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Note

Bilateral commitment ☆

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Abstract

We consider non-cooperative environments in which two players have the power to gradually and unilaterally rule out some of their actions. Formally, we embed a strategic-form game into a multi-stage game, in which players can restrict their action spaces in all but the final stage, and select among the remaining actions in the last stage. We say that an action profile is implementable by commitment if this action profile is played in the last stage of a subgame-perfect equilibrium path. We provide a complete characterization of all implementable action profiles and a simple method to find them. It turns out that the set of implementable profiles does not depend on the length of the commitment process. We show, furthermore, that commitments can have social value in the sense that in some games there are implementable action profiles that dominate all Nash equilibria of the original game.

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1. Introduction

In a variety of situations (e.g., advertising or R&D), economic agents are able to repeatedly and irreversibly rule out some of their actions before payoffs accrue. Consider two companies that compete in a research tournament developing a prototype for the defense industry. Assume that each company has a fixed budget of $3 millions in the year before the submission deadline for the prototype. Say each has to decide how much to invest in the development of the prototype and in alternative uses over the course of several periods. The likelihood to win the competition depends on the total amount of money spent on the development of the prototype. Each company may decide to *commit* its entire budget in the first period (e.g., $1 million to the development of the prototype and $2 millions to other uses). Alternatively, the same company may decide to retain some flexibility, allocating for instance $1 million each to either one of the uses. This company would retain the option to allocate the remaining million in later periods. Initial and subsequent allocations constrain the spending on the development of the prototype. In other words, initial and subsequent allocations commit the company to choose its total contribution within a certain range. Other examples include lobbies making multiple contributions to influence the choice of a policy, or firms increasing stocks and/or installing capacities. The question this paper addresses is: how does the possibility of multiple rounds of commitments before payoffs accrue affect equilibrium payoffs?

To answer this question, we embed a strategic-form game G into a multi-stage game, in which players can restrict their action spaces in all but the final stage (the commitment stages), and play the game induced by their commitments in the final stage. We call this multi-stage game *a commitment game*. In our example, a company's initial action space is $[0, 3]$ i.e., it can allocate up to $3 millions on the development of a prototype. If a company allocates $1 million to the development of a prototype and $1 million other uses in the first period, it effectively restricts its action space to the interval $[1, 2]$ i.e., the total amount of money devoted to the research tournament is at least $1 million and at most $2 millions. The company might further decrease its leeway in the ensuing periods by committing more funds to either one of the two uses. For instance, if it further spends $0.2 million on the development of the prototype, it restricts its action set to $[1.2, 2]$, etc.

We are interested in the complete characterization of the action profiles of G that are implementable by commitments, that is, the action profiles played in the final stage in any subgame perfect equilibrium. We restrict attention to two-player games with unidimensional action spaces and strictly quasi-concave utilities. These assumptions are met by many economic models, such as Cournot competition, differentiated Bertrand duopoly games, rent-seeking games, or games of tax competition. The introductory example can also be formalized as a game meeting these criteria. With respect to commitments, we assume that players commit to convex restrictions at each period. In other words, we assume that any player who keeps the option to play actions x and x' also retains the option to play action $\lambda x + (1 - \lambda)x'$ for any $\lambda \in [0, 1]$. This assumption seems natural in many contexts. For instance, in the above example, it is hard to imagine that a company retains the option to allocate either 1 or 2 millions to the research tournament, but irreversibly commits to never spend 1.5 millions. It is similarly hard to imagine a donor or a lobby committing to contribute either 1 or 2 millions, but not 1.5. In international economics we actually observe commitments to convex intervals: monetary authorities frequently commit to inflation or exchange rates to be in given bands. This is not to say that our assumption is the only sensible assumption. There might be examples in which it is natural to also consider non-convex commitments. In Section 6, we present some results for non-convex commitments.

The main result of this paper is that an action profile is implementable in a commitment game with multiple rounds *if, and only if*, it is implementable by a "simple commitment" in a two-round game. In such a simple commitment, one player commits to a single action, and the other player truncates his action space at either the top or the bottom. Moreover, the truncation is at the latter player's (original) best-reply to the single action his opponent is committed to. It follows that for any action profile, there are only *four* such simple commitments. This result drastically reduces the complexity of our problem. In Section 5, we consider several economic applications and explain how to apply this result.

The idea that the power to commit oneself can be beneficial has received a great deal of attention in economics. A (very) partial list of contributions includes applications in industrial organization (e.g., Dixit [5]), international trade (e.g., Brander and Spencer [3]), political economy (e.g., Yildirim [16]), to name just a few. Most of these applications can be seen as special cases of our theory, in the sense that commitments made in an initial stage restrict the set of actions available in a later stage. Our paper is most closely related to Romano and Yildirim [13].[1] They characterize the set of implementable action profiles when players have the opportunity to restrict their action spaces from below in a pre-play stage. A first major difference between their model and ours is that we assume that players can restrict their action spaces from the bottom *and* the top at each round. We show that many of their results extend to this more general commitment technology. Furthermore, while Romano and Yildirim show that if a profile is implementable in T rounds of commitment ($T \geqslant 2$), then it is also implementable in two rounds of commitment, we show that the converse also holds true. A second important difference is that we do not assume that the original game has a unique interior Nash equilibrium, differentiable payoff functions or monotonic best replies, all assumptions made by Romano and Yildirim. We show that these assumptions are not just technical, but substantial. For instance, non-monotonic best replies are often necessary for the implementation of Pareto improvements upon the Nash equilibria of a game.

2. Commitment games

The initial situation we consider is a two-player strategic-form game $G := \langle N, (Y_i, u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ the set of players, Y_i the set of actions available to player i, and $u_i : Y_1 \times Y_2 \,(=: Y) \to \mathbb{R}$ the payoff function of player i. As a convention, we call the opponent of player i, player j. We assume that action spaces are $Y_i = [0, 1]$, $i = 1, 2$. The payoff functions u_i are assumed to be continuous in all its arguments and strictly quasi-concave in y_i. We furthermore assume that upper contour sets are path-connected.[2] This assumption is met if u_i is quasi-concave in y_j or if indifference curves are path connected (see Milgrom and Shannon [10]). A commitment is formally defined as a *restriction* of a player's action space.

Definition 1. A (*bilateral*) *commitment* is a pair (X_1, X_2) where for each $i \in \{1, 2\}$, X_i is a closed interval $[\underline{x}_i, \overline{x}_i] \subset [0, 1]$.

[1] See also the literature on endogenous timing in games—e.g., Hamilton and Slutsky [6] or Romano and Yildirim [13] and the references therein.

[2] A set U is *path-connected* if for any two points $u, u' \in U$ there exists a continuous mapping $f : [0, 1] \to U$ (the path) such that $f(0) = u$ and $f(1) = u'$.

It is important to note that a commitment does not necessarily prescribe the choice of an action. In the words of Hart and Moore [7], "in a bilateral commitment, the players commit not to consider actions not on the list (X_1, X_2), i.e., these actions are ruled out. Ex-post, the players are free to choose from the list of actions specified in the commitment i.e., actions are not ruled in."

We say that the bilateral commitment (X_1, X_2) *induces* the game $G(X) := \langle N, (X_i, u_i^X)\rangle$, where $X = X_1 \times X_2$, and for $i \in \{1, 2\}$, $u_i^X(x) = u_i(x)$ for all $x \in X$. Abusing notation, we will drop the superscript X in the sequel. The *induced* game $G(X)$ is thus obtained from the game G by restricting the action sets of the players. We shall use the term '*mother*' to make reference to the original game G. For instance, we shall use the expressions *mother game*, *mother best-reply*, *mother action set*, etc. Similarly, the term '*induced*' will refer to the best reply, action sets, etc., in $G(X)$. For any set Z_i, we denote by $\mathcal{Z}_i$ the collection of all non-empty, compact and convex subsets of Z_i, and define $\mathcal{Z} := \bigtimes_{i\in\{1,2\}} \mathcal{Z}_i$. In a commitment game, players have the opportunity to gradually and unilaterally restrict their action sets prior to choosing their actions. Formally, given the strategic-form game G, the *commitment game* $\Gamma^T(G)$ is a T-stage game with almost perfect information, in which:

Stage 1. Both players simultaneously choose action sets $X_i^1 \in \mathcal{Y}_i$.
Stage $1 < t < T$. Both players simultaneously choose action sets $X_i^t \in \mathcal{X}_i^{t-1}$.
Stage T. Players play the induced strategic-form game $G(X_1^{T-1} \times X_2^{T-1})$.

For simplicity, we write Γ^T for $\Gamma^T(G)$, and Γ for the two-period commitment game $\Gamma^2(G)$ which will play an important role in our analysis. A *pure strategy* s_i for player i in the game Γ^T prescribes a restriction $X_i^1 \in \mathcal{Y}_i$ in period 1, contains rules that map any pairs of restrictions in the prior $t-1$ periods to a restriction $X_i^t \in \mathcal{X}_i^{t-1}$, and prescribes the choice of an action $x_i \in X_i^{T-1}$ which can depend on the entire history of restrictions $(X^1, \ldots, X^{T-1})$, where $X^t = (X_1^t, X_2^t)$. Payoffs only depend on the action profiles chosen in the final stage of the game. If the action profile x is chosen in the final stage of the game, player i's payoff is $u_i(x)$ where u_i is the player's payoff in the mother game G. As a shorthand, we let $u_i(s) := u_i(x)$ for player i's payoff when x is the action profile chosen in period T under the strategy profile s. We use subgame perfection as a solution concept, and focus on equilibria in *pure strategies*. A central concept of the paper is the concept of implementation by commitment, which we now define.

Definition 2. An action profile x^* is *implementable in* Γ^T if there exists a subgame-perfect equilibrium s^* of Γ^T such that the action profile x^* is played in period T on the equilibrium path.

Lastly, in a two-period commitment game Γ, we say that x^* is implementable by the bilateral commitment X^* if there exists an equilibrium s^* that prescribes the commitment to X^* in period 1 and the play of x^* in period 2 if X^* was chosen in the first period.

3. Preliminaries

Denote $BR_i : [0,1] \to [0,1]$, the (*mother*) *best-reply* map of player i in the game G, and $br_i^X : X_j \to X_i$ the best-reply map of player i in the game $G(X)$ induced by the commitment $X_i \times X_j$. For simplicity, we write $br_i^{X_i}$ for $br_i^{X_i\times[0,1]}$, that is, $br_i^{X_i}$ is the restricted best-reply of player i when he is committed to X_i and player j can choose any action in [0, 1]. Note that best-reply maps are non-empty, single valued and continuous. Furthermore, the strict quasi-concavity of payoff functions enables us to easily characterize the mapping br_i^X as a function of BR_i and X.

Lemma 1. *Player i's best-reply function in $G(X)$, $br_i^X : X_j \to X_i$, is*

$$br_i^X(x_j) = \begin{cases} \underline{x}_i & \text{if } BR_i(x_j) < \underline{x}_i, \\ BR_i(x_j) & \text{if } \underline{x}_i \leqslant BR_i(x_j) \leqslant \overline{x}_i, \\ \overline{x}_i & \text{if } \overline{x}_i < BR_i(x_j). \end{cases}$$

In words, the best-reply map br_i^X of the restricted game $G(X)$ agrees with the best-reply map BR_i of the mother game G on the set $\{x_j \in X_j \colon BR_i(x_j) \in X_i\}$, and is either $\underline{x}_i$ or $\overline{x}_i$, otherwise. The proof of Lemma 1 can be found in Bade et al. [1].

Note that any induced game $G(X)$ is a proper subgame of Γ^T (last period subgame). Subgame perfection in pure strategies therefore requires that any $G(X)$ possesses at least one pure Nash equilibrium. Since any $G(X)$ has non-empty, compact and convex action sets and continuous and strictly quasi-concave payoff functions, it has a pure Nash equilibrium—see Osborne and Rubinstein [11, p. 20]. The existence of a subgame-perfect equilibrium in pure strategies of Γ is not, *a priori*, guaranteed, for the cardinality of each player's strategy set in Γ is uncountable. It turns out, however, that the issue of equilibrium existence in our case is easily solved.

Proposition 1. *The commitment game Γ^T has an equilibrium.*

Proof. Choose y^*, a pure Nash equilibrium of G, and consider the following strategy profile s^*. In the first period (and every period after that), each player i restricts his action space to $\{y_i^*\}$. If players commit to any different restrictions $X_i^t \times X_j^t$ in period t, the strategy s_i^* prescribes to play $\{x_i^{t'}\}$ in every period $t \leqslant t' < T$, where (x_i^t, x_j^t) is a pure Nash equilibrium of the induced game $G(X_i^t \times X_j^t)$. Since Γ^T is a finite horizon game, we can use the one-shot deviation property to check that a profile is an equilibrium—see Osborne and Rubinstein [11, p. 103]. By construction, no player can (profitably) change his final-stage action or his commitments in the periods after the first. Moreover, given the first period restriction of player i to $\{y_i^*\}$, player j cannot profitably deviate since y_j^* is player j's mother best-reply to y_i^*. $\square$

When does the set of implementable profiles coincide with the set of Nash equilibria of the original one-shot game? From the proof of Proposition 1, we know that the set of Nash equilibria of the original one-shot game is a subset of the set of implementable profiles.[3] Furthermore, the lead-follow profiles $(l_i^*, BR_j(l_i^*))$ with $l_i^* \in \arg\max_{x_i \in X_i} u_i(x_i, BR_j(x_i))$ are implementable by the commitment $(\{l_i^*\}, [0,1])$. Therefore, if at least one lead-follow profile differs from the Nash equilibria of the original one-shot game, then the possibility of commitment strictly enlarges the set of implementable profiles.[4] In the next section, we completely characterize the set of all implementable profiles.

[3] In a related paper, Jackson and Wilkie [8] propose a model in which players can commit to utility transfers conditional on actions being played. They notably show that Nash equilibria of the game without transfer, the mother game, might not be implementable, while they are in our paper. An essential difference between their paper and ours is that players can undo the commitments of others in their paper (by transferring back money), while it is not possible in our paper.

[4] See Renou [12] for a class of games where commitment does not enlarge the set of equilibrium payoffs.

4. Implementation: a complete characterization

The main result of the paper is that an action profile is implementable in the commitment game Γ^T if and only if it is implementable by one of a very small number of bilateral commitments, those that we call *simple*, and in *two* periods only. This result is remarkable as in a commitment game Γ^T, players cannot only use an infinite number of commitments at each period, but there is also an infinite number of possible sequences of commitments over the T periods.

Definition 3. A bilateral commitment X is *simple* if it has the form $(\{x_i\}, [0, BR_j(x_i)])$ or $(\{x_i\}, [BR_j(x_i), 1])$.

In a simple commitment, one player commits to a single action. The other player truncates his action space either from below or from above, but not both. Moreover, the truncation is at his (mother) best-reply to the only action in his opponent's extreme commitment. We are now ready to formally state our main result.

Theorem 1. *An action profile x^* is implementable in Γ^T if and only if it is implementable by a simple bilateral commitment in Γ.*

Before proving this characterization result, let us comment on the implications of this theorem. If we want to check whether a particular profile x^* can be implemented in a T-period commitment game, we only need to check whether it can be implemented in two periods and by one of the following simple commitments: $([0, BR_1(x_2^*)], \{x_2^*\})$, $([BR_1(x_2^*), 1], \{x_2^*\})$, $(\{x_1^*\}, [0, BR_2(x_1^*)])$, $(\{x_1^*\}, [BR_2(x_1^*), 1])$.

How would one check whether the action profile x^* is implementable in Γ by the simple commitment $(\{x_1^*\}, [0, BR_2(x_1^*)])$? Observe that in the second stage, neither player has an incentive to deviate (player 2 will be playing his mother best-reply to player 1's action, and player 1 has a single action). Furthermore, given that player 1 commits to $\{x_1^*\}$, player 2 does not have an incentive to alter his commitment since the mother best-reply to x_1^* is already contained in $[0, BR_2(x_1^*)]$. Therefore, we only need to check whether player 1 has an incentive to deviate in the first stage of the game. Since any Nash equilibrium $(x_1, br_2^{[0,BR_2(x_1^*)]}(x_1))$ of $G(X_1 \times [0, BR_2(x_1^*)])$ is also a Nash equilibrium of $G(\{x_1\} \times [0, BR_2(x_1^*)])$, it is sufficient to consider deviations to singletons $\{x_1\}$. Consequently, x^* is implementable by the simple commitment $(\{x_1^*\}, [0, BR_2(x_1^*)])$ if, and only if, x_1^* solves the optimization program

$$\max_{x_1 \in [0,1]} u_1\big(x_1, br_2^{[0,BR_2(x_1^*)]}(x_1)\big). \tag{1}$$

In a commitment game, players need to carefully trade off the costs and benefits of committing early. The benefit lies in credibly ruling out certain actions while the cost lie in relinquishing flexibility to punish the other player, should he deviate from his plan. To see this, suppose that x^* is implementable by the simple commitment $(\{x_1^*\}, [0, BR_2(x_1^*)])$, and consider the commitment of player 2 to $\{BR_2(x_1^*)\}$. The induced outcome is still x^*. However, if $BR_1(BR_2(x_1^*)) \neq x_1^*$, player 1 would now like to change his first period commitment to $\{BR_1(BR_2(x_1^*))\}$. Since x^* is implementable by the simple commitment $(\{x_1^*\}, [0, BR_2(x_1^*)])$, this deviation of player 1 is not profitable when player 2 commits to $[0, BR_2(x_1^*)]$: player 2 has the ability to react to such a deviation and "punish" player 1 for trying to play his (mother) best-reply to $BR_2(x_1^*)$. On the other hand, a commitment cannot be "too" flexible. For instance, if player 1 commits to $[x_1^* - \varepsilon, x_1^* + \varepsilon]$ with $\varepsilon > 0$ instead, then the action profile x^* is not even a Nash equilibrium of $G([x_1^* - \varepsilon, x_1^* + \varepsilon] \times [0, BR_2(x_1^*)])$, unless x^* is a Nash equilibrium of G. Yet, our result on simple commitments shows that, without loss of generality, only one player has to retain enough flexibility to deter his opponent from deviating from the single action he is committed to.

Finally, it is important to note that the number of commitment stages does not change the set of implementable outcomes. This result differs sharply from the results in Lockwood and Thomas [9], and Caruana and Einav [4]. In Lockwood and Thomas, players derive payoffs at each period of the (dynamic) game, hence time enters into the (discounted) payoff. Caruana and Einav assume that players can always change their commitment, but paying an extra cost that increases over time. The intuition that can help to implement action profiles that would not be implementable with only one stage of commitment does not apply in our model. This is so because at least one player has to play his mother best-reply to the action of his opponent, and this player gains nothing by making an early commitment or by delaying his commitment, since it is sufficient to choose a commitment that contains the mother best-reply to his opponent's action.

We now present the main steps leading to Theorem 1 and give intuitions for these intermediate results. Detailed proofs can be found in Appendix A or in Bade et al. [2].

Lemma 2. *If the action profile x^* is implementable in Γ^T, then $x_i^* = BR_i(x_j^*)$ for at least one player $i \in \{1, 2\}$.*

To see the intuition behind Lemma 2, consider the two-period case. Suppose that a profile x^* is implementable by the commitment X^*, but neither player is using his mother best-reply. It follows from Lemma 1 that player i's action x_i^* is on the boundary of X_i^*, $i \in \{1, 2\}$. Consider now the commitment $\{x_j'\}$ such that x_j' is closer to $BR_j(x_i^*)$, player j's mother best-reply to x_i^*, than x_j^*. If x_j' sufficiently close to x_j^*, player i's restricted best-reply $br_i^{X_i^*}(x_j')$ remains x_i^*. Consequently, the Nash equilibrium in the second stage is (x_i^*, x_j'). The strict quasi-concavity of player j's payoff function then implies that (x_i^*, x_j') is strictly preferred to x^*, a contradiction. A similar intuition applies to the case of more than 2 periods. The next lemma states that an action profile can be implemented in T periods if and only if it can be implemented in 2 periods.

Lemma 3. *Let x^* be an action profile such that $x_j^* = BR_j(x_i^*)$. The action profile x^* is implementable in Γ^T if and only if it is implementable in Γ.*

To develop some intuition for this result, consider a profile s^* that implements the action profile x^* with $x_j^* = BR_j(x_i^*)$ in Γ^T. We show that this implies that x^* is also implementable by the commitment $(\{x_i^*\}, X_j^1)$ in Γ, where X_j^1 is the first-period commitment of player j under s^*. If it were not the case, then player i would have a profitable first-stage deviation $\{x_i'\}$. We show that this entails that $\{x_i'\}$ is also (part of) a profitable deviation in Γ^T, a contradiction. Conversely, assume that the action profile x^* is implementable in Γ by the commitment X^* (with $x_j^* = BR_j(x_i^*)$). In the proof we show that x^* is implementable by a strategy profile in which the players commit to $\{x_i^*\}$ and X_j^* in the first and all subsequent periods. In short, our proof very much relies on the observation that any first period commitment in an equilibrium in Γ^T can also serve as an equilibrium commitment in Γ. Lemma 3 allows us to focus our attention on the characterization of implementable profiles in Γ.

Lemma 4. *Let x^* be implementable in Γ by the commitment X^* with $x_j^* = BR_j(x_i^*)$. Then x^* is also implementable by the commitment $X' := (\{x_i^*\}, X_j^*)$.*

The intuition behind Lemma 4 is simple. Let $x_j^* = BR_j(x_i^*)$ and suppose that player i commits to the singleton $\{x_i^*\}$ instead of X_i^*. Since player j can still play $BR_j(x_i^*)$ in the second stage, player j has no incentive to deviate in the first stage given player i's commitment to $\{x_i^*\}$. If player i has a profitable deviation X_i from the commitment $\{x_i^*\}$, then the commitment X_i is also a profitable deviation from the commitment X_i^*, a contradiction with the implementation of x^* by X^*.

Finally, to show that any action profile x^* implementable in Γ by the commitment $(\{x_i^*\}, X_j^*)$, with $x_j^* = BR_i(x_i^*)$, is also implementable by a simple commitment, it remains to show that X_j^* can be changed to either $[0, x_j^*]$ or $[x_j^*, 1]$ without altering the equilibrium incentives. The intuition for this result is that while a modification of X_j^* to either $[0, x_j^*]$ or $[x_j^*, 1]$ makes different action profiles reachable to player i by unilateral deviations from $\{x_i^*\}$, none of these additional profiles is preferred by player i.

Lemma 5. *Let x^* be implementable by the commitment $(\{x_i^*\}, X_j^*)$ with $x_j^* = BR_j(x_i^*)$ in Γ. Then x^* is also implementable by a commitment X' such that $X_i' = \{x_i^*\}$ and either $X_j' = [BR_j(x_i^*), 1]$ or $X_j' = [0, BR_j(x_i^*)]$.*

Combining Lemmata 2, 4 and 5, we get that any action profile x^* that is implementable in Γ is implementable by a simple commitment. Conversely, if an action profile is implementable by a simple commitment in Γ, it is clearly implementable in Γ. These observations, together with Lemma 3, prove Theorem 1.

5. Examples

We first consider some of Romano and Yildirim's [13] examples to highlight the differences between their approach and our more general approach. Consider their model of private contributions to a public good. Player i allocates his income, normalized to 1, between the contribution to the public good and private consumption. The payoff $v_i(z_i, x_i + x_j)$ to player i depends on the total contribution $x_i + x_j$ to the public good and total private consumption z_i, and is increasing in all its arguments. Since the budget constraint has to bind, i.e., $z_i = 1 - x_i$, the payoff can be rewritten as $u_i(x_i, x_j) := v_i(1 - x_i, x_i + x_j)$. Romano and Yildirim's assumptions on the player's payoffs are maintained.[5] Romano and Yildirim show that the (unique) Nash equilibrium of the mother game, x^N, is the *only* implementable profile if players can only constrain the total contribution towards the public good from below.

Different from Romano and Yildirim we let private consumption and contributions to the public good in the first period reduce the budget available in the second period and, thus, con-

[5] That is: (1) a unique and interior Nash equilibrium, (2) $u_i(\cdot, BR_j(\cdot))$ strictly quasi-concave, (3) unique and interior lead-follow profiles, (4) monotonic best-replies, and (5) u_i strictly quasi-concave in x_i and x_j. Assumption 5 implies the strict quasi-concavity of u_i in x_i and path-connectedness of upper contour sets.

strain the total contribution towards the public good form above and below. If a player privately consumes $1 - \bar{x}_i$ and contributes $\underline{x}_i$ to the public good in the first period (with $\bar{x}_i \geqslant \underline{x}_i$), the total contribution towards the public good is bounded within $[\underline{x}_i, \bar{x}_i]$. We now show that a wider range of action profiles are implementable in this case.[6] The lead-follow profiles $(l_i^*, BR_j(l_i^*))$ and $(BR_i(l_j^*), l_j^*)$ are implementable. Intuitively, if player i contributes l_i^* and consumes $1 - l_i^*$ in the first period, he commits himself to $\{l_i^*\}$, and the best for player j is to best-reply to l_i^*. Player j can achieve this by contributing $BR_j(l_i^*)$ to the public good and not consuming in the first period (a simple commitment), and not contributing in the second period. However, this can also be achieved by neither consuming nor contributing in the first period and contributing $BR_j(l_i^*)$ in the second period. We can also show that any action profile $(x_i^*, BR_j(x_i^*))$ with $l_i^* \leqslant x_i^* \leqslant x_i^N$ is implementable by the simple commitment $(\{x_i^*\}, [0, BR_j(x_i^*)])$ if best-replies are decreasing and $(\{x_i^*\}, [BR_j(x_i^*), 1])$ if best-replies are increasing.[7] Assume best-replies are decreasing. Clearly, player j has no incentive to deviate as the equilibrium outcome is $(x_i^*, BR_j(x_i^*))$. What about player i? Observe that the restricted best-reply of player j is given by $BR_j(x_i)$, if $x_i \geqslant x_i^*$, and $BR_j(x_i^*)$ otherwise (since BR_j is decreasing). Assume that player i deviates and commits to $x_i > x_i^*$. The induced payoff is then $u_i(x_i, BR_j(x_i)) < u_i(x_i^*, BR_j(x_i^*))$ by strict quasi-concavity of $u_i(\cdot, BR_j(\cdot))$, not a profitable deviation. Assume now that player i deviates to $x_i < x_i^*$. The induced payoff is $u_i(x_i, BR_j(x_i^*))$. If $u_i(x_i, BR_j(x_i^*)) > u_i(x_i^*, BR_j(x_i^*))$, then x_i is closer to $BR_i(BR_j(x_i^*))$ than x_i^* by strict quasi-concavity of u_i in x_i. Thus, we have that $BR_i(BR_j(x_i^*)) - x_i^* < 0$ and $BR_i(BR_j(0)) - 0 > 0$. The Intermediate Value Theorem implies that there exists another Nash equilibrium of the mother game, a contradiction.

More generally, the technical assumptions imposed in Romano and Yildirim [13] lead to a very simple characterization of the set of implementable profiles given by[8]:

$$\bigcup_{i=1,2,\ j\neq i} \bigl\{\bigl(x_i, BR_j(x_i)\bigr)\colon \min\bigl(l_i^*, x_i^N\bigr) \leqslant x_i \leqslant \max\bigl(l_i^*, x_i^N\bigr)\bigr\}.$$

In all their applications but the model of quantity competition with substitutes (Cournot), our richer commitment technology therefore allows for a wider range of action profiles to be implemented. In particular, this implies that their Proposition 2, which states conditions for the implementation of only the original equilibrium, does not hold in our environment unless lead-follow profiles coincide with the unique Nash equilibrium.

Our paper differs from Romano and Yildirim's in two main respects: we allow for a richer commitment technology and we analyze a more general class of games. We now revisit the introductory example to show that commitments might be considered more relevant in this more general class. Suppose that the action sets of the two research companies are $[0, 3]$ and $[0.001, 2]$ respectively, suppose their payoff functions are $u_1(x_1, x_2) = x_1/(x_1 + x_2^2) - x_1/4$ and $u_2(x_1, x_2) = 2(x_2 - x_1) - x_2^2$.[9] This game has a unique Nash equilibrium $(1, 1)$ with equilibrium payoffs $(1/4, -1)$ and two lead-follow profiles $(1, 1)$ and $(0, 2)$ with payoffs $(1/4, -1)$ and

[6] Varian [15] showed that in this case total contributions may be lower, which suggests that contributions should be collected in one period only.

[7] Provided that $l_i^* \leqslant x_i^N$, which obtains under mild conditions (see Romano and Yildirim [13, p. 80]). If $x_i^N < l_i^*$, use the commitment $(\{x_i^*\}, [BR_j(x_i^*), 1])$ if best-replies are decreasing and $(\{x_i^*\}, [0, BR_j(x_i^*)])$ if best-replies are increasing, instead.

[8] See the geometric characterization in Bade et al. [2]. Note that it also holds for their rent-seeking game.

[9] These functional forms might derive from the assumption that the value of the "alternative uses"—or opportunity cost, might differ for the two companies: for company 1 the cost of spending x is $.25x$, for the competitor the cost if x^2, and

$(0, 0)$, respectively. Neither of the lead-follow profiles Pareto dominates the unique Nash equilibrium. However, players can implement a Pareto improvement. To see this, suppose that player 1 commits to $[3/4, 1]$ and player 2 to $\{1/2\}$. The induced game has a unique equilibrium $(3/4, 1/2)$ with a payoff of $(9/16, -3/4)$, a Pareto improvement upon the Nash equilibrium. Moreover, $3/4$ is player 1's best-reply to $1/2$ and, therefore, player 1 has no incentive to deviate. Furthermore, $-3/4$ is the highest payoff player 2 can obtain when player 1 is committed to $[3/4, 1]$.[10] Hence, player 2 has no incentive to deviate. Observe that it is important for player 1 to not restrict his action space further. For instance, suppose that player 1 commits to $[3/4, 4/5]$. This commitment further limits his ability to react in the second period and player 2 can take advantage of that. If player 2 would deviate to 1 he could obtain a payoff of $-3/5$ since player 1's best-reply to 1 would then be $4/5$.[11] A similar argument applies if player 1 commits to $\{3/4\}$. As already argued, it is important to retain enough flexibility in order to react to eventual deviations of an opponent.

The above example illustrates the possibility of implementing an action profile that represents an *improvement upon the status quo*, which is formally defined as an action profile x^*, such that $u_i(x^*) \geqslant u_i(y^*)$ for all $i \in \{1, 2\}$, and $u_i(x^*) > u_i(y^*)$ for at least one player, for all mother Nash equilibria y^*. In short, an action profile that Pareto dominates all mother Nash equilibria is an improvement upon the status quo. Of course, if a lead-follow profile improves upon the status quo, the existence of an implementable improvement is trivial since lead-follow profiles are always implementable in our model. So, the important question is: Can players use irreversible commitments to implement improvements *even if* the lead-follow profiles do not represent such improvements? We know from the above example that this is indeed possible. Our next result states that the non-monotonicity of at least one best-reply is a necessary condition if player payoffs are monotonic in the action of the opponent, a feature that the above example shares with all the examples provided in Romano and Yildirim.[12] We say that a game features *constant externalities* if u_i is monotonic in y_j for $i = 1, 2$.

Proposition 2. *Let G be a game with constant externalities such that the lead-follow profiles do not improve upon the status quo. An implementable improvement upon the status quo exists only if at least one best-reply map is non-monotonic.*

Therefore, for games with monotonic best-replies and constant externalities (the games analyzed by Romano and Yildirim all belong to this class of games), improvements upon the status quo exist if, and only if, at least one lead-follow profile is already such an improvement.

6. Discussion

6.1. Non-convex restrictions

An important assumption of our model is that players commit to *convex* action sets. Although natural in many contexts, this restriction of players' choice sets might be problematic in other

the assumption that company 1 cares more about coming out ahead, whereas company 2 cares more about the research quality differential. In either case, the utility of a company is strictly decreasing in the spending of the competitor.

10 The restricted best-reply of player 1 is $2x_2 - x_2^2$ if $x_2 \in [1/2, 3/2]$ and $3/4$, otherwise.

11 Player 1's restricted best-reply is $4/5$ if $x_2 \in [1 - 0.2\sqrt{5}, 1 + 0.2\sqrt{5}]$, $3/4$ if $x_2 \in [0, 1/2] \cup [3/2, 2]$, and $2x_2 - x_2^2$, otherwise.

12 The proof is in Bade et al. [2].

contexts since players might find it optimal to commit to non-convex sets. First, we show that Theorem 1 partially characterizes the action profiles that are implementable in a general commitment game (i.e., without the convexity assumption). Second, we isolate a class of games for which it is *without loss of generality* to assume convex commitments. We define a *general commitment game* just as a commitment game $\Gamma^T(G)$ with the difference that we allow for all compact commitments and restrict attention to games G that exhibit strategic complementarities.[13] The assumptions of compact restrictions as well as of strategic complementarities are imposed to avoid problems of non-existence of Nash equilibria in pure strategies.

Proposition 3. *Let G be a game with strategic complementarities. If x^* is implementable in a commitment game, then x^* is implementable in a general commitment game.*

To derive some intuition for the case of two periods let x^* be an action profile implementable by the simple commitment $([0, BR_1(x_2^*)], \{x_2^*\})$.[14] More flexibility in player 1's commitment technology does not help him: he already obtains the highest possible payoff given the commitment of player 2. As for player 2, the additional flexibility in his commitment technology implies that he can induce a larger set of proper subgames. However, any pure Nash equilibrium of these subgames can also be obtained as an equilibrium of a game in which player 2 commits to a singleton (a convex restriction). We next identify a class of games for which allowing for non-convex commitment does not affect the set of implementable outcomes.

Proposition 4. *Assume that the game G features constant externalities, the map $u_i(\cdot, BR_j(\cdot))$ is strictly quasi-concave and $T = 2$. If x^* is implementable by the general commitment X^*, then it is implementable by a simple commitment.*

While a careful analysis of general commitment games deserves future research, Proposition 4 suggests that assuming convexity is without loss of generality in several important economic applications (including all applications considered in Romano and Yildirim).

6.2. More than two players

While a full-fledged analysis awaits future research, we can offer a preliminary observation. Contrary to the two-player case, lead-follow profiles may not be implementable in games with three or more players. To see this, consider a Cournot triopoly game with payoff $(1 - x_i - x_j - x_k)x_i$ for firm i and one commitment stage. If firms 1, 2 and 3 move in that order, the unique equilibrium outcome is $(1/2, 1/4, 1/8)$. Suppose that there exists a commitment (X_1, X_2, X_3) implementing the profile $(1/2, 1/4, 1/8)$. Since the (mother) best-reply of firm 2 to $(1/2, 1/8)$ is $3/16$, we should have that the lower bound of X_2 is $1/4$ by Lemma 1. Similarly, $1/2$ has to be the lower bound of X_1. We show next that it would be profitable for firm 1 to deviate to $X_1' = \{1/2 - \epsilon\}$ for some $\epsilon > 0$ chosen small enough so that the lower bound of firm 2's action space remains binding in the second period of the

[13] The game G is a game with strategic complementarities (see e.g., Topkis [14]) if for $y_i \geqslant y_i'$ (resp. $y_i \leqslant y_i'$) and $y_j \geqslant y_j'$ (resp. $y_j \leqslant y_j'$), we have $u_i(y_i, y_j) - u_i(y_i', y_j) \geqslant u_i(y_i, y_j') - u_i(y_i', y_j')$ (resp. $u_i(y_i, y_j) - u_i(y_i', y_j) \leqslant u_i(y_i, y_j') - u_i(y_i', y_j')$). Cournot duopoly, differentiated Bertrand as well as all games considered by Romano and Yildirim [13] belong to this class of games.

[14] The formal proof can be found in Bade et al. [13].

game. The unique second stage equilibrium is $(1/2-\epsilon, 1/4, \min\{\bar{x}_3, 1/8+\epsilon/2\})$. Therefore $u_1(1/2-\epsilon, 1/4, 1/8+\epsilon/2) = (1/2-\epsilon)(1/8+\epsilon/2)$ constitutes a lower bound on player 1's profit given the deviation. So X_1' represents a profitable deviation for some small $\epsilon > 0$.

Appendix A

Proof of Lemma 3. Let x^* be an action profile implementable in Γ by the commitment X^*, and suppose that $x_2^* = BR_2(x_1^*)$. We show that we can implement x^* in Γ^T. To this end, consider the strategies in Γ^T such that player 1 chooses the restriction $\{x_1^*\}$ in the first stage (and, hence in all subsequent stages) and player 2 restricts to X_2^* at the initial history and does not change his commitment at all subsequent histories h^t of length $t < T$. Let player 2 choose x_2^* in period T. Clearly, any profile satisfying this requirement yields the result x^*. Assume, furthermore, that for any alternative restrictions in the first period (X_1', X_2'), the strategy profile is subgame perfect and leads the players to play a Nash equilibrium of the restricted game $G(X_1' \times X_2')$ in period T. Proposition 1 guarantees the existence of such a strategy profile.

Observe that by construction the strategy profile under consideration induces a subgame-perfect equilibrium after any deviation. Since $x_2^* = BR_2(x_1^*)$, and given that player 1 restricts to the singleton $\{x_1^*\}$, player 2 has no incentive to deviate at any period. As for player 1, observe that he can only deviate at the first stage. Suppose that player 1 deviates to X_1'. By construction, the strategy profile specifies that the players will play a Nash equilibrium x' of $G(X_1' \times X_2^*)$ in period T. Note that since x' is the Nash equilibrium of $G(X_1' \times X_2^*)$, we have $x_2' = br_2^{X_2^*}(x_1')$. Consequently, a deviation to $\{x_1'\}$ in the game Γ also induces the players to play x' in period 2. Therefore, if player 1 has a profitable deviation to X_1' in Γ^T at the initial history, he also has a profitable deviation to $\{x_1'\}$ in Γ, a contradiction.

($\Rightarrow$). Let s^* be a subgame-perfect equilibrium of Γ^T that implements the profile x^*, and denote by (X_1^1, X_2^1) the restriction played in the first stage of Γ^T. From Lemma 2, it follows that $x_i^* = BR_i(x_j^*)$ for at least one player $i \in \{1, 2\}$. W.l.o.g., suppose that $x_2^* = BR_2(x_1^*)$. We claim that the commitment $(\{x_1^*\}, X_2^1)$ implements x^* in Γ. Player 2 has no incentive to deviate given the commitment of player 1 to $\{x_1^*\}$ as $x^* = BR_2(x_1^*)$. Now suppose that player 1 has a profitable deviation X_1' from his commitment $\{x_1^*\}$. Following player 1's deviation, the induced game is $G(X_1' \times X_2^1)$, and let x' be a Nash equilibrium of $G(X_1' \times X_2^1)$ with $u_1(x') > u_1(x^*)$. Notice that $x_2' = br_2^{X_2^1}(x_1')$ since x' is a Nash equilibrium of $G(X_1' \times X_2^1)$. This implies that $\{x_1'\}$ is also a profitable deviation for player 1 in Γ. We now show that the existence of such a deviation in Γ contradicts the fact that s^* is a subgame-perfect equilibrium of Γ^T. To see this, consider the strategy s_1' in which player 1 plays $\{x_1'\}$ in the first period of Γ^T and play according to s_1^* at any other history. Consider the subgame starting after this deviation by player 1. We then have the game $\Gamma^{T-1}(G(\{x_1'\} \times X_2^1))$. Given x_1', $br_2^{X_2^1}(x_1')$ is the best result that player 2 can induce; hence, the profile of strategies (s_1', s_2^*) leads to a unique equilibrium result, $(x_1', br_2^{X_2^1}(x_1'))$. It follows that s_1' is a profitable deviation for player 1 given the strategy s_2^* of player 2, which implies that (s_1^*, s_2^*) cannot be an equilibrium of Γ^T, a contradiction. We conclude that x^* must also be implementable in Γ. $\square$

We first introduce additional notations. A *strategy* for player i in the game Γ is a vector $s_i = (X_i, \sigma_i)$ where $X_i \in \mathcal{Y}_i$, and σ_i is a mapping from $\mathcal{Y}$ to $[0, 1]$ such that $\sigma_i(X) \in X_i$, for all $X \in \mathcal{Y}$. That is, a strategy prescribes a restriction X_i in the first period and for each possible choice of a restriction for *both* players in the first period, an action $x_i \in X_i$. The outcome of a strategy profile $(X_i, \sigma_i)_{i\ \{1,2\}}$ is the pair (X, x) where $x_i = \sigma_i(X)$. If (X, x) is the outcome of the strategy profile s, we call x the result.

Proof of Lemma 5. Let $s^* = ((\{x_i^*\}, \sigma_i^*), (X_j^*, \sigma_j^*))$ be an equilibrium of Γ with result x^*, $X_j^* = [\underline{x}_j, \bar{x}_j]$, and $x_j^* = BR_j(x_i^*)$. Define $s_j' = ([x_j^*, 1], \sigma_j^*)$ and $s_j'' = ([0, x_j^*], \sigma_j^*)$. We claim that either (s_i^*, s_j') or (s_i^*, s_j'') is an equilibrium of Γ with result x^*. First, observe that both strategy profiles under consideration have x^* as their result. To see this, note that player i has only one action x_i^*, and player j's mother best response to x_i^*, $BR_j(x_i^*)$, is contained in his restricted action space in either case. Second, note that player j does not have an incentive to change his restricted action space given player i's commitment to $\{x_i^*\}$ as his restricted action space contains his mother best-reply $BR_j(x_i^*)$ to the single action in player 1's restricted action space.

It remains to show that player i has no profitable deviation from his commitment to $\{x_i^*\}$ given the commitment of player j to either $[x_j^*, 1]$ or $[0, x_j^*]$. Since s^* is an equilibrium of Γ, the set of action profiles that give player i a payoff strictly higher than $u_i(x^*)$, $\{x\colon u_i(x) > u_i(x^*)\}$, does not intersect the graph of the restricted best-reply $br_j^{[\underline{x}_j, \bar{x}_j]}$ of player j. For otherwise, player i would have a strictly profitable deviation from s_1^*. It follows from the path connectedness of the strict upper contour sets $\{x\colon u_i(x) > u_i(x^*)\}$ that for all $x' \in \{x\colon u_i(x) > u_i(x^*)\}$, we have either

$$\text{(A1):}\quad br_j^{[\underline{x}_j, \bar{x}_j]}(x_i') - x_j' > 0 \quad \text{or} \quad \text{(A2):}\quad br_j^{[\underline{x}_j, \bar{x}_j]}(x_i') - x_j' < 0.$$

For otherwise, from the Intermediate Value Theorem, there exists an action profile $(\tilde{x}_i, \tilde{x}_j) \in \{x\colon u_i(x) > u_i(x^*)\}$ such that $\tilde{x}_j = br_j^{[\underline{x}_j, \bar{x}_j]}(\tilde{x}_j)$, a contradiction with the implementation of x^*. We can also observe that for all $x_i \in [0, 1]$, $br_j^{[\underline{x}_j, \bar{x}_j]}(x_i) \leqslant br_j^{[x_j^*, \bar{x}_j]}(x_i) \leqslant br_j^{[x_j^*, 1]}(x_i)$, and $br_j^{[\underline{x}_j, \bar{x}_j]}(x_i) \geqslant br_j^{[\underline{x}_j, x_j^*]}(x_i) \geqslant br_j^{[0, x_j^*]}(x_i)$. Suppose that (A1) holds. It follows from the above observation that for all $x' \in \{x\colon u_i(x) > u_i(x^*)\}$, $br_j^{[x_j^*, 1]}(x_i') - x_j' > 0$. This implies that given the commitment of player j to $[x_j^*, 1]$, player i cannot obtain a payoff strictly higher than $u(x^*)$. Therefore, player i has no profitable deviation from s_i^* given s_j', hence (s_i^*, s_j') is an equilibrium of Γ. If (A1) does not hold, then (A2) must hold. If (A2) holds, we can use the same arguments to show that x^* is implementable by $(\{x_i^*\}, [0, x_j^*])$. □

Proof of Proposition 2. Assume that the game features positive externalities (the arguments are similar if we assume negative externalities). Let x^* be implementable by the general commitment (X_i^*, X_j^*) and assume that $x_j^* < BR_j(x_i^*)$. Let $X_j^{**} := [\min X_j^*, x_j^*]$, it is convex. We first show that x^* is implementable by the commitment (X_i^*, X_j^{**}). By strict quasi-concavity, we have that x^* is a Nash equilibrium of $G(X_i^* \times X_j^{**})$ (see Lemma 1). Moreover, since x^* is implementable by X^*, player j has no incentive to deviate from X_j^{**}. To see this, suppose that player j has a profitable deviation i.e., there exists a commitment X_j such that *all* equilibria of $G(X_i^* \times X_j)$ gives player j a payoff strictly higher than $u_j(x^*)$. Then, we have a contradiction with x^* being implementable by X^* since subgame perfection requires to play a Nash equilibrium of $G(X_i^* \times X_j)$. All these equilibria would give to player j a payoff strictly higher than $u_j(x^*)$. Similarly, the graph of $br_j^{X_j^*}$ is included in the lower contour set $LC_i(x^*)$ of player i at x^* with $LC_i(x^*) := \{x \in [0, 1]^2\colon u_i(x) \leqslant u_i(x^*)\}$. It remains to show that the graph of $br_j^{X_j^{**}}$ is also included in $LC_i(x^*)$. Loosely speaking, we want to show that player i cannot "move" along the graph of player j's restricted best-reply and find a profile that strictly improves his payoff over $u_i(x^*)$. For otherwise, he would have a profitable deviation.

First, for all $x_i \in [0,1]$ such that $br_j^{X_j^{**}}(x_i) = br_j^{X_j^*}(x_i)$, we clearly have $(x_i, br_j^{X_j^{**}}(x_i)) \in LC_i(x^*)$. Second, for all $x_i \in [0,1]$ such that $br_j^{X_j^{**}}(x_i) < br_j^{X_j^*}(x_i)$, we have that

$$u_i\big(x_i, br_j^{X_j^{**}}(x_i)\big) \leqslant u_i\big(x_i, br_j^{X_j^*}(x_i)\big) \leqslant u_i(x^*),$$

where the first inequality comes from positive externalities and the second from the fact that the graph of $br_j^{X_j^*}$ is in $LC_i(x^*)$. Third, consider all $x_i \in [0,1]$ such that $br_j^{X_j^{**}}(x_i) > br_j^{X_j^*}(x_i)$. We have that

$$u_i\big(x_i, br_j^{X_j^*}(x_i)\big) \leqslant u_i\big(x_i, br_j^{X_j^{**}}(x_i)\big) \leqslant u_i\big(x_i, x_j^*\big),$$

where the last equality comes from $x_j^* \geqslant br_j^{X_j^{**}}(x_i)$ for all $x_i \in [0,1]$. Suppose that there exists a $\tilde{x}_i$ with $br_j^{X_j^{**}}(\tilde{x}_i) > br_j^{X_j^*}(\tilde{x}_i)$ such that $u_i(\tilde{x}_i, x_j^*) > u_i(x_i^*, x_j^*)$. Since BR_j is increasing and $x_j^* < BR_j(x_i^*)$, there either exists a $\hat{x}_i < x_i^*$ such that $BR_j(\hat{x}_i) = x_j^*$ or $BR_j(x_i) > x_j^*$ for all $x_i \in [0,1]$. Consider the former case. We have that $\tilde{x}_i < \hat{x}_i$. To see this, note that $br_j^{X_j^{**}}(x_i) = x_j^*$ for all $x_i \geqslant \hat{x}_i$ since best-replies are increasing. Moreover, assume that there exists a $\tilde{x}_i \geqslant \hat{x}_i$ such that $br_j^{X_j^{**}}(\tilde{x}_i) > br_j^{X_j^*}(\tilde{x}_i)$. This implies that $BR_j(\tilde{x}_i) \geqslant BR_j(\hat{x}_i) = x_j^* > br_j^{X_j^*}$, a contradiction with the strict quasi-concavity of the payoff function and $x_j^* \in X_j^*$. Consequently, the strict quasi-concavity of u_i implies that $u_i(x_i, x_j^*) > u_i(x^*)$ for all $x_i \in [\hat{x}_i, x_j^*]$, a contradiction since these points belong to the graph of $br^{X_j^*}$. Consider now the latter case i.e., $BR_j(x_i) > x_j^*$ for all $x_i \in [0,1]$. In this case, we show that $br_j^{X_j^{**}}(x_i) \leqslant br_j^{X_j^*}(x_i)$ for all $x_i \in [0,1]$. To see this, suppose there exists a $\hat{x}_i$ such that $br_j^{X_j^{**}}(\hat{x}_i) > br_j^{X_j^*}(\hat{x}_i)$. However, strict quasi-concavity implies that $br_j^{X_j^{**}}(x_i) = x_j^*$ for all x_i, henceforth $BR_j(\hat{x}_i) > br_j^{X_j^{**}}(\hat{x}_i) = x_j^* > br_j^{X_j^*}(\hat{x}_i)$, a contradiction since $x_j^* \in X_j^*$ and u_i is strictly quasi-concave in x_i. Henceforth, x^* is implementable by the commitment (X_i^*, X_j^{**}).

If $x_j^* > BR_j(x_i^*)$, apply the same arguments as above with $X_j^{**} := [x_j^*, \max X_j^*]$.

If $x_j^* = BR_j(x_i^*)$, strict quasi-concavity of the map $x_i \mapsto u_i(x_i, BR_j(x_i))$ implies that x^* is implementable by either $(\{x_i^*\}, [0, BR_j(x_i^*)])$ or $(\{x_i^*\}, [BR_j(x_i^*), 1])$, two simple commitments. For instance, suppose that the lead-follow profile $(l_i, BR_j(l_i))$ is higher than $(x_i^*, BR_j(x_i^*))$, then x^* is implementable by the simple commitment $(\{x_i^*\}, [0, BR_j(x_i^*)])$. To see this, observe that $br_j^{[0,BR_j(x_i^*)]}(x_i) = BR_j(x_i)$ for any $x_i \leqslant x_i^*$ and $br_j^{[0,BR_j(x_i^*)]}(x_i) = x_j^*$ for any $x_i > x_i^*$ since BR_j is increasing. Clearly, strict quasi-concavity of the map $x_i \mapsto u_i(x_i, BR_j(x_i))$ implies that player i has no profitable deviation to $\tilde{x}_i \leqslant x_i^*$. Suppose he has a profitable deviation to $\tilde{x}_i > x_i^*$. First, if $(\tilde{x}_i, x_j^*)$ belongs to the graph of $br_j^{X_j^*}$, we have a contradiction. Second, if $(\tilde{x}_i, x_j^*)$ does not belong to the graph of $br_j^{X_j^*}$, then $br_j^{X_j^*}(\tilde{x}_i) > x_j^*$. It follows from positive externalities that

$$u_i\big(\tilde{x}_i, br_j^{X_j^*}(\tilde{x}_i)\big) \geqslant u_i\big(\tilde{x}_i, x_j^*\big) > u_i(x^*),$$

again a contradiction with the implementation of x^* by the general commitment X^*. To complete the proof, repeat the same arguments as above but starting with x^* being implementable by the commitment (X_i^*, X_j^{**}).

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