Notes, Comments, and Letters to the Editor

More strategies, more Nash equilibria

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Abstract

We show in this paper that for the class of two-player games with compact real intervals as strategy spaces and continuous and strictly quasi-concave payoff functions there exists a monotone relation between the size of strategy spaces and the number of Nash equilibria. These sufficient conditions for our theorem to hold are shown to be tight.

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1. Introduction

An old and central problem in Game Theory is the enumeration of all Nash equilibria of a game.\textsuperscript{1} This problem turns out to be a difficult and tedious one, and to date, we do not even have a complete answer for “small” games like $m \times m$ bi-matrix games.\textsuperscript{2} However, a close look at this literature shows that there is, in general, a monotone relation between the number of pure strategies and the \textit{maximal} number of (pure and mixed) Nash equilibria. Similarly, several papers

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\textsuperscript{1} See McKelvey and McLennan [6] or von Stengel [12] for recent surveys.

\textsuperscript{2} For instance, after proving that when $m = 3$ there is at most 7 equilibria (provided the game satisfies some regularity conditions), Quint and Shubik [10] conjectured that $2^m - 1$ would be an upper bound. This bound turns out to be correct when $m = 4$ but not when $m \geqslant 6$ (see [8,11]). The conjecture remains open for $m = 5$.
tackle the issue of the mean number of equilibria (e.g., [3,7]) and, again, show that there is a monotone relation between the number of strategies and the (mean) number of equilibria. 3

In this paper, we are interested in a slightly different question: can we isolate a class of games for which there exists a monotone relation between the size of strategy spaces and the exact number of Nash equilibria? The answer we propose is the class of two-player games with non-empty compact real intervals as strategy spaces, and continuous and strictly quasi-concave payoff functions, assumptions met by many economic models. For any two games belonging to this class, such that both players have smaller strategy sets in one of the two games, we show that any Nash equilibrium of the game with the smaller strategy sets can be mapped injectively to a Nash equilibrium of the other game. The core of our proof consists in the construction of this map. Finally, we provide a comprehensive series of counterexamples to show that the conditions for our theorem to hold are tight.

The paper is organized as follows. Section 2 presents the model and states our main result. Section 3 proves our main result while Section 4 discusses the tightness of our sufficient conditions. Section 5 concludes.

2. Definitions and main result

Let \( G := \langle N, (Y_i, u_i)_{i \in N} \rangle \) be a strategic-form game with \( N \) the set of players, \( Y_i \) the set of strategies available to player \( i \), and \( u_i : \times_{i \in N} Y_i \rightarrow \mathbb{R} \) the payoff function of player \( i \). Denote \( Y := \times_{i \in N} Y_i \), \( Y_{-i} := \times_{j \in N \setminus \{i\}} Y_j \), and \( y_{-i} \) a generic element of \( Y_{-i} \). Let \( \mathcal{G} \) be the class of strategic-form games such that \( N = \{1, 2\} \), and for each player \( i \in N \), \( Y_i \) is a non-empty compact real interval (i.e., a compact and convex subset of the real line), the payoff function \( u_i \) is continuous in all its arguments and strictly quasi-concave in \( y_i \). 4

Following Gossner [4], we say that the game \( g := \langle N, (X_i, v_i)_{i \in N} \rangle \) is a restriction of \( G \) if for each player \( i \in N \), \( X_i \subseteq Y_i \), and \( v_i(x) = u_i(x) \) for all \( x \in X \). In words, a game \( g \) is a restriction of a game \( G \) if \( g \) is obtained from \( G \) by restricting the strategy spaces of the players.

We denote the set of pure Nash equilibria of \( g \) and \( G \) by \( N(g) \) and \( N(G) \), respectively. The main result of the paper is stated in the following theorem:

Theorem 1. Let \( g \) and \( G \) be two games in \( \mathcal{G} \) such that \( g \) is a restriction of \( G \). There exists an injective map from \( N(g) \) to \( N(G) \).

A direct implication of Theorem 1 is that, if \( g \) and \( G \) are two games in \( \mathcal{G} \) such that both players have smaller strategy sets in \( g \), then \( G \) has more Nash equilibria than \( g \). In particular, Theorem 1 implies that if \( G \) has a unique Nash equilibrium, then \( g \) also has a unique equilibrium.

3. Proof

Let \( g \) and \( G \) be two games in \( \mathcal{G} \) with \( g \) a restriction of \( G \). As a first observation, note that \( N(g) \) and \( N(G) \) are non-empty sets (see, e.g., [5, p. 34]). Second, observe that we can easily construct

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3 For instance, using techniques from statistical mechanics, Berg and McLennan show that the mean number of Nash equilibria in bi-matrix games with \( m \) pure strategies for each player is \( \exp(m[0.281644 + O(\log(m)/m)]) \).

4 See Aliprantis and Border [1, p. 176] for the definition of strict quasi-concavity.

5 Moulin [9] calls \( \mathcal{G} \) the class of nice games. For nice games, point-rationalizability coincides with rationalizability, and with the iterated deletion of strictly dominated strategies. (see [2] for a proof.)
an injective mapping from the set of strategy profiles, which are equilibria in both the game \( G \) and its restriction \( g \), to the set of equilibria of \( G \); the identity mapping will do. Therefore, the crucial part of the proof consists in showing that there exists an injective mapping that associates to every equilibrium in \( N(g) \setminus N(G) \) an equilibrium in \( N(G) \setminus N(g) \).

To characterize the set \( N(g) \setminus N(G) \), we first establish a relationship between the best-reply maps in \( G \) and \( g \). Let \( \text{BR}_i : Y_{-i} \rightarrow Y_i \) be the best-reply map of player \( i \) in the game \( G \) such that for all \( y_{-i} \in Y_{-i} \),

\[
\text{BR}_i(y_{-i}) := \{ y_i \in Y_i : u_i(y_i, y_{-i}) \geq u_i(y_i', y_{-i}) \text{ for all } y_i' \in Y_i \}.
\]

Analogously, we denote the best-reply map of player \( i \) in the game \( g \) by \( b_{ri} : X_{-i} \rightarrow X_i \). From Berge’s Maximum Theorem [1, p. 539] and strict quasi-concavity of payoff functions, it follows that best-reply maps are non-empty, single-valued and continuous. For any non-empty compact real interval \( Z \), we denote \( \overline{z} \) its maximum and \( \underline{z} \) its minimum.

**Lemma 1.** Player \( i \)'s best-reply function \( b_{ri} : X_{-i} \rightarrow X_i \) in \( g \) is given by

\[
b_{ri}(x_{-i}) = \begin{cases} \underline{x}_i & \text{if } \text{BR}_i(x_{-i}) < \underline{x}_i, \\ \text{BR}_i(x_{-i}) & \text{if } \underline{x}_i \leq \text{BR}_i(x_{-i}) \leq \overline{x}_i, \\ \overline{x}_i & \text{if } \text{BR}_i(x_{-i}) > \overline{x}_i. \end{cases}
\]

**Proof.** First, we clearly have that \( b_{ri}(x_{-i}) = \text{BR}_i(x_{-i}) \) for any \( x_{-i} \) such that \( \text{BR}_i(x_{-i}) \in X_i \). Second, choose a \( x_{-i} \in X_{-i} \) such that \( \text{BR}_i(x_{-i}) < \underline{x}_i \), and suppose that \( b_{ri}(x_{-i}) > \overline{x}_i \). The single-valuedness of the best-reply maps implies that \( u(\text{BR}_i(x_{-i}), x_{-i}) > u(\overline{x}_i, x_{-i}) \) and \( u(b_{ri}(x_{-i}), x_{-i}) > u(\underline{x}_i, x_{-i}) \). It follows that \( (\text{BR}_i(x_{-i}), x_{-i}) \) and \( (b_{ri}(x_{-i}), x_{-i}) \) both belong to the strict upper contour set of \( (\overline{x}_i, x_{-i}) \). Since \( \text{BR}_i(x_{-i}) < \overline{x}_i \), we have a contradiction with the strict quasi-concavity of \( u_i \). Analogous arguments hold if we choose a \( x_{-i} \in X_{-i} \) such that \( \text{BR}_i(x_{-i}) > \overline{x}_i \), and suppose that \( b_{ri}(x_{-i}) < \underline{x}_i \). \( \square \)

In words, the best-reply map \( b_{ri} \) of the restricted game \( g \) agrees with the best-reply map \( \text{BR}_i \) of the game \( G \) on the set \( \{ x_{-i} \in X_{-i} : \text{BR}_i(x_{-i}) \in X_i \} \), and is either \( \underline{x}_i \) or \( \overline{x}_i \), otherwise.

**Lemma 2.** If \( y^* \in N(G) \cap X \), then \( y^* \in N(g) \).

**Lemma 2** states that any equilibrium of \( G \), which belongs to the restricted set of strategies \( X \), is also an equilibrium of \( g \). The proof is obvious and left to the reader. The converse is not true. However, we can prove that any equilibrium of \( g \), which is not on the boundary \( \partial_Y X \) of \( X \) in \( Y \), is also an equilibrium of \( G \).\(^6\)

**Lemma 3.** If \( x^* \in N(g) \setminus \partial_Y X \), then \( x^* \in N(G) \).

**Proof.** Let \( x^* \in N(g) \setminus \partial_Y X \). First, since \( x^* \in N(g) \), we have that \( x_i^* = b_{ri}(x_{-i}^*) \) for all \( i \in N \). Second, since \( x^* \notin \partial_Y X \), we either have \( x_j < x_i^* < \overline{x}_i \) for all \( i \in N \) or \( x_i^* = y_j \) or \( x_i^* = \overline{y}_j \) for

\(^6\) Let \((Y,d)\) be a metric space and \(X \subset Y\). A point \(x\) is a boundary point of \(X\) in \(Y\) if each open neighborhood \(U\) of \(x\) satisfies \(U \cap X \neq \emptyset\) and \(U \cap (Y \setminus X) \neq \emptyset\). The set of all boundary points of \(X\) in \(Y\) is \(\partial_Y X\). For instance, if \(Y = [0,1]\), \(\partial_Y [0, \frac{1}{2}] = \{\frac{1}{2}\}\) while \(\partial_Y [\frac{1}{2}, \frac{3}{2}] = \{\frac{1}{2}, \frac{3}{2}\}\).
Lemma 4.

of elementary restrictions.

and only from above. Once this result established, Theorem 1 follows by successive applications mapping for an elementary restriction the identity mapping injectively maps injective mapping to construct an has a unique best reply to have that

at least one player \( i \in N \). In the former case, it follows from Lemma 1 that \( \text{br}_i(x^*_i) = \text{BR}_i(x^*_i) \) for all \( i \in N \), hence \( x^* \in N(G) \). In the latter case, suppose \( x^*_i = \bar{y}_i \). It then follows from Lemma 1 and compactness of \( Y_i \) that \( \bar{y}_i = \text{br}_i(x^*_i) \geq \text{BR}_i(x^*_i) \geq \bar{y}_i \), hence \( x^* \in N(G) \). □

From Lemmas 2 and 3, it immediately follows that the equilibria of \( g \), which are not equilibria of \( G \), are on the boundary of \( X \) in \( Y \). The next proposition formally states this result.

Proposition 1. If \( x^* \in N(g) \setminus N(G) \), then \( x^* \in \partial_Y X \).

After this preliminary work on the characterization of \( N(g) \setminus N(G) \), we are now in position to construct an injective mapping from \( N(g) \) to \( N(G) \). In the next lemma, we construct such a mapping for an elementary restriction \( g \) of \( G \), namely when only player 1’s strategy set is truncated and only from above. Once this result established, Theorem 1 follows by successive applications of elementary restrictions.

Lemma 4. Let \( g \) and \( G \) be two games in \( \mathcal{G} \) such that \( g \) is a restriction of \( G \) with \( X_1 = [\underline{y}_1, \bar{x}_1] \) and \( X_2 = Y_2 \). There exists an injective map from \( N(g) \) to \( N(G) \).

Proof. First, if \( N(g) \setminus N(G) \) is empty, then all equilibria of \( g \) are also equilibria of \( G \), hence the identity mapping injectively maps \( N(g) \) to \( N(G) \). Second, suppose that \( N(g) \setminus N(G) \) is non-empty. From Proposition 1, it follows that any equilibrium in \( N(g) \setminus N(G) \) must lie on \( \partial_Y X = \{(x_1, x_2) \in Y : x_1 = \bar{x}_1\} \). Observe that there is exactly one equilibrium in that set, since player 2 has a unique best reply to \( \bar{x}_1 \). Therefore, \( N(g) \setminus N(G) \) is the singleton \( \{(\underline{y}_1, \text{br}_2(\underline{y}_1))\} \).

Define \( F: Y_1 \to \mathbb{R} \) with \( F(y_1) = \text{BR}_1(\text{BR}_2(y_1)) - y_1 \). Observe that \( F(\bar{y}_1) \leq 0 \). Moreover, since \( (\bar{x}_1, \text{br}_2(\bar{x}_1)) \notin N(G) \), we have \( F(\bar{x}_1) \neq 0 \). Furthermore, since \( (\bar{x}_1, \text{br}_2(\bar{x}_1)) \in N(g) \), we have that \( \bar{x}_1 = \text{br}_1(\text{br}_2(\bar{x}_1)) \). By Lemma 1, \( \text{br}_2 \) is the restriction of \( \text{BR}_2 \) to the domain \( X_1 \subseteq Y_1 \), hence \( \text{br}_2(\bar{x}_1) = \text{BR}_2(\bar{x}_1) \). From the previous arguments, it then follows that \( F(\bar{x}_1) > 0 \). For otherwise, we would have \( \text{BR}_1(\text{BR}_2(\bar{x}_1)) < \bar{x}_1 = \text{br}_1(\text{BR}_2(\bar{x}_1)) \), a contradiction with Lemma 1.

Since \( F \) is continuous, it follows from the Intermediate Value Theorem that there exists a \( y^*_1 \) in \( (\underline{x}_1, \bar{y}_1] \) such that \( F(y^*_1) = 0 \), hence we can associate to \( (\bar{x}_1, \text{br}_2(\bar{x}_1)) \in N(g) \) the equilibrium \( y^* = (y^*_1, \text{br}_2(y^*_1)) \) in \( N(G) \setminus N(G) \). We can therefore construct a mapping that projects all elements of \( N(g) \cap N(G) \) onto themselves and projects the unique element of \( N(g) \setminus N(G) \) to an element of \( N(G) \setminus N(G) \). □

To complete the proof of Theorem 1, first observe that Lemma 4 is readily adapted to the other elementary restrictions, that is, the case of a truncation from below of player 1’s strategy space or the cases of a truncation from below and from above of player 2’s strategy space. Second, observe that any restriction \( g \) of \( G \) can be obtained by four successive elementary restrictions. Hence, an injective mapping from \( N(g) \) to \( N(G) \) is easily constructed as the composition of four injective mappings as described in Lemma 4 (and its adaptations to the other relevant cases).

4. Tightness of Theorem 1

To recapitulate, the sufficient conditions for Theorem 1 to hold are that both \( g \) and \( G \) are two-player games with compact real intervals as strategy spaces and continuous and strictly quasi-concave payoff functions. We first show that Theorem 1 does neither extend to \( n \)-player games \( (n \geq 3) \), nor to games with multi-dimensional strategy spaces, nor to games with payoff
functions that are not strictly quasi-concave. Interestingly enough, in all three cases, our proof of Lemma 4 breaks down at the same point: there might be multiple equilibria on the boundary of \( X \) in \( Y \) i.e., the set \( N(g) \setminus N(G) \) is not necessarily a singleton. Hence, the mapping from \( N(g) \) to \( N(G) \) need not be an injection.

**Example 1 (More than two-player games).** Consider the following game \( G \), which is not in \( \mathcal{G} \) since \( G \) has three players.\(^7\) Assume that the strategy spaces are \( Y_1 = Y_2 = Y_3 = [0, 1] \). The payoffs of the three players are given by

\[
\begin{align*}
    u_1(y_1, y_2, y_3) &= y_1, \\
    u_2(y_1, y_2, y_3) &= -0.1(y_2)^2 + (1 - y_1)y_3y_2, \\
    u_3(y_1, y_2, y_3) &= -(y_3 - y_2)^2.
\end{align*}
\]

Player 1 has a strictly dominant strategy: 1. The unique Nash equilibrium of \( G \) is \((1, 0, 0)\). Now consider the restriction \( g \) of \( G \) with \( X_1 = [0, \frac{1}{2}] \), \( X_2 = Y_2 \), \( X_3 = Y_3 \). Player 1’s strictly dominant strategy is now \( \frac{1}{2} \). The restricted game \( g \) has multiple Nash equilibria, both \((\frac{1}{2}, 0, 0)\) and \((\frac{1}{2}, 1, 1)\) belong to \( N(g) \).

**Example 2 (Strategy spaces are not subsets of the real line).** Consider the following game \( G \), which does not belong to \( \mathcal{G} \) because player 1’s strategy space is multi-dimensional. Player 1’s strategy space is the unit square \( Y_1 = [0, 1] \times [0, 1] \), player 2’s strategy space is \([0, 1]\), and the payoffs of the two players are given by

\[
\begin{align*}
    u_1((y_1^1, y_1^2), y_2) &= y_1^1 - (y_2 - y_1^2)^2, \\
    u_2((y_1^1, y_1^2), y_2) &= -0.1(y_2)^2 + (1 - y_1^1)y_1^2y_2.
\end{align*}
\]

Clearly, this example is a very close relative of the previous example, and has similar equilibria. The strategy profile \(((1, 0), 0)\) is the unique Nash equilibrium of the game \( G \). However, the game \( g \) with \( X_1 = [0, \frac{1}{2}] \times [0, 1] \) and \( X_2 = Y_2 \) has more than one equilibrium: \(((\frac{1}{2}, 0), 0)\) and \(((\frac{1}{2}, 1), 1)\) are both equilibrium profiles.

**Example 3 (Lack of strict quasi-concavity).** Consider a game \( G \), which does not belong to \( \mathcal{G} \) since player 2’s payoff function is not strictly quasi-concave. We have \( Y_1 = Y_2 = [0, 1] \) with \( u_1(y_1, y_2) = y_1 \) and \( u_2(y_1, y_2) = (\frac{1}{2} - y_1)y_2 \). \( G \) has a unique Nash equilibrium \((1, 0)\). Now, let us restrict player 1’s strategy space to \( X_1 = [0, \frac{1}{2}] \) while keeping player 2’s strategy space. The game \( g \) has infinitely many equilibria: any profile \((\frac{1}{2}, \lambda)\) with \( \lambda \in [0, 1] \) is a Nash equilibrium.

If we drop any one of the other sufficient conditions i.e., convexity or compactness of strategy spaces or continuity of payoff functions, we might simply not be able to construct an injective mapping from \( N(g) \) to \( N(G) \).\(^8\) To see this, consider any game \( G \) with \( N(G) = \emptyset \) and restrict both players’ strategy spaces to singletons \( X_1 = \{x_1\} \) and \( X_2 = \{x_2\} \). The strategy profile \((x_1, x_2)\) is an equilibrium of \( g \), which clearly cannot be mapped to an equilibrium in \( N(G) \) since this set is empty. However, even if \( N(G) \) is non-empty, our proof still fails as the following examples show.

\(^7\) We thank Andrew McLennan for having suggested to us this example.

\(^8\) Moreover, \( N(G) \) might be empty.
Example 4 (Lack of convexity). Consider the finite games in Fig. 1. The game \(G\) has a unique equilibrium \((1,1)\) while \(g\) has two (pure) equilibria \((0,0)\) and \((2,2)\). Observe that strict quasi-concavity is satisfied as \(u_i(1, y_{-i}) > \min(u_i(0, y_{-i}), u_i(2, y_{-i}))\) for all \(y_{-i} \in \{0, 1, 2\}\), for all \(i \in \{1, 2\}\).

Example 5 (Lack of compactness). Consider a game \(G\), which does not belong to \(G\) because strategy spaces are not compact. For all \(i \in \{1, 2\}\), \(Y_i = \mathbb{R}\) and \(u_i(y_i, y_{-i}) = (1 - 2y_{-i})y_i - \frac{1}{2}y_i^2\). The game \(G\) has a unique Nash equilibrium \((\frac{1}{3}, \frac{1}{3})\). Now, consider the restriction \(g\) of \(G\), with \(X = [0, \frac{1}{2}] \times \mathbb{R}\). It is easy to see that \(g\) has three Nash equilibria: \((\frac{1}{3}, \frac{1}{3}), (0, 1)\) and \((\frac{1}{2}, 0)\).

5. A discrete counterpart of the main result

As a final remark, we can mention a discrete counterpart of Theorem 1.\(^9\) Let \(\mathcal{G}^*\) be the class of strategic-form games such that \(N = \{1, 2\}\), and for each player \(i \in N\), \(Y_i\) is finite i.e., \(Y_i = \{1, 2, \ldots, m_i\}\), the payoff function \(u_i\) is strictly quasi-concave in \(y_i\), generic i.e., \(u_i(y) \neq u_i(y')\) for any \(y \neq y'\), and has increasing differences in \((y_i, y_{-i})\).\(^{10}\) We say that the game \(g := \langle N, (X_i, u_i)_{i \in N} \rangle\) is a restriction of \(G\) if for each player \(i \in N\), \(X_i\) is an interval of \(Y_i\), and \(v_i(x) = u_i(x)\) for all \(x \in X\).

Theorem 2. Let \(g\) and \(G\) be two games in \(\mathcal{G}^*\) such that \(g\) is a restriction of \(G\). There exists an injective map from \(N(g)\) to \(N(G)\).

The proof is almost identical to the proof of Theorem 1. First, observe that since \(G\) as well as \(g\) are games with strategic complementarities, \(N(G)\) and \(N(g)\) are non-empty. Second, the characterization of \(N(g)\) is the same as in the proof of Theorem 1. The essential novelty is that best-reply maps are monotone non-decreasing. Third, the essential difference in the proof of Lemma 4 is that we need to use the map \(f : [X_1, Y_1] \to [X_1, Y_1] \) with \(f(x_1) = BR_1(BR_2(x_1))\) to show that for any equilibrium of \(g\) in \(N(g)\), there exists an equilibrium of \(G\) in \(N(G)\). The existence result follows from the monotonicity of \(f\) and Tarski fixed-point Theorem (see e.g. [1, p. 15]). Finally, Theorem 2 follows from the application of four successive elementary restrictions.

\(^9\)The discrete counterpart of Theorem 1 answers a question asked by Joseph Abdou during a talk in Paris. We thank Joseph Abdou for his thoughtful question. A complete proof is available upon request.

\(^{10}\)See Fudenberg and Tirole [5, p. 490] for a definition.
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