists play the game $G^B = \left\{(A_i, u^i)_{i=1,2}\right\}$, where $A_i = \mathbb{R}_+$ is the price space, and

$$u^i := (\min\left\{D^i(p_i, p_j), K\right\}(p_i - c), v^i(p_i, p_j))$$

with $v^i(p_i, p_j) := \min\left\{D^i(p_i, p_j), K\right\}$ as long as $p_i \geq c$ and $v^i(p_i, p_j) := -1$ otherwise. Observe that these preferences over price profiles are not convex, so we cannot apply Theorem 2.

The main result of this section is that there is a unique equilibrium in this Bertrand-game:

$$N(G^B) = \{(1 - 2K + c, 1 - 2K + c)\}. \tag{**}$$

Let us first show that in any Nash equilibrium we have $p_i = p_j$. Suppose $p_i < p_j$ in equilibrium. If $1 - p_i + c \leq K$, then $D^j(p_i p_j) = 0$, so by dropping its price to $p_i$ firm $j$ could get positive profits and sales. If, on the other hand, $1 - p_i + c > K$, then firm $i$ sells $K$ units of the good on the market, whereas it could also sell $K$ at any higher price $p_i'$ for which $p_i' < p_j$ and $1 - p_i' + c > K$. Charging such a price $p_i'$ firm $i$ could increase its profits while keeping its quantity sold constant. So, in any equilibrium, both firms must charge the same price, say $p$.

[10] Maskin (1986) shows that for a very large range of such capacity constrained Betrand games mixed strategy equilibria exist. However, such mixed strategy equilibria are generally very difficult to calculate. Moreover, the supports of the firms' equilibrium strategies tend to be extremly large (see, for example, Osborne and Pitchik, 1986).

If $p < 1 - 2K + c$, then either firm could increase its profits, while keeping its quantity sold constant, by increasing its price by a small amount. If $p > 1-2K+c$, then there exists some $p'_i < p$ at which firm $i$ sells a higher quantity yielding higher profits, given that firm $j$ continues to play $p$. Thus, *if an equilibrium exists*, it must equal $(1 - 2K + c, 1 - 2K + c)$.

What is more, any deviation of a firm either lowers this firm's profits or its quantity sold. The essential difference between the classical model and the model advanced here is this last step. Certain deviations from $(1-2K+c, 1-2K+c)$ may raise profits, but only at the expense of sales. Under the classical profit maximization hypothesis these deviations are of course beneficial, and lead to the non-existence of equilibrium. Since sales decrease as a consequence, these deviations are, however, not beneficial in our multi-objective model: (**) holds. One can similarly show that the equilibrium of the model is $(c, c)$ when $K > \frac{1}{2}$ Thus if we denote the present Bertrand model with capacity constraint $K$ by $G^B(K)$, then we have $N(G^B(K)) = (p_K^*, p_K^*)$ where $p_K^* := \max\{1 - 2K + c, c\}, K \geq 0$.[11]

It is important to note that this result is robust in the sense that it remains valid even when a certain level of trade-off between profits and sales is allowed. To see this, instead of assuming that firms cannot rank any two price profiles, one with higher sales and the other with higher (positive) profits, let us change the firms utilities to

$$u_\kappa^i := (\pi^i, \kappa\pi^i + v^i) \qquad i = 1, 2,$$

where $\kappa > 0$. According to this alternative model, a change that considerably increases profits, while decreasing sales only to a small extent, is preferred by the firm if $\kappa$ is big enough.

For instance, take the profit and sales profile $(\pi^*, v^*)$ in Figure 2. In our initial formulation of the preferences, a firm prefers only those profiles $(\pi, v)$,for which we have $(\pi, v) \geq (\pi^*, v^*)$ holds. According to our new formulation, the objective profile of firm $i$ is $u_\kappa^i := (\pi^i, \kappa\pi^i + v^i)$ that is firm $i$ strictly prefers any $(\pi, v)$ in Figure 2 that lies left of the line $\pi^*$ and above the line $AB$ (the line through $(\pi^*, v^*)$ with slope $-\kappa$). Observe that the cone of options that cannot be ranked according to $u_\kappa^i(p)$ (the striped areas in Figure 2) decreases with $\kappa$, in the limit as $\kappa \to \infty$ the firms preferences converge to the standard scenario. Let us simplify the analysis by assuming $c = 0$ (all the results reported below hold for any $c > 0$ as well), and let us find an upper bound on $\kappa$ such that a firm with the utility function $u_\kappa^i(p)$ has no incentive to deviate from the equilibrium of $G^B(K)$. We restrict our attention to all linear completions of $u_\kappa^i(p)$, that is, all functions $\gamma\pi^i(p) + v^i(p)$ with $\gamma \in (\kappa, \infty)$. We distinguish three cases in which $K$ belongs to $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{1}{2}]$ or $(\frac{1}{2}, \infty)$.

If $K \leq \frac{1}{3}$, any deviation from the equilibrium price $p_K^* := \max\{1 - 2K + c, c\}$ lowers profits and sales at the same time. If $\frac{1}{3} < K \leq \frac{1}{2}$, then there does not exist a preferred deviation from $p_K^*$, if for all $p > 1 - 2K$, we have $\gamma(1-2K)K + 1 - 2K \geq \gamma(1 - K - p)p + 1 - K - p$. The latter inequality holds

[11] The argument establishing the unique equilibrium for multi-objective competition easily extends to the case of different capacities $K_1, K_2 > 0$, any strictly decreasing continuous demand $D : \mathbb{R}_+ \to \mathbb{R}_+$, and different sharing rules. The unique equilibrium is $(p^*, p^*)$ with $p^* = \max\{c, D^{-1}(K_1 + K_2)\}$.

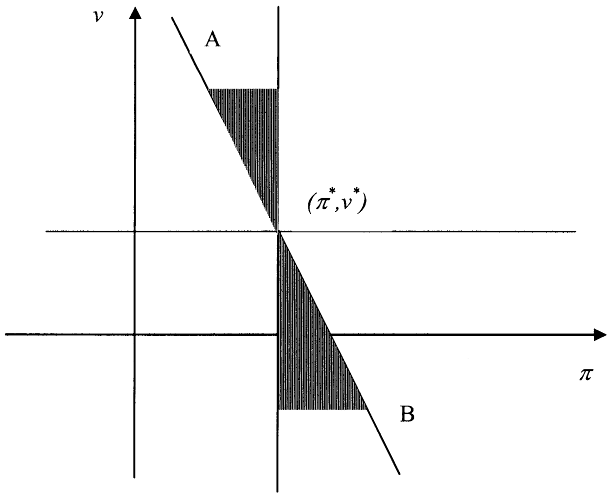


**Figure 2**

only if $\frac{1}{3K-1} \geq \gamma$. So if $2 \geq \gamma$, then there does not exist any profitable deviation from $(p^*_K, p^*_K)$ for any $K \leq \frac{1}{2}$. Let us now show that if we impose $2 \geq \gamma$ there also does not exist any preferred deviation from $(p^*_K, p^*_K) = (0, 0)$ for $K > \frac{1}{2}$. Given that the other firm charges $p_j = 0$ the set of feasible profit-sales for firm $i$ is smaller when $K > \frac{1}{2}$ than it is when $K = \frac{1}{2}$. We just showed that, given $\gamma \leq 2$, the profit sales profile $(0, \frac{1}{2})$ is maximal in the larger set when $\gamma \leq 2$. But $(0, \frac{1}{2})$ is also contained (and therefore maximal) in the smaller set, so $p_i = 0$ is a best response to $p_j = 0$, when $K > \frac{1}{2}$. We conclude that if $\gamma \in (0, 2]$, $(p^*_K, p^*_K)$ remains an equilibrium of $G^B(K)$ for *any* $K \geq 0$. In other words $(p^*_K, p^*_K)$ is the unique equilibrium of the present multi-objective Bertrand game with capacity constraint $K \geq 0$ as long as both firms assign a "weight" lower than $\frac{2}{3}$ to profits.[12]

## *5.3 The Kreps-Scheinkman model*

There exists a marked tension between the models of Cournot and Bertrand competition. While the mechanism assumed in the Bertrand setup (i.e. price competition), has more empirical support, the prediction of the Cournot model (i.e. oligopolistic rents exist), seems more realistic. In a famous article, Kreps and Scheinkman (1983) reconcile these conflicting intuitions in a sequential setup, where they provide a model in which firms reap oligopolistic rents even though they compete in prices. In this section we introduce the incomplete preferences discussed in the previous two subsections into the Kreps-Scheinkman model. We show that the Kreps-Scheinkman result is robust to this modification. There is, however, a considerable

[12] If we allow the capacites of both firms to differ, then we have

$$N(G^B(K_1, K_2)) = \{(\max\{1 - K_1 - K_2 + c, c\}, \max\{1 - K_1 - K_2 + c, c\})\}$$

is a Nash equilibrium provided that the weight on profits is less than one half (that is $\kappa < 1$).

advantage of the present approach: While some complicated off-the-equilibrium path mixing is necessary to solve the original game of Kreps and Scheinkman (1983) the incomplete preferences version of the model has straightforward pure strategy equilibria.

In the first stage of the game the two competing firms simultaneously build their capacities, incurring a cost $c$ for each unit. Subsequently, each firm can produce a homogenous good up to its capacity level at zero marginal cost. Having observed the capacities both firms compete in the market by setting prices. The market demand is $1 + c - p$ as long as this expression is positive and 0 otherwise. The firms share the demand as explained above in the discussion of Bertrand competition. The only difference between this model and that of Kreps and Scheinkman (1983) is that the firms are not only motivated by profits but also by sales and revenues; they have the same type of incomplete preferences as discussed above. We denote the resulting game as $G^{KS}$, and a subgame that obtains after firm $i$ chooses capacity $K_i$, by $G^{KS}(K_1, K_2)$. A strategy of firm $i = 1, 2$ in this game consists of a capacity $K_i \geq 0$ and a function $f_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ that specifies a price for every $G^{KS}(K_1, K_2)$. As is standard, we call a strategy profile a *subgame perfect* Nash equilibrium, if it induces a Nash equilibrium in every subgame.

**Proposition 1.** *The set of all subgame perfect Nash equilibria of $G^{KS}$ consists of all $(K_1, f_1, K_2, f_2)$ such that*

$$\frac{1}{2}(1 - K_j) \leq K_i \leq 1 - K_j \quad \textit{and} \quad f_i(K_i, K_j) = \max\{1 - K_i - K_j + c, c\},\ i, j = 1, 2,\ j \neq i.$$

Before proving this proposition we need to remark that in games with incomplete preferences there can be action profiles that survive backward induction even though they are not subgame perfect Nash equilibria. For instance, consider the game presented in Figure 3.

In this game player 1 has two objectives (and hence incomplete preferences) while player 2 has complete preferences. Observe that the strategy profile $raR$ survives backward induction, but that $raR$ is not a Nash equilibrium, for player 1 is better off playing $lL$. Since the focus of the present paper is on static games, we will not pursue this matter further here. Suffice it to say that Proposition 1 cannot be proved by backward induction; we need to check directly if the strategy profiles in Proposition 1 induce Nash equilibria in every subgame.[13]

*Proof of Proposition 1.* Let us first show that indeed all action profiles named in Proposition 1 are subgame perfect Nash equilibria. Pick any $(K_1, f_1, K_2, f_2)$ that satisfies the conditions given in Proposition 1. We need to show that by deviating from its choice of $K_1, f_1$ firm 1 can neither raise sales nor profits without lowering the other for the given $K_2, f_2$. Suppose an alternative strategy $K', f'$ that raises sales or profits while keeping the other at least constant existed. Since

[13] The game in discussion actually belongs to a certain class of games with incomplete preferences in which backward induction can be used to solve for subgame perfect Nash equilibria, to define this class and give the proof would however go beyond the scope of this paper. Questions like this one will be dealt with in a companion paper on dynamic games.

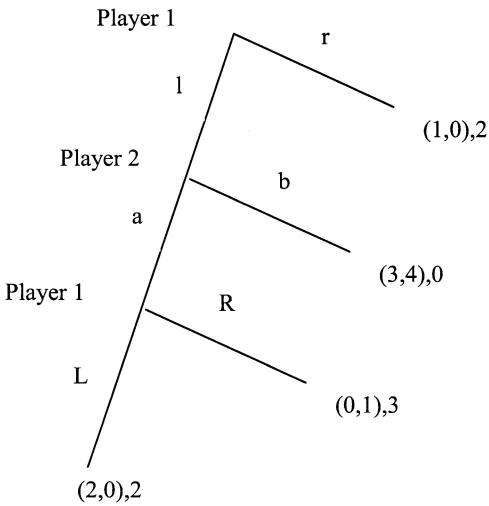


**Figure 3**

$v^1(K_1, f_1, K_2, f_2) = K_1$[14] we need to have $K' \geq K_1$. Keeping firm 2's strategy fixed this again implies that $f_2(K', K_2) = \max(c, 1 - K' - K_2 + c) \leq 1 - K_1 - K_2 + c = f_2(K_1, K_2)$ or in words firm 2 will not raise its price given firm 1's altered strategy. We proceed by distinguishing three cases: The price firm 1 would charge in subgame $G^{KS}(K', K_2) : f'(K', K_2)$ might be higher equal or lower than $f^1(K_1, K_2)$, the price firm 1 charges according to the given strategy profile $(K_1, f_1, K_2, f_2)$.

*Case 1.* $f'(K', K_2) > f^1(K_1, K_2)$. Since $f^1(K_1, K_2) = f^2(K_1, K_2) \geq f_2(K', K_2)$ firm 1 charges a higher price than firm 2 and only faces the residual demand $1 - K_2 - f'(K^{'}, K_2) < K_1$. So under this deviation sales decrease and we can rule out such a deviation. *Case 2*: If $f'(K', K_2) = f^1(K_1, K_2)$ the firm spends more on building capacity while selling the same quantity, so we can rule out this case. *Case 3*: Observe that if $f'(K', K_2) < f^1(K_1, K_2)$ for all deviations $f', K'$ there exists a deviation $f'', K''$ that is at least as good for firm 1 and has $f''(K'', K_2) = 1 - K'' - K_2 + c$ (if this equation does not already hold for $f', K'$ price can either be raised or capacity reduced to increase profits while keeping the quantity sold constant). But this means we need to find some $K'' \leq 1 - K_2$ such that $(K'', (1 - K'' - K_2)K'') > (K_1, (1 - K_1 - K_2)K_1)$. In our discussion of Cournot equilibria we saw that no such deviation exists. So we conclude that indeed all strategy profiles named in Proposition 1 are Nash equilibria. We can draw the argument that any of these strategy profiles induces a Nash equilibrium in any proper subgame $G^{KS}(K_1, K_2)$ from our discussion of Bertrand competition.[15] So we conclude that indeed all the strategy profiles defined in Proposition 1 subgame perfect equilibria.

[14] Observe the slight change in notation: the functions $v^i$ and $\pi^i$ now map vectors of two real variables $(K_1, K_2)$ and two functions $f_1, f_2$ to the reals.

[15] This was actually only shown for $K_1 = K_2 \in [0, 1]$; the generalization needed here is easy.

We only need to check that we did not overlook any subgame perfect Nash equilibria. First, again by our calculation of Bertrand equilibria we do know that in any proper subgame $G^{KS}(K_1, K_2)$ there is a *unique* Nash equilibrium$\{(p_1, p_2) : p_i = \max\{1 - K_i - K_j, c\}\}$. It follows that for any alternative candidate of a subgame perfect Nash equilibrium $(K_1, f_1, K_2, f_2)$ we must have that $f_i(K_i, K_j) = \max\{1 - K_i - K_j + c, c\}$. Taking this into account the firms' utility vectors reduce to the same payoff vectors we had specified in the Cournot game. And we saw above that $\left\{(K_1, K_2) : \frac{1}{2}(1 - K_j) \leq K_i \leq 1 - K_j\right\}$ is the set of Cournot equilibria. So indeed Proposition 1 describes the set of all Nash equilibria. □

Proposition 1 shows that the Kreps-Scheinkman compromise between quantity and price competition translates to the case where firms have multiple objectives, profits and sales. As in the case of firms motivated only by profits, the *outcomes* of the Cournot model of Section 5.1. and the two stage game of capacity and price setting are equivalent given the present variation of the firms objectives. One advantage of the incomplete preference formulation of the Kreps-Scheinkman model is that it has a pure strategy equilibrium, while the solution of the original model depends on some rather complex mixing off-the-equilibrium path.

### *5.4 Owners and mangers in the Kreps-Scheinkman model*

Following the literature on the objectives of the firm more closely we can actually single out one of all these equilibrium outcomes, the classical Cournot equilibrium. We investigate the option that the quantity and the price of a firm need not be set by the same agent. They might be set by different agents with different preferences. Actually, a large part of the literature on the goals of firms focuses on the possible differences between the goals of owners and managers. It is generally held that owners wish to maximize profits, while there remains much larger debate around the goals of managers. It is furthermore typically assumed that owners set up the game for managers which then make the day to day decisions. Following this tradition, it seems warranted that we introduce two different agents in the above formulation of the Kreps-Scheinkman model. We assume that capacities are set by profit maximizing owners while managers with incomplete preferences over profits and sales choose prices.[16] It is easy to see that in this modified game the set of subgame perfect Nash equilibria reduces to

$$\left\{(K_1, f_1, K_2, f_2) : K_i = \frac{1}{3} \text{ and } f_i(K_i, K_j) = \max\{1 - K_i - K_j + c, c\},\ i, j = 1, 2,\ j \neq i\right\}.$$

[16] In this model we ignore the contractual stage of the game. This should be more carefully adressed elsewhere where strategic delegation matters would be discussed.

The Nash equilibrium outcome is then found as:

$$(K_1, f_1(K_1, K_2), K_2, f_2(K_1, K_2)) = \left(\frac{1}{3}, \frac{1}{3} + c, \frac{1}{3}, \frac{1}{3} + c\right),$$

which is also the Nash equilibrium outcome in the original Kreps-Scheinkman model and of the classical Cournot model. So, we obtain the same prediction as Kreps and Scheinkman (1983), albeit in a slightly different setup. By introducing two different decisionmakers for the two stages of the game and by assuming that the second decision maker, the manager, has incomplete preferences (in the sense of having two objectives, profits and sales), we find that the *unique* pure strategy subgame perfect equilibrium of the resulting game induces the classical Cournot equilibrium outcome.[17]

## 6 Remarks on the existence of equilibrium

A few remarks on the existence of equilibria for games with incomplete preferences are in order. In this section we compare and discuss the two prevalent approaches in the literature for establishing conditions under which games with incomplete preferences have equilibria. We start by the observation that given Theorem 1, we can simply import existence results for games with complete preferences to the theory of games with incomplete preferences. If we can show that some completion of a game has an equilibrium, then the game itself has an equilibrium. With representable preferences, linear completions provide a natural starting point, since they are easy to calculate. And indeed this technique is applied in the literature on multicriteria decision-making to establish conditions under which equilibria exist.[18]

To obtain completions that are covered by some suitable fixed point theorem, these papers assume in general that there exists some $\beta \in \Delta_+$ such that, for each player $i$, the function $\beta^i u^i$ is quasiconcave in her own action. This condition guarantees that, in the completion $G_\beta$, the best response correspondence of each player is convex-valued, which again makes standard fixed point arguments applicable. In general, however, this condition is not easy to verify, for the set of quasiconcave functions is not closed under addition, and hence it does not suffice to check that every component of the utility of each player is quasiconcave in her own action. A more restrictive but substantially more operational requirement is, on the other hand, that all component utilities $u^i$ are concave in the players' own action, for then $\beta^i u^i$ must be concave (and therefore quasiconcave) in the players' own action for all $\beta^i \gg 0$.

We should also note that there is an alternative approach to establish existence of equilibria. Shafer and Sonnenschein (1975), for instance, focus directly on games

[17] Our prior discussion on the extent of incompleteness of a managers preferences that is necessary for $(\max\{1 - 2K + c, c\}, \max\{1 - 2K + c, c\})$ to be an equilibrium, applies also here. Again, it is not necessary that managers cannot rank *any* two price profiles such that one yields the firm higher sales the other higher profits.

[18] Ding (2000), Wang (1991), and Yu and Yuan (1998) among others apply this technique in their existence proofs.

with incomplete preferences in which the preferences of each player are convex in her own action.[19] Below we illustrate the difference between the two approaches to the existence of equilibrium by means of two examples. The first is an example of a game to which the Shafer-Sonnenschein existence theorem applies, while no linear completion of this game has a Nash equilibrium. On the other hand, the Shafer-Sonnenschein existence theorem does not cover the game considered in the second example, but we can establish the existence of an equilibrium using the linear completion method.

*Example 1.* Consider a two player game $G = \{(A_i, u_i)_{i=1,2}\}$ where $A_i := \{x \in [0,1]^2 : x_1 + x_2 \leq 1\}$ for $i = 1, 2$, and

$$u^1(x,y) := \begin{cases} \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, & \text{if } y_1 = \frac{1}{2} \\ \begin{pmatrix} -|\frac{1}{2} - x_1| \\ 0 \end{pmatrix}, & \text{otherwise,} \end{cases}$$

$$u^2(x,y) := \begin{cases} \begin{pmatrix} -|\frac{1}{2} - y_1| \\ 0 \end{pmatrix}, & \text{if } x_1 = \frac{1}{2} \\ \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Clearly both $u^1$ and $u^2$ are continuous and quasiconcave in the respective own action, and $A_1$ and $A_2$ are both convex and compact, so by the Shafer-Sonnenschein existence theorem there exists an equilibrium of $G$. (Indeed, $((.3, .7), (\frac{1}{2}\frac{1}{2}))$ constitutes a Nash equilibrium.) Observe, however, that for no $\beta \in \Delta_+$ is $\beta^1 u^1$ quasiconcave in $x$ and we even have that $N(G_\beta)$ is empty for all $\beta \in \Delta_+$.[20]

*Example 2.* Take a two player game $G = \{(A_i, u_i)_{i=1,2}\}$ with $A_1 := [0,1]^2$, $A_2 := [0,1]$,

$$u^1(x,y) := \begin{pmatrix} x_1^2 + x_2^2 + y \\ 2x_1x_2 \end{pmatrix} \quad \text{and} \quad u^2(x,y) := |x_1 + x_2 - y|.$$

Observe that $u^1$ is not quasiconcave in $x$. Holding player 2's action fixed observe that player 1's utility from choosing either $(1,0)$ or $(0,1)$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, whereas the utility of playing the linear combination $(\frac{1}{2}, \frac{1}{2})$ is $\begin{pmatrix} \frac{1}{2} + y \\ \frac{1}{2} \end{pmatrix}$. On the other hand, for $\beta = (\frac{1}{2}, \frac{1}{2})$, the function $\beta u^1(\cdot, y)$ is quasiconcave for all $y$. Therefore $G_\beta$, and hence $G$, has an equilibrium.

[19] Actually their requirement is even more general than that, but for the purpose of our discussion we only cover the case of convex preferences.

[20] To see this, observe that for any $\beta \in \Delta_+$ the game $G_\beta$ reduces to some kind of matching-pennies game. For any $\beta \in \Delta_+$, the $x_1, y_1-$ best responses of both players map to the set $\{0, \frac{1}{2}, 1\}$. But player 1 prefers an extreme action (0 or 1) if, and only if, player two plays the middle $(\frac{1}{2})$. Player 2, on the other hand, prefers extreme actions if, and only if, also player 1 plays an extreme action.

## 7 Mixed extensions

### *7.1 Mixed strategy equilibria*

In this section we investigate the mixed strategy extension of normal form games with incomplete preferences. We will see that, with some additional arguments, Theorem 2 in fact yields a full characterization of the set of all mixed strategy equilibria.

As in the standard context of games with complete preferences, we define the *mixed strategy extension* of a game $G = \{(A_i, u^i)_{i\in I}\}$ (with each $A_i$ being a non-empty subset of a metric space and each $u_i$ being bounded and Borel measurable) as $G^{mix} := \{(\triangle A_i, U^i)_{i\in I}\}$, where $\Delta A_i$ is the set of all Borel probability measures on $A_i$ and $U^i$ is a real function on $\triangle A := \underset{i\in I}{\times} \triangle A_i$ defined by $U^i(\sigma) = \int_A u^i d\sigma$ for some $u^i : A \to \mathbb{R}^{m_i}$. A strategy profile $\sigma$ is a *mixed strategy Nash equilibrium* of $G = \{(A_i, u^i)_{i\in I}\}$, if it is a Nash equilibrium of $G^{mix} = \{(\triangle A_i, U^i)_{i\in I}\}$.[21] We denote the set of mixed strategy Nash equilibria of a game $G$ by $N^{mix}(G)$.

**Theorem 4 (Shapley, 1959; Aumann, 1962).** *Let $G = \{(A_i, u^i)_{i\in I}\}$ be a finite game. Then*

$$\bigcup \{N^{mix}(G_\beta) : \beta \in \Delta_+\} = N^{mix}(G).$$

*Proof.* By Theorem 2 we readily have $\bigcup \{N^{mix}(G_\beta) : \beta \in \Delta_+\} \subseteq N^{mix}(G)$. Moreover, since all $A_i$ are finite, for all maximal points $x$ in $u^i(A_i, a_j)$, there exists a $\beta^i \gg 0$ such that $x \in \max_{y\in u^i(A_i,a_j)} \beta^i y$. Consequently $\bigcup \{N^{mix}(G_\beta) : \beta \in \Delta_+\} = N^{mix}(G)$. □

Theorem 4 is closely related to the earlier results of Shapley (1959) and Aumann (1962). Aumann shows that $\sigma$ is a mixed strategy Nash equilibrium of a finite game $G = \{(A_i, \succsim_i)_{i\in I}\}$ if, and only if, $\sigma$ is a Nash equilibrium of a completion of $G$, such that each completed preference relation admits a von Neumann-Morgenstern representation.[22] This result stands in between Theorems 1 and 4. Given that Aumann assumes certain properties about each player's preferences, he does not need to look at *all* completions (as in Theorem 1) but only at those that obey the von Neumann-Morgenstern axioms to determine the set of all Nash equilibria. Theorem 4 imposes more assumptions on the preferences of the players, in particular we assume finite dimensional representability, and thus a smaller set of completions suffices to capture all Nash equilibria: we find that an action profile is a Nash equilibrium in a game with incomplete preferences if, and only, if it is a Nash equilibrium in a *linear* completion of that game. Shapley's (1959) result only covers

[21] Aumann's (1962) definition of a mixed strategy equilibrium in a game with incomplete preferences reduces to the present one, when the preferences of each player can be represented by a finite dimensional utility function.

[22] Aumann 's (1962) result is actually more general than that, the result, for instance, does not assume finite dimensional representability of the incomplete preference relations. The precise statement of his result would go beyond the scope of this paper.

two-person zero-sum matrix games. He furthermore assumes a peculiar kind of preferences that directly admit finite dimensional representation, so that also in his case linear completions of the game suffice to describe the set of all Nash equilibria.

We conclude by noting that a version of Theorem 4 still holds when we relax the finiteness assumption:

**Theorem 5.** *Let $G = \left\{(A_i, u^i)_{i\in I}\right\}$ be a game such that each $A_i$ is a convex subset of some finite dimensional Euclidean space. Then*

$$\bigcup\left\{N^{mix}(G_\beta) : \beta \in \Delta_+\right\} \subseteq N^{mix}(G) \subseteq \bigcup\left\{N^{mix}(G_\beta) : \beta \in \Delta\right\}.$$

The proof of Theorem 5 is analogous to the proof of Theorem 2, and is therefore omitted here.

### *7.2 Trembling hand perfect equilibria*

Since Nash equilibrium sets of games with incomplete preferences can be large, it is of interest to consider refinements of equilibria such as "trembling hand perfection". As in the standard theory, we say that a strategy profile $\sigma$ in a finite strategic game $G = \{(A_i, u^i)_{i\in I}\}$ is a *trembling hand perfect equilibrium*, if there exists a sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles that converges to $\sigma$ and $\sigma_i \in BR_{u^i}(\sigma^k_{-i})$ for all $k \in \mathbb{N}$ and all $i \in I$.

While in the context of games with complete preferences it is trivial to show that all trembling hand perfect equilibria are Nash equilibria, this implication does not hold true for games with incomplete preferences. Consider, for instance, the following two-player-game:

| | $L$ | $R$ |
|---|---|---|
| $A$ | $\binom{1}{1}, 1$ | $\binom{\frac{1}{2}}{0}, 0$ |
| $B$ | $\binom{0}{1}, 1$ | $\binom{1}{1}, 0$ |

The row player (1) has incomplete preferences over the outcomes; her preferences are represented by the two-component-vectors. The preferences of the column player (player 2), on the other hand, are complete. It is easily checked that the only Nash equilibrium in this game is $(A, L)$. However $(B, L)$ is trembling hand perfect according to the above definition. To see this, take any sequence of completely mixed strategies $(\sigma^k)_{k=0}^{\infty}$ that converges to $\sigma$, where $\sigma_1(B) = 1 = \sigma_2(L)$. Given that player 2 plays the completely mixed strategy $\sigma_2^k$ player 1 compares $U^1(A, \sigma_2^k) = (1 - 0.5\sigma_2^k(R), 1 - \sigma_2^k(R))$ to $U^1(B, \sigma_2^k) = (\sigma_2^k(R), 1)$. For no $\sigma_2^k(R) > 0$ can these utilities be ranked. Therefore we have that $B \in BR_{u^1}(\sigma_2^k)$ (and of course $\{L\} = BR_{u^2}(\sigma_1^k)$). We conclude that $(B, L)$ is a trembling hand perfect equilibrium, while it is not even a Nash equilibrium.

Since we are interested in the concept of trembling hand perfection only as a refinement of Nash equilibrium, in what follows we restrict our attention to trembling hand perfect Nash equilibria. To state the following result we define a game

$G' := \{(A_i, v^i)_{i\in I}\}$ a *representable completion* of a game $G = \{(A_i, u^i)_{i\in I}\}$ if for every player $i$ the utility $v^i : A \to \mathbb{R}$ represents a completion of the preferences represented by $u^i$.

**Lemma 3.** *Let $G = \{(A_i, u^i)_{i\in I}\}$ be a finite game. If $\sigma$ is a trembling hand perfect equilibrium of some representable completion of $G$, then $\sigma$ is a trembling hand perfect Nash equilibrium in $G$.*

*Proof.* Let the strategy profile $\sigma$ be a trembling hand perfect equilibrium of some representable completion $G' := \{(A_i, v^i)_{i\in I}\}$ of $G$. By Theorem 1 $\sigma$ is a Nash equilibrium of the game $G$. On the other hand, since $\sigma$ is a trembling hand perfect equilibrium of $G'$, there exists a sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles such that $\sigma_i \in BR_{u^{i\prime}}(\sigma^k_{-i})$ for all $i, k$ and $\sigma^k \to \sigma$. But since $v^i$ a completion of $u^i$, we have $\sigma_i \in BR_{u^i}(\sigma^k_{-i})$ for all $i$ and $k$. So $\sigma$ is a trembling hand perfect Nash equilibrium of $G$. □

Corollary 1 is a direct consequence of Lemma 3.

**Corollary 1.** *Let $G = \{(A_i, u^i)_{i\in I}\}$ be a finite game. Then $G$ has a trembling hand perfect Nash equilibrium.*

Unfortunately, however, the converse of Lemma 3 is not true: Trembling hand perfect Nash equilibria in games with incomplete preferences need not be trembling hand perfect equilibria in any completion of that game. Consider, for instance, the strategy profile $\sigma_1(B) = \frac{1}{2}, \sigma_2(L) = 1$ in the following game:

| | $L$ | $R$ |
|---|---|---|
| $A$ | $\binom{0}{2}, 1$ | $\binom{0}{3}, 0$ |
| $B$ | $\binom{1}{1}, 1$ | $\binom{0}{2}, 0$ |

First, notice that $\sigma$ is a Nash equilibrium in this game. Secondly, it is also trembling hand perfect. For, since $U^1(A, \sigma^k_2) = \big(0, 2 + \sigma^k_2(R)\big)$ and $U^1(B, \sigma^k_2) = (1 - \sigma^k_2(R), 1 + \sigma^k_2(R))$ are not comparable we have $A, B \in BR_{u^1}(\sigma^k_2)$ for any completely mixed strategy profile $\sigma^k_2$, and hence $\sigma$ is a trembling hand perfect Nash equilibrium. We now argue that in any completion of $G$ for which $\sigma$ is a Nash equilibrium playing $B$ is a weakly dominated strategy for player 1. For any such completion we must have $(A, L) \sim'_1 (B, L)$ since player 1 is playing $A$ and $B$ with positive probability in the Nash equilibrium $\sigma$. On the other hand, $u^1(A, R) > u^1(B, R)$ so for any completion of the preferences of player 1 must have $(A, R) \succ'_1 (B, R)$. We conclude that $B$ indeed is a weakly dominated strategy in any completion of $G$ for which $\sigma$ is a Nash equilibrium. Therefore, there is no completion of $G$ such that $\sigma$ is trembling hand perfect in that completion.

Observe that this example arises only because definition of mixed strategy equilibria for games with incomplete preferences does not require that players be indifferent between all actions that they play with a positive probability in equilibrium. All these actions need to be best responses to all other players strategies, but in the case of incomplete preferences, $a_i, a_i' \in BR(a_{-i})$ does not imply $a_i \sim_i a_i'$, for the two actions might be unranked.

As in the context of complete preferences, it can be shown for games with incomplete preferences that a strategy profile in a finite two-player game is a trembling hand perfect Nash equilibrium if, and only if, it is a Nash equilibrium in which neither player plays a weakly dominated strategy.[23] This is proved in the same way as it is proved under the assumption of completeness of preferences.

## 8 Conclusion

The goal of this paper was to develop an operational theory of games with incomplete preferences, and to demonstrate the applicability of this theory by means of some economic examples. We started out by showing a fundamental similarity between the theory of games with incomplete preferences and the existing theory of games with complete preferences. In Theorem 1 we showed that for every Nash equilibrium of a game with incomplete preferences, there exists a completion of that game such that this action profile is a Nash equilibrium of the completed game. This result permits us to move back and forth between games with complete and incomplete preferences. For the calculation of equilibrium sets, we can choose whichever form of the game is more convenient, with complete or incomplete preferences. Equivalently, we can without loss of generality model the preferences of some players as incomplete, if we do not know them precisely.

Restricting our attention to games with preferences that can be represented by vector valued utility functions, we can obtain more operational results. In particular, we showed here that under certain restrictions on the players action spaces and preferences, linear completions of a game suffice to characterize the set of all equilibria. Finite mixed strategy games do, for example, fit these restrictions, thereby allowing one to characterize the set of all equilibria in mixed strategies of a finite game in terms of its linear completions. Unfortunately, it turns out that with respect to some concepts it does not suffice to look at the completions of a given game. For instance we cannot characterize all trembling hand perfect equilibria by using the completion method.

We discussed our results at the hand of three well-known models of oligopolistic competition in which firms have incomplete preferences over profits and sales. We established a maximal set of equilibria in the case of quantity competition between firms and we showed that the classical nonexistence problem of capacity constrained Bertrand competition can be solved by assuming that firms cannot rank all sales and output profiles. Finally, we showed that our pure strategy equilibrium solution of capacity constrained Bertrand competition yields the same result as the mixed strategy solution in the context of the celebrated sequential model of quantity and price competition by Kreps and Scheinkman (1983).

[23] The definition of weak domination is a direct application of the standard definition to games with incomplete preferences. We say that a strategy $\sigma_i$ weakly dominates another strategy $\sigma_i'$ if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_{-i}$ and $u_i(\sigma_i, \sigma_{-i}') > u_i(\sigma_i', \sigma_{-i}')$ for at least one $\sigma_{-i}'$. A strategy is called weakly dominated if there exists another strategy that weakly dominates it.

With the exception of our discussion of the incomplete preferences version of the model by Kreps and Scheinkman (1983), the present study only covered normal-form games. In our treatment of the latter model, we pointed out in section 5.3. that a set of peculiar problems arise in extensive form games with incomplete preferences. In particular, in this context we can no longer use backward induction to solve for subgame perfect equilibria. Further investigation of extensive form games is needed to make the theory of games with incomplete preferences applicable to a wider range of economic problems. It would also be interesting to study games with incomplete preferences and incomplete information.

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