Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 135 (2007) 551–557
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Notes, Comments, and Letters to the Editor

More strategies, more Nash equilibria

Sophie Bade[a,*], Guillaume Haeringer[b], Ludovic Renou[c]

[a]Department of Economics, Penn State University, 518 Kern Graduate Building, University Park, PA 16802-3306, USA
[b]Departament d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
[c]Department of Economics, Room G52, Napier Building, University of Adelaide, Adelaide 5005, Australia

Received 10 February 2005; final version received 19 August 2005
Available online 30 March 2006

Abstract

We show in this paper that for the class of two-player games with compact real intervals as strategy spaces and continuous and strictly quasi-concave payoff functions there exists a monotone relation between the size of strategy spaces and the number of Nash equilibria. These sufficient conditions for our theorem to hold are shown to be tight.

JEL classification: C72

Keywords: Strategic-form games; Strategy spaces; Nash equilibrium; Number of Nash equilibria

1. Introduction

An old and central problem in Game Theory is the enumeration of all Nash equilibria of a game.[1] This problem turns out to be a difficult and tedious one, and to date, we do not even have a complete answer for "small'' games like $m \times m$ bi-matrix games.[2] However, a close look at this literature shows that there is, in general, a monotone relation between the number of pure strategies and the *maximal* number of (pure and mixed) Nash equilibria. Similarly, several papers

* Corresponding author. Fax: +1 814 863 4775.
E-mail addresses: sub18@psu.edu (S. Bade), Guillaume.Haeringer@uab.es (G. Haeringer), ludovic.renou@adelaide.edu.au (L. Renou).

[1] See McKelvey and McLennan [6] or von Stengel [12] for recent surveys.

[2] For instance, after proving that when $m = 3$ there is at most 7 equilibria (provided the game satisfies some regularity conditions), Quint and Shubik [10] conjectured that $2^m - 1$ would be an upper bound. This bound turns out to be correct when $m = 4$ but not when $m \geqslant 6$ (see [8,11]). The conjecture remains open for $m = 5$.

0022-0531/\$ - see front matter
doi:10.1016/j.jet.2005.08.009

tackle the issue of the *mean number* of equilibria (e.g., [3,7]) and, again, show that there is a monotone relation between the number of strategies and the (mean) number of equilibria. [3]

In this paper, we are interested in a slightly different question: can we isolate a class of games for which there exists a monotone relation between the size of strategy spaces and the *exact* number of Nash equilibria? The answer we propose is the class of two-player games with non-empty compact real intervals as strategy spaces, and continuous and strictly quasi-concave payoff functions, assumptions met by many economic models. For any two games belonging to this class, such that both players have smaller strategy sets in one of the two games, we show that any Nash equilibrium of the game with the smaller strategy sets can be mapped injectively to a Nash equilibrium of the other game. The core of our proof consists in the construction of this map. Finally, we provide a comprehensive series of counterexamples to show that the conditions for our theorem to hold are tight.

The paper is organized as follows. Section 2 presents the model and states our main result. Section 3 proves our main result while Section 4 discusses the tightness of our sufficient conditions. Section 5 concludes.

2. Definitions and main result

Let $G := \langle N, (Y_i, u_i)_{i\in N}\rangle$ be a strategic-form game with N the set of players, Y_i the set of strategies available to player i, and $u_i : \times_{i\in N} Y_i \to \mathbb{R}$ the payoff function of player i. Denote $Y := \times_{i\in N} Y_i$, $Y_{-i} := \times_{j\in N\setminus\{i\}} Y_j$, and y_{-i} a generic element of Y_{-i}. Let $\mathcal{G}$ be the class of strategic-form games such that $N = \{1, 2\}$, and for each player $i \in N$, Y_i is a non-empty compact real interval (i.e., a compact and convex subset of the real line), the payoff function u_i is continuous in all its arguments and strictly quasi-concave in y_i. [4, 5]

Following Gossner [4], we say that the game $g := \langle N, (X_i, v_i)_{i\in N}\rangle$ is a **restriction** of G if for each player $i \in N$, $X_i \subseteq Y_i$, and $v_i(x) = u_i(x)$ for all $x \in X$. In words, a game g is a restriction of a game G if g is obtained from G by restricting the strategy spaces of the players.

We denote the set of pure Nash equilibria of g and G by $N(g)$ and $N(G)$, respectively. The main result of the paper is stated in the following theorem:

Theorem 1. *Let g and G be two games in $\mathcal{G}$ such that g is a restriction of G. There exists an injective map from $N(g)$ to $N(G)$.*

A direct implication of Theorem 1 is that, if g and G are two games in $\mathcal{G}$ such that both players have smaller strategy sets in g, then G has more Nash equilibria than g. In particular, Theorem 1 implies that if G has a unique Nash equilibrium, then g also has a unique equilibrium.

3. Proof

Let g and G be two games in $\mathcal{G}$ with g a restriction of G. As a first observation, note that $N(g)$ and $N(G)$ are non-empty sets (see, e.g., [5, p. 34]). Second, observe that we can easily construct

[3] For instance, using techniques from statistical mechanics, Berg and McLennan show that the mean number of Nash equilibria in bi-matrix games with m pure strategies for each player is $\exp(m[0.281644 + O(\log(m)/m)])$.

[4] See Aliprantis and Border [1, p. 176] for the definition of strict quasi-concavity.

[5] Moulin [9] calls $\mathcal{G}$ the class of *nice* games. For nice games, point-rationalizability coincides with rationalizability, and with the iterated deletion of strictly dominated strategies. (see [2] for a proof.)

an injective mapping from the set of strategy profiles, which are equilibria in both the game G and its restriction g, to the set of equilibria of G; the identity mapping will do. Therefore, the crucial part of the proof consists in showing that there exists an injective mapping that associates to every equilibrium in $N(g)\backslash N(G)$ an equilibrium in $N(G)\backslash N(g)$.

To characterize the set $N(g)\backslash N(G)$, we first establish a relationship between the best-reply maps in G and g. Let $BR_i : Y_{-i} \to Y_i$ be the best-reply map of player i in the game G such that for all $y_{-i} \in Y_{-i}$,

$$BR_i(y_{-i}) := \{y_i \in Y_i : u_i(y_i, y_{-i}) \geqslant u_i(y_i', y_{-i}) \text{ for all } y_i' \in Y_i\}.$$

Analogously, we denote the best-reply map of player i in the game g by $br_i : X_{-i} \to X_i$. From Berge's Maximum Theorem [1, p. 539] and strict quasi-concavity of payoff functions, it follows that best-reply maps are non-empty, single-valued and continuous. For any non-empty compact real interval Z, we denote $\overline{z}$ its maximum and $\underline{z}$ its minimum.

Lemma 1. *Player i's best-reply function $br_i : X_{-i} \to X_i$ in g is given by*

$$br_i(x_{-i}) = \begin{cases} \underline{x}_i & \text{if } BR_i(x_{-i}) < \underline{x}_i, \\ BR_i(x_{-i}) & \text{if } \underline{x}_i \leqslant BR_i(x_{-i}) \leqslant \overline{x}_i, \\ \overline{x}_i & \text{if } BR_i(x_{-i}) > \overline{x}_i. \end{cases}$$

Proof. First, we clearly have that $br_i(x_{-i}) = BR_i(x_{-i})$ for any x_{-i} such that $BR_i(x_{-i}) \in X_i$. Second, choose a $x_{-i} \in X_{-i}$ such that $BR_i(x_{-i}) < \underline{x}_i$, and suppose that $br_i(x_{-i}) > \underline{x}_i$. The single-valuedness of the best-reply maps implies that $u(BR_i(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i})$ and $u(br_i(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i})$. It follows that $(BR_i(x_{-i}), x_{-i})$ and $(br_i(x_{-i}), x_{-i})$ both belong to the strict upper contour set of $(\underline{x}_i, x_{-i})$. Since $BR_i(x_{-i}) < \underline{x}_i < br_i(x_{-i})$, we have a contradiction with the strict quasi-concavity of u_i. Analogous arguments hold if we choose a $x_{-i} \in X_{-i}$ such that $BR_i(x_{-i}) > \overline{x}_i$, and suppose that $br_i(x_{-i}) < \overline{x}_i$. $\square$

In words, the best-reply map br_i of the restricted game g agrees with the best-reply map BR_i of the game G on the set $\{x_{-i} \in X_{-i} : BR_i(x_{-i}) \in X_i\}$, and is either $\underline{x}_i$ or $\overline{x}_i$, otherwise.

Lemma 2. *If $y^* \in N(G) \cap X$, then $y^* \in N(g)$.*

Lemma 2 states that any equilibrium of G, which belongs to the restricted set of strategies X, is also an equilibrium of g. The proof is obvious and left to the reader. The converse is not true. However, we can prove that any equilibrium of g, which is not on the *boundary $\partial_Y X$ of X in Y*, is also an equilibrium of G.[6]

Lemma 3. *If $x^* \in N(g)\backslash\partial_Y X$, then $x^* \in N(G)$.*

Proof. Let $x^* \in N(g)\backslash\partial_Y X$. First, since $x^* \in N(g)$, we have that $x_i^* = br_i(x_{-i}^*)$ for all $i \in N$. Second, since $x^* \notin \partial_Y X$, we either have $\underline{x}_i < x_i^* < \overline{x}_i$ for all $i \in N$ or $x_i^* = \underline{y}_i$ or $x_i^* = \overline{y}_i$ for

[6] Let (Y, d) be a metric space and $X \subset Y$. A point x is a boundary point of X in Y if each open neighborhood U of x satisfies $U \cap X \neq \emptyset$ and $U \cap (Y\backslash X) \neq \emptyset$. The set of all boundary points of X in Y is $\partial_Y X$. For instance, if $Y = [0, 1]$, $\partial_Y[0, \frac{1}{2}] = \{\frac{1}{2}\}$ while $\partial_Y[\frac{1}{3}, \frac{2}{3}] = \{\frac{1}{3}, \frac{2}{3}\}$.

at least one player $i \in N$. In the former case, it follows from Lemma 1 that $br_i(x^*_{-i}) = BR_i(x^*_{-i})$ for all $i \in N$, hence $x^* \in N(G)$. In the latter case, suppose $x^*_i = \underline{y}_i$. It then follows from Lemma 1 and compactness of Y_i that $\underline{y}_i = br_i(x^*_{-i}) \geqslant BR_i(x^*_{-i}) \geqslant \underline{y}_i$, hence $x^* \in N(G)$. $\square$

From Lemmas 2 and 3, it immediately follows that the equilibria of g, which are not equilibria of G, are on the boundary of X in Y. The next proposition formally states this result.

Proposition 1. *If* $x^* \in N(g)\backslash N(G)$, *then* $x^* \in \partial_Y X$.

After this preliminary work on the characterization of $N(g)\backslash N(G)$, we are now in position to construct an *injective mapping* from $N(g)$ to $N(G)$. In the next lemma, we construct such a mapping for an elementary restriction g of G, namely when only player 1's strategy set is truncated and only from above. Once this result established, Theorem 1 follows by successive applications of elementary restrictions.

Lemma 4. *Let g and G be two games in $\mathcal{G}$ such that g is a restriction of G with $X_1 = [\underline{y}_1, \overline{x}_1]$ and $X_2 = Y_2$. There exists an injective map from $N(g)$ to $N(G)$.*

Proof. First, if $N(g)\backslash N(G)$ is empty, then all equilibria of g are also equilibria of G, hence the identity mapping injectively maps $N(g)$ to $N(G)$. Second, suppose that $N(g)\backslash N(G)$ is non-empty. From Proposition 1, it follows that any equilibrium in $N(g)\backslash N(G)$ must lie on $\partial_Y X = \{(x_1, x_2) \in Y : x_1 = \overline{x}_1\}$. Observe that there is exactly one equilibrium in that set, since player 2 has a unique best reply to $\overline{x}_1$. Therefore, $N(g)\backslash N(G)$ is the *singleton* $\{(\overline{x}_1, br_2(\overline{x}_1))\}$.

Define $F : Y_1 \to \mathbb{R}$ with $F(y_1) = BR_1(BR_2(y_1)) - y_1$. Observe that $F(\overline{y}_1) \leqslant 0$. Moreover, since $(\overline{x}_1, br_2(\overline{x}_1)) \notin N(G)$, we have $F(\overline{x}_1) \neq 0$. Furthermore, since $(\overline{x}_1, br_2(\overline{x}_1)) \in N(g)$, we have that $\overline{x}_1 = br_1(br_2(\overline{x}_1))$. By Lemma 1, br_2 is the restriction of BR_2 to the domain $X_1 \subseteq Y_1$, hence $br_2(\overline{x}_1) = BR_2(\overline{x}_1)$. From the previous arguments, it then follows that $F(\overline{x}_1) > 0$. For otherwise, we would have $BR_1(BR_2(\overline{x}_1)) < \overline{x}_1 = br_1(BR_2(\overline{x}_1)$, a contradiction with Lemma 1.

Since F is continuous, it follows from the Intermediate Value Theorem that there exists a y^*_1 in $(\overline{x}_1, \overline{y}_1]$ such that $F(y^*_1) = 0$, hence we can associate to $(\overline{x}_1, br_2(\overline{x}_1)) \in N(g)\backslash N(G)$ the equilibrium $y^* = (y^*_1, BR_2(y^*_1))$ in $N(G)\backslash N(g)$. We can therefore construct a mapping that projects all elements of $N(g) \cap N(G)$ onto themselves and projects the unique element of $N(g)\backslash N(G)$ to an element of $N(G)\backslash N(g)$. $\square$

To complete the proof of Theorem 1, first observe that Lemma 4 is readily adapted to the other elementary restrictions, that is, the case of a truncation from below of player 1's strategy space or the cases of a truncation from below and from above of player 2's strategy space. Second, observe that any restriction g of G can be obtained by four successive elementary restrictions. Hence, an injective mapping from $N(g)$ to $N(G)$ is easily constructed as the composition of four injective mappings as described in Lemma 4 (and its adaptations to the other relevant cases).

4. Tightness of Theorem 1

To recapitulate, the sufficient conditions for Theorem 1 to hold are that both g and G are two-player games with compact real intervals as strategy spaces and continuous and strictly quasi-concave payoff functions. We first show that Theorem 1 does neither extend to n-player games ($n \geqslant 3$), nor to games with multi-dimensional strategy spaces, nor to games with payoff functions that are not strictly quasi-concave. Interestingly enough, in all three cases, our proof of Lemma 4 breaks down at the same point: there might be multiple equilibria on the boundary of X in Y i.e., the set $N(g)\backslash N(G)$ is *not* necessarily a singleton. Hence, the mapping from $N(g)$ to $N(G)$ need not be an injection.

Example 1 (*More than two-player games*). Consider the following game G, which is not in $\mathcal{G}$ since G has three players.[7] Assume that the strategy spaces are $Y_1 = Y_2 = Y_3 = [0, 1]$. The payoffs of the three players are given by

$$u_1(y_1, y_2, y_3) = y_1,$$
$$u_2(y_1, y_2, y_3) = -0.1(y_2)^2 + (1 - y_1)y_3y_2,$$
$$u_3(y_1, y_2, y_3) = -(y_3 - y_2)^2.$$

Player 1 has a strictly dominant strategy: 1. The unique Nash equilibrium of G is $(1, 0, 0)$. Now consider the restriction g of G with $X_1 = [0, \frac{1}{2}]$, $X_2 = Y_2$, $X_3 = Y_3$. Player 1's strictly dominant strategy is now $\frac{1}{2}$. The restricted game g has multiple Nash equilibria, both $(\frac{1}{2}, 0, 0)$ and $(\frac{1}{2}, 1, 1)$ belong to $N(g)$.

Example 2 (*Strategy spaces are not subsets of the real line*). Consider the following game G, which does not belong to $\mathcal{G}$ because player 1's strategy space is multi-dimensional. Player 1's strategy space is the unit square $Y_1 = [0, 1] \times [0, 1]$, player 2's strategy space is $[0, 1]$, and the payoffs of the two players are given by

$$u_1((y_1^1, y_1^2), y_2) = y_1^1 - (y_2 - y_1^2)^2,$$
$$u_2((y_1^1, y_1^2), y_2) = -0.1(y_2)^2 + (1 - y_1^1)y_1^2y_2.$$

Clearly, this example is a very close relative of the previous example, and has similar equilibria. The strategy profile $((1, 0), 0)$ is the unique Nash equilibrium of the game G. However, the game g with $X_1 = [0, \frac{1}{2}] \times [0, 1]$ and $X_2 = Y_2$ has more than one equilibrium: $((\frac{1}{2}, 0), 0)$ and $((\frac{1}{2}, 1), 1)$ are both equilibrium profiles.

Example 3 (*Lack of strict quasi-concavity*). Consider a game G, which does not belong to $\mathcal{G}$ since player 2's payoff function is not strictly quasi-concave. We have $Y_1 = Y_2 = [0, 1]$ with $u_1(y_1, y_2) = y_1$ and $u_2(y_1, y_2) = (\frac{1}{2} - y_1)y_2$. G has a unique Nash equilibrium $(1, 0)$. Now, let us restrict player 1's strategy space to $X_1 = [0, \frac{1}{2}]$ while keeping player 2's strategy space. The game g has infinitely many equilibria: any profile $(\frac{1}{2}, \lambda)$ with $\lambda \in [0, 1]$ is a Nash equilibrium.

If we drop any one of the other sufficient conditions i.e., convexity or compactness of strategy spaces or continuity of payoff functions, we might simply not be able to construct an injective mapping from $N(g)$ to $N(G)$.[8] To see this, consider any game G with $N(G) = \emptyset$ and restrict both players' strategy spaces to singletons $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$. The strategy profile (x_1, x_2) is an equilibrium of g, which clearly cannot be mapped to an equilibrium in $N(G)$ since this set is empty. However, even if $N(G)$ is non-empty, our proof still fails as the following examples show.

[7] We thank Andrew McLennan for having suggested to us this example.

[8] Moreover, $N(G)$ might be empty.

	0	1	2
0	1, 1	2, 4	−1, −1
1	4, 0	4, 4	4, 1
2	0, 0	1, 4	3, 3

G

	0	2
0	1, 1	−1, −1
2	0, 0	3, 3

g

Fig. 1. Finite games.

Example 4 (*Lack of convexity*). Consider the finite games in Fig. 1. The game G has a unique equilibrium $(1, 1)$ while g has two (pure) equilibria $(0, 0)$ and $(2, 2)$. Observe that strict quasi-concavity is satisfied as $u_i(1, y_{-i}) > \min(u_i(0, y_{-i}), u_i(2, y_{-i}))$ for all $y_{-i} \in \{0, 1, 2\}$, for all $i \in \{1, 2\}$.

Example 5 (*Lack of compactness*). Consider a game G, which does not belong to $\mathcal{G}$ because strategy spaces are not compact. For all $i \in \{1, 2\}$, $Y_i = \mathbb{R}$ and $u_i(y_i, y_{-i}) = (1 - 2y_{-i})y_i - \frac{1}{2}y_i^2$. The game G has a unique Nash equilibrium $(\frac{1}{3}, \frac{1}{3})$. Now, consider the restriction g of G, with $X = [0, \frac{1}{2}] \times \mathbb{R}$. It is easy to see that g has three Nash equilibria: $(\frac{1}{3}, \frac{1}{3})$, $(0, 1)$ and $(\frac{1}{2}, 0)$.

5. A discrete counterpart of the main result

As a final remark, we can mention a discrete counterpart of Theorem 1.[9] Let $\mathcal{G}^\star$ be the class of strategic-form games such that $N = \{1, 2\}$, and for each player $i \in N$, Y_i is finite i.e., $Y_i = \{1, 2, \ldots, m_i\}$, the payoff function u_i is strictly quasi-concave in y_i, generic i.e., $u_i(y) \neq u_i(y')$ for any $y \neq y'$, and has increasing differences in (y_i, y_{-i}).[10] We say that the game $g := \langle N, (X_i, v_i)_{i \in N} \rangle$ is a restriction of G if for each player $i \in N$, X_i is an interval of Y_i, and $v_i(x) = u_i(x)$ for all $x \in X$.

Theorem 2. *Let g and G be two games in $\mathcal{G}^\star$ such that g is a restriction of G. There exists an injective map from $N(g)$ to $N(G)$.*

The proof is almost identical to the proof of Theorem 1. First, observe that since G as well as g are games with *strategic complementarities*, $N(G)$ and $N(g)$ are non-empty. Second, the characterization of $N(g)\backslash N(G)$ is the same as in the proof of Theorem 1. The essential novelty is that best-reply maps are monotone non-decreasing. Third, the essential difference in the proof of Lemma 4 is that we need to use the map $f : [\overline{x}_1, \overline{y}_1] \to [\overline{x}_1, \overline{y}_1]$ with $f(x_1) = BR_1(BR_2(x_1))$ to show that for any equilibrium of g in $N(g)\backslash N(G)$, there exists an equilibrium of G in $N(G)\backslash N(g)$. The existence result follows from the monotonicity of f and Tarski fixed-point Theorem (see e.g. [1, p. 15]). Finally, Theorem 2 follows from the application of four successive elementary restrictions.

[9] The discrete counterpart of Theorem 1 answers a question asked by Joseph Abdou during a talk in Paris. We thank Joseph Abdou for his thoughtful question. A complete proof is available upon request.

[10] See Fudenberg and Tirole [5, p. 490] for a definition.

Acknowledgments

We thank Joseph Abdou, Pierpaolo Battigalli, Myeonghwan Cho, Vijay Krishna, Andy McLennan, Bernhard von Stengel, the associate editor, two referees and seminar participants at the Institut Henri Poincaré for constructive comments. Haeringer acknowledges a financial support from the Spanish Ministry of Science and Technology through grant BEC2002- 02130 and from the Barcelona Economics-CREA program.

References

[1] C. Aliprantis, K.C. Border, Infinite Dimensional Analysis: A Hitchhiker's Guide, Second ed., Springer, Berlin, 1999.
[2] P. Battigalli, Rationalizability in infinite, dynamic games of incomplete information, Res. Econ. 57 (2003) 1–38, doi:10.1016/S1090-9443(02)00054-6
[3] J. Berg, A. McLennan, Asymptotic expected number of nash equilibria of a normal form game, Games Econ. Behav. 51 (2005) 264–295, doi:10.1016/j.geb.2004.10.008
[4] O. Gossner, Ability and Knowledge, Mimeo, June 2004, ⟨http://www.enpc.fr/ceras/labo/AbilKnow.pdf⟩.
[5] D. Fudenberg, J. Tirole, Game Theory, MIT Press, Cambridge, MA, 1991.
[6] R.D. McKelvey, A. McLennan, Computation of Equilibria in Finite Games, in: H.M. Aumann, D.A. Kendrick, J. Rust (Eds.), Handbook of Computational Economics, vol. I, Elsevier, Amsterdam, 1996, pp. 87–142.
[7] A. McLennan, The expected number of nash equilibria of a normal form game, Econometrica 73 (2005) 141–174.
[8] A. McLennan, I.U. Park, Generic 4×4 games have at most 15 Nash equilibria, Games Econ. Behav. 26 (1999) 111–130, doi:10.1006/game.1998.0640
[9] H. Moulin, Dominance solvability and cournot stability, Math. Soc. Sci. 7 (1984) 83–102, doi:10.1016/0165-4896(84)90090-8
[10] T. Quint, M. Shubik, A Theorem on the number of nash equilibria in a bimatrix game, Internat. J. Game Theory (26) 1997 353–359, doi:10.1007/s001820050038
[11] B. von Stengel, New maximal numbers of equilibria in bimatrix games, Discrete Comput. Geom. 21 (1999) 557–568.
[12] B. von Stengel, Computing equilibria for two-person games, in: R. Aumann, S. Hart (Eds.), Handbook of Game Theory, vol. 3, North-Holland, Amsterdam, 2002, pp. 1723–1759.