13

Skein polynomials and the Tutte polynomial when $x = y$

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Synopsis

This chapter surveys some graph polynomials that are based on medial graph constructions. While none of these polynomials are specializations of the Tutte polynomial, all of them coincide with the Tutte polynomial for special classes of graphs or along special curves. We give these relations to the Tutte polynomial, as well as a number of combinatorial interpretations that derive from them.

- A brief review of vertex and graph states, and skein relations.
- Some graph and link polynomials arising from skein relations, including: the Martin, or circuit partition, polynomial; the Penrose polynomial; the Kauffman bracket; and transition polynomials.
- Evaluations of the Tutte polynomial when $x = y$ that come from medial graph and skein polynomial connections.

13.1 Introduction

We consider graph polynomials that can be defined through *skein relations*, which in this chapter are recursive reductions that involve splitting a graph at a vertex. Jaeger's transition polynomial and the generalized transition polynomial unify skein polynomials such as the Martin or circuit partition polynomials, the Penrose polynomial, and the Kauffman bracket, as well as their 'topological' extensions to graphs in surfaces. Moreover, medial graph constructions

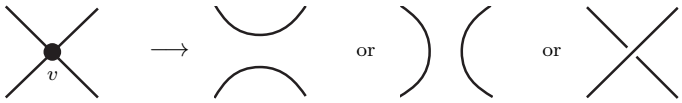


FIGURE 13.1: The three vertex states of a degree-four vertex v of an abstract graph (shown after splicing). In all four diagrams, the rest of the graph is the same outside the configurations shown.

for both plane and embedded graphs relate the Tutte polynomial and its generalizations to various transition polynomials, albeit along restricted curves or surfaces. This connection between deletion–contraction invariants and skein invariants has proved highly fruitful, as new understandings in one setting transfer to the other, with the result that identities and combinatorial interpretations propagate through the different types of polynomials.

We give an overview here of the operations of skein reductions, as well as a selection of skein and transition polynomials with their properties and evaluations. We focus on the connections of these ideas to the Tutte polynomial and the new information that results.

13.2 Vertex states, graph states, and skein relations

Vertex states and graph states lead to state model formulations for the skein polynomials covered in this chapter. A *vertex state* at a degree-four vertex v in an abstract graph G is a partition, into pairs, of the edges incident with v. If edges (a, v) and (b, v) are paired, and (c, v) and (d, v) are paired, then these edges are removed from the graph and replaced with edges (a, b) and (c, d) as in Figure 13.1. We refer to this replacement as *splicing*. In the case of a loop, say $a = b$ in the above, then the added edge (a, b) is a *free loop* which is an edge with no vertex, and is considered as a closed curve.

A *graph state* s of a 4-regular graph results from choosing a vertex state at each of its vertices. Through splicing, a graph state s gives rise to a set of disjoint closed curves called the *components* of the state.

We begin by defining vertex states for vertices of degree-four and and graphs states for 4-regular graphs since these are involved in many applications. However, vertex states, graph states, and splicing can be defined for Eulerian graphs in general, as follows.

Definition 13.1. A *vertex state* at a vertex v is a partition, into pairs, of the edges incident with v. A *graph state* is a choice of vertex state at each of its vertices. Through splicing, a graph state s gives rise to a set of disjoint closed curves, called the *components* of the state. We denote the number of components of s by $c(s)$.

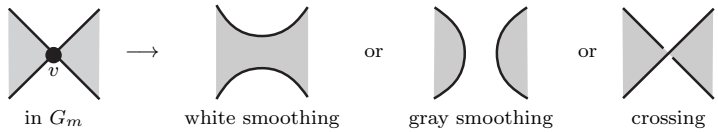


FIGURE 13.2: The three vertex states (shown after splicing) of a vertex v of a checkerboard colored embedded graph.

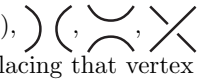
A graph state with exactly k components may also be called an *Eulerian k-partition* of the graph (for example in [748, 825]), and vertex and graph states also correspond to the *transitions* and *transition systems* of Fleischner from [488, 489].

Computing the skein polynomials given later in this chapter often necessitates distinguishing among the possible vertex states and there are some common ways to do this. Recall from Section 1.2.17 the construction from an embedded graph G of its canonically checkerboard colored medial graph G_m, here using the convention that the medial graph of an isolated vertex is a free loop. We may use the canonical checkerboard coloring of G_m (or any checkerboard colored 4-regular graph) to distinguish among the three vertex states at a degree-four vertex v, calling them a *white smoothing*, a *gray smoothing*, and a *crossing* as indicated in Figure 13.2.

Edge orientations of 4-regular digraphs also distinguish one vertex state that is different from the other two. At each vertex of a 4-regular digraph, there is one state that pairs the two incoming edges and also pairs the two outgoing edges. This vertex state is said to be *inconsistent* with the underlying orientation of the digraph, while the remaining two, which pair incoming with outgoing edges, are called *consistent*.

Each graph state consists of a collection of closed curves. Various graph polynomials arise by assigning a weight to each graph state of a graph, then summing these weights over the set of all possible graph states of the graph. (The weights may include indeterminates, and here depend upon the number of components and the vertex states of the graph states.) Alternatively, various graph polynomials arise by reducing a graph via a weighted sum of vertex states. This gives a linear recursion relation for a graph polynomial, called a *skein relation*, a term appropriated from knot theory.

In this chapter a *skein relation* for a graph polynomial (or graph invariant) is a linear relation that expresses the polynomial of a graph in terms of the polynomials of graphs that result from replacing some vertex with the results of splicing its possible vertex states. For example, the top line of Figure 13.3 shows a skein relation for the Martin polynomial $M(G;x)$. In the figure, the images  ,  ,  , and  represent four graphs that are identical except in the local region shown, where they differ as illustrated. Thus  shows a graph G locally at a vertex (in this particular example it must not be

a cut-vertex),  represent the three graphs G_1, G_2 and G_3 that arise by replacing that vertex with its three vertex states. The skein relation says that the value of the polynomial on the graph G is equal to the the sum of its values on the graphs G_1, G_2 and G_3.

Repeated application of a skein relation to a graph results in an expression where the graphs are all closed curves or, more generally, are graphs that are "simpler" than the original in some sense. In such cases a polynomial or an element of a ring can be obtained by providing rules for sending a the closed curves (or the "simpler" graphs) to a ring element. If the value of such a calculation is independent of any choices made in the computation, then we obtain a graph invariant called a *skein invariant*, or a *skein polynomial* if it is polynomial-valued.

For example, repeated application of the skein relation for the Martin polynomial given in the top line of Figure 13.3 to non-cut-vertices results in an expression involving graphs whose vertices are all cut-vertices. Repeated application of the relation in the middle line of the figure will reduce this expression to one involving polynomials of closed loops. The third relation in Figure 13.3 then states that the polynomial of a closed loop is 1. Taken together these three relations provide a recursive way to calculate the Martin polynomial for any connected 4-regular graph. Such a recursive definition involving a skein relation is called a *skein definition* of a polynomial or invariant.

13.3 Skein polynomials

Graph states and skein relations give rise to a family of skein polynomials. We present several of its important members here. Note that as graph states are also called transition systems, skein polynomials are also called *transition polynomials*, for example in the seminal work by Jaeger [651]. The are also a motivating example in Yetter's discussion of graph invariants given by linear recursion relations in [1181].

13.3.1 The Martin and circuit partition polynomials

The Martin polynomials, from Martin's 1977 dissertation [825], were the first skein graph polynomials. The Martin polynomials were originally formulated for 4-regular graphs and digraphs, but Las Vergnas subsequently showed that they extend naturally to all Eulerian graphs and digraphs and have a generating function formulation [744, 748, 752]. Recall that an *Eulerian digraph* has, at each vertex, the same number of incoming as outgoing edges, and an *Eulerian orientation* of an Eulerian graph is an assignment of directions to the edges that results in an Eulerian digraph. The Martin polynomials, and

the circuit partition polynomials which are reformulations of the Martin polynomials, encode information about cycles in Eulerian graphs and digraphs.

The term *Eulerian graph* is used somewhat loosely in this section, as the graphs need not be connected. We only require that all vertices are of even degree. We follow the analogous convention for Eulerian digraphs. Recall that an Eulerian k-partition is a graph state with exactly k components.

Definition 13.2. The *Martin polynomial*, $M(G;x)$, of a connected Eulerian graph G is

$$M(G;x) = \sum_{k\geq 0} f_{k+1}(G)(x-2)^k,$$

and the *Martin polynomial*, $m(\vec{G};x)$, of a connected Eulerian digraph $\vec{G}$ is

$$m(\vec{G};x) = \sum_{k\geq 0} f_{k+1}(\vec{G})(x-1)^k,$$

where $f_k(G)$ and $f_k(\vec{G})$ are the numbers of Eulerian k-partitions of G and $\vec{G}$, respectively.

Example 13.3. If G is the connected 4-regular graph with two loops and two parallel edges, then $M(G;x) = 4(x-2)^0 + 4(x-2)^1 + (x-2)^2$.

The circuit partition polynomials, $J(G;x)$ and $j(\vec{G};x)$, defined in [448] and so-named in [153], have the following generating function formulations.

Definition 13.4. The *circuit partition polynomial*, $J(G;x)$, of an Eulerian graph G is

$$J(G;x) = \sum_{k\geq 1} f_k(G)\, x^k,$$

and the *circuit partition polynomial*, $j(\vec{G};x)$, of an Eulerian digraph $\vec{G}$ is

$$j(\vec{G};x) = \sum_{k\geq 1} f_k(\vec{G})\, x^k,$$

where $f_k(G)$ and $f_k(\vec{G})$ are the numbers of Eulerian k-partitions of G and $\vec{G}$, respectively.

Note that vertices of degree zero or two do not have any effect on either the Martin or circuit partition polynomials, so may be ignored.

For a connected graph G and connected digraph $\vec{G}$ the Martin polynomials and circuit partition polynomials are related as follows:

$$J(G;x) = x\, M(G;x+2), \qquad \text{and} \qquad j(\vec{G};x) = x\, m(\vec{G};x+1).$$

This identity can be used to extend the Martin polynomial to non-connected

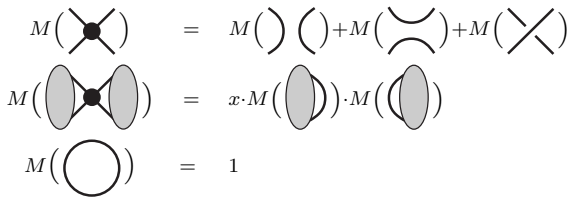


FIGURE 13.3: Skein definition for the Martin polynomial of 4-regular Eulerian graphs. The first equation is for non-cut-vertices, the second is for cut-vertices, and the third is for a free loop.

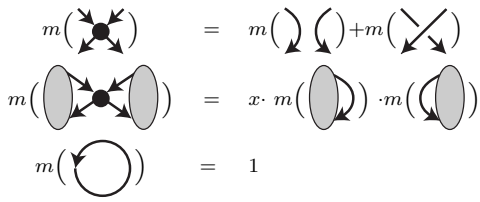


FIGURE 13.4: Skein definition for the Martin polynomial of 4-regular Eulerian digraphs. The first equation is for non-cut-vertices, the second is for cut-vertices, and the third is for a free loop.

graphs (see also Section 14.4 for the extension). Furthermore, for any Eulerian graph G,

$$J(G;x) = \sum j\Big(\vec{G}; \frac{x}{2}\Big),$$

where the sum is over all Eulerian orientations $\vec{G}$ of G (see [748]).

The Martin and circuit partition polynomials satisfy skein relations. Figures 13.3 and 13.4 show the skein relations for degree-four vertices for the Martin polynomial for Eulerian graphs, and for Eulerian digraphs, respectively. For higher degree vertices it is generally more convenient to use the generating function model from Definition 13.2, or to work with the circuit partition polynomial. Figure 13.5 shows a skein definition for the circuit partition polynomial for general Eulerian digraphs. A skein definition for Eulerian graphs is given by summing over all vertex states, not just consistent ones.

Example 13.5. Figure 13.6 illustrates the computation of $j(\vec{G};x) = x^3 + 2x^2 + x$ via the skein relation of Figure 13.5, where $\vec{G}$ is the Eulerian digraph consisting of a digon with a loop at each vertex.

The circuit partition polynomials are multiplicative on disjoint unions. They also have splitting formulas analogous to Tutte's identity for the chromatic polynomial shown in Example 11.4, Item 7.

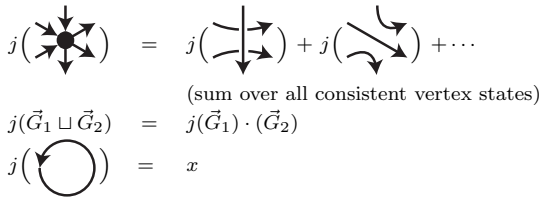


FIGURE 13.5: Circuit partition polynomial skein definition for Eulerian digraphs.

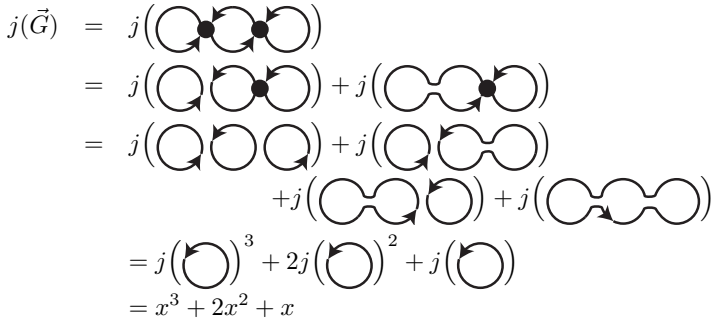


FIGURE 13.6: Computing the circuit partition polynomial of an Eulerian digraph.

Theorem 13.6. *Using the notation $J(A;x)$ for $J(G[A];x)$ and A^c for $E \setminus A$,*

$$J(G;x+y) = \sum J(A;x)\ J(A^c;y),$$

where the sum is over all subsets A of $E(G)$ such that G restricted to both A and A^c is Eulerian. Also,

$$j(\vec{G};x+y) = \sum j(\vec{A};x)\, j(\vec{A}^c;y),$$

where the sum is over all subsets A of $E(\vec{G})$ such that $\vec{G}$ restricted to both A and A^c is an Eulerian digraph.

Theorem 13.6 follows from the fact that $J(G;x)$ is a Hopf algebra map from a Hopf algebra of graphs to the binomial bialgebra, and hence $J(G;1 \otimes x + x \otimes 1) = \sum J(A;x) \otimes J(A^c;x)$ (see [448]). Substituting x for $1 \otimes x$ and y for $x \otimes 1$ gives the result. Alternative simple combinatorial proofs as well as generalizations may be found in [153, 451].

Martin [825] found a connection between the Martin, or circuit partition, polynomial of certain digraphs and the Tutte polynomial of a plane graph G using the *directed medial graph* $\vec{G}_m$ (see Section 1.2.17 for a definition of $\vec{G}_m$).

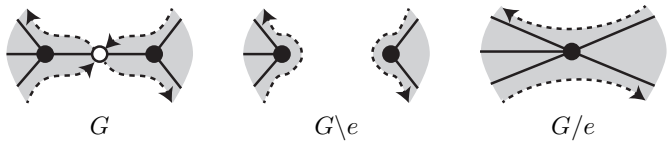


FIGURE 13.7: Relating vertex states of $\vec{G}_m$ to deletion and contraction in G.

Theorem 13.7. *Let G be a connected plane graph, and let $\vec{G}_m$ be its directed medial graph. Then*

$$T(G; x, x) = m(\vec{G}_m; x) = \frac{1}{x-1} j(\vec{G}_m; x-1). \tag{13.1}$$

This connection to Tutte polynomial follows from a fundamental observation relating deletion and contraction of an edge e in G with the appropriate vertex states at the vertex corresponding to e in $\vec{G}_m$, as shown in Figure 13.7. In the figure, G has black vertices and solid edges, and $\vec{G}_m$ has a white vertex and dashed edges, and we see that contraction in G corresponds to a white smoothing in $\vec{G}_m$ while deletion in G corresponds to a gray smoothing in $\vec{G}_m$.

Example 13.8. The digraph in Example 13.6 is the directed medial graph of the P_2, the path with two edges. Since $T(P_2; x, x) = x^2$, then indeed $xT(P_2; x+1, x+1) = x^3 + 2x^2 + x = j((\vec{P}_2)_m, x)$.

Both Martin [826] and Las Vergnas [752] found combinatorial interpretations for some small integer evaluations of the Martin polynomials, while combinatorial interpretations for all integer values as well as some combinatorial evaluations of some derivatives for the Martin and circuit partition polynomials were given in [153, 449, 450, 451]. The derivative formulas give insights into the derivatives of the Tutte polynomial in Section 13.4.1. These evaluations, combined with induction and the splitting formulas in Theorem 13.6 yield a number of combinatorial interpretations. We give two representative examples below, and more may be found in Martin [825, 826], Las Vergnas [744, 748, 752], Bollobás [153], and also [448, 449, 450, 451].

Write $Eul(G)$ for the number of Eulerian orientations of an Eulerian graph G, i.e. the number of ways to direct the edges so there are the same number of incoming as outgoing edges at each vertex. Let $A_n(G)$ denote the set $\{(A_1, \ldots, A_n)\}$ of ordered partitions of $E(G)$ into n subsets such that $G[A_i]$ is Eulerian for all i, recalling the convention that these graphs need not be connected. For digraphs, $\vec{A}_n(\vec{G})$ is defined similarly. From [153, 450, 451],

$$J(G; 2n) = \sum_{A_n(G)} \prod_{i=1}^{n} \left(\mathrm{Eul}(A_i) \prod_{v \in V(A_i)} \left(\frac{\deg_{A_i}(v)}{2} \right)! \right),$$

where $\mathrm{Eul}\,(G)$ is the number of Eulerian orientations of G; and from [450],

$$j(\vec{G};n) = \sum_{\vec{A}_n(\vec{G})} \prod_{i=1}^{n} \left(\prod_{v \in V(\vec{A}_i)} \left(\frac{\deg_{\vec{A}_i}(v)}{2} \right)! \right).$$

These and similar identities and evaluations of the Martin and partition polynomials combined with the relationship of Theorem 13.7 gives the foundation for many of the combinatorial interpretation of the Tutte polynomial along the line $y = x$ described in Section 13.4.

Regts and Sevenster [957] use such identities and Theorem 13.6 to express the Martin polynomial at even negative integers as a skew-partition function of skew-symmetric tensors. Furthermore, the theory of isotropic systems, which unifies essential properties of 4-regular graphs and pairs of dual binary matroids, provides a framework to considerably extend the relation between the Tutte and Martin polynomials. Details of this can be found in Chapter 14.

13.3.2 The Penrose polynomial

The Penrose polynomial $P(G;\lambda)$ of a plane graph G appeared implicitly in work of Penrose [917] in the context of tensor diagrams in physics. Aigner developed it in a purely graph theoretical context in [10], and the following definition reveals that it can be viewed as skein polynomial of medial graphs (see also Definition 13.15, Item 3).

Definition 13.9. Let G be a plane graph and G_m be its canonically checkerboard colored medial graph. Then the *Penrose polynomial* is

$$P(G;\lambda) = \sum_s (-1)^{\mathrm{cr}(s)} \lambda^{c(s)}, \tag{13.2}$$

where the sum is over the graph states s of G_m that consist entirely of white smoothing and crossing vertex states (see Figure 13.2). Here $\mathrm{cr}(s)$ is the number of vertices with crossing states in s, and $c(s)$ its number of components.

Example 13.10. If G is the plane theta-graph, then $P(G;\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda$, as in Figure 13.8.

The Penrose polynomial has some surprising properties, particularly with respect to graph coloring. The four color theorem is equivalent to showing that every planar, cubic, connected graph can be properly edge-colored with three colors. The Penrose polynomial encodes exactly this information (see [917]).

Theorem 13.11. *Let G be a plane, cubic, connected graph. Then,*

$$P\,(G;3) = \left(\frac{-1}{4}\right)^{\frac{|V|}{2}} P\,(G;-2) = \textit{the number of edge-3-colorings of } G.$$

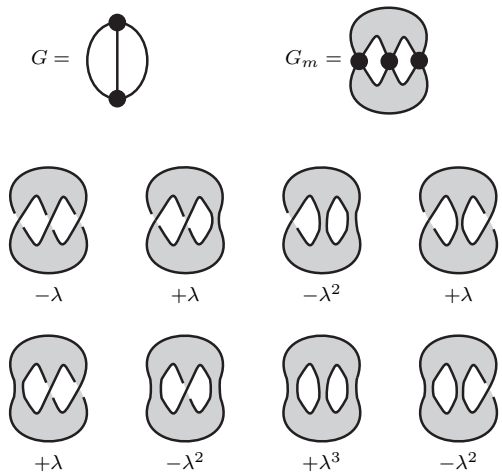


FIGURE 13.8: Computing the Penrose polynomial, $P(G;\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda$.

A good resource for other properties and evaluations of the Penrose polynomial is [10].

Since the medial graph of an embedded graph may also be checkerboard colored, the definition of the Penrose polynomial given in (13.2) extends immediately to graphs embedded in surfaces. The resulting polynomial, often called the *topological Penrose polynomial*, was first considered in [458]. The main advantage of considering the Penrose polynomial of non-plane graphs is that in this class it satisfies a recursive deletion–contraction-type definition (see Theorem 5.2 of [458]):

$$P(G;\lambda) = P(G/e,\lambda) - P(G^{\tau(e)}/e;\lambda),$$

or equivalently

$$P(G^{\delta\tau(e)};\lambda) = P(G\backslash e;\lambda) - P(G/e;\lambda).$$

Here δ and τ are the partial dual and twist of an edge defined in Section 1.2.16. Furthermore, the class of non-plane graphs allows identities that cannot be realized for plane graphs, such as the following connection with the chromatic polynomial $\chi(G;\lambda)$ from [456].

Theorem 13.12. *Let G be a ribbon graph. Then*

$$P(G;\lambda) = \sum_{A\subseteq E(G)} (-1)^{|A|}\chi((G^{\tau(A)})^*;\lambda).$$

Here $G^{\tau(A)}$ is the result of taking the partial Petrial with respect to A, i.e. giving a half-twist to each of edges in A, again as in Subsection 1.2.16.

An alternative extension of the Penrose polynomial for embedded graphs appears in [453]. The Penrose polynomial has also been extended to both matroids and delta-matroids. In [12]. Aigner and Mielke defined the Penrose polynomial of a binary matroid $M = (E, \mathcal{F})$ as

$$P(M;\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\dim(B_M(X))}, \tag{13.3}$$

where $B_M(X)$ is the binary vector space formed of the incidence vectors of the sets in the collection $\{A \in \mathcal{C}(M) : A \cap X \in \mathcal{C}^*(M)\}$. Brijder and Hoogeboom defined the Penrose polynomial in greater generality for vf-safe delta-matroids in [206]. If the delta-matroid corresponds to an embedded graph then their definition agrees with the topological Penrose polynomial up to a factor of $\lambda^{k(G)}$ (see [318]).

13.3.3 The Tutte polynomial

The Tutte polynomial of a plane graph may be expressed, along a restricted curve, in terms of the states of its medial graph. The connection was first observed by Martin in his 1977 thesis [825] and in [826]. We give a more general version here.

Theorem 13.13. *Let G be a plane graph and G_m be its canonically checkerboard colored medial graph. Then*

$$\sum_s a^{\mathrm{gr}(s)} b^{\mathrm{wh}(s)} t^{c(s)} = t^{k(G)} a^{n(G)} b^{r(G)} T\Big(G; \frac{at}{b} + 1, \frac{bt}{a} + 1\Big), \tag{13.4}$$

where the sum is over all graph states s of G_m that have no crossing states, where $\mathrm{gr}(s)$ is the number of gray states in s and $\mathrm{ws}(s)$ is the number of white states in s, and where $c(s)$ its number of components.

The identity in Theorem 13.13 does not hold for non-plane graphs in general. However, an analogous formula holds for graphs on surfaces by using the checkerboard colored medial graph of an embedded graph and the Bollobás–Riordan polynomial $R(G)$ (see Chapter 27) instead of the Tutte polynomial.

Theorem 13.14. *Let G be a graph cellularly embedded in any surface and G_m be its canonically checkerboard colored medial graph. Then*

$$\sum_s a^{\mathrm{gr}(s)} b^{\mathrm{wh}(s)} t^{c(s)} = t^{k(G)} a^{n(G)} b^{r(G)} R\Big(G; \frac{at}{b} + 1, \frac{bt}{a}, \frac{1}{t}, 1\Big),$$

where the sum is over all graph states s of G_m that have no crossing states, where $\mathrm{gr}(s)$ is the number of gray states in s and $\mathrm{ws}(s)$ is the number of white states in s, and where $c(s)$ its number of components.

Theorem 13.14 may be found in [462], and Theorem 13.13 is an immediate consequence of it. See [457] for an overview of these and similar identities.

13.3.4 The Kauffman bracket

The Kauffman bracket is a polynomial valued function on knot and link diagrams. See Chapter 18 for notation and background on it. Consider a link diagram as a 4-regular plane graph in which the crossings are vertices with additional information to record the crossing. The A- and B-splicings of Figure 18.6 can then be regarded as vertex states. Moreover, the crossing type of the diagram at that vertex distinguishes between the vertex states, again as in Figure 18.4. With this

$$\sum_s A^{\alpha(s)} B^{\beta(s)} d^{c(s)} = d[D](A, B, d), \tag{13.5}$$

and

$$\sum_s A^{\alpha(s)} A^{-\beta(s)} (-A^2 - A^{-2})^{c(s)} = (-A^2 - A^{-2}) \langle D \rangle (A) \tag{13.6}$$

where the sum is over all graph states s, $\alpha(s)$ is the number of A-splicings, and $\beta(s)$ is the number of B-splicings.

The Kauffman bracket of a virtual link digram (again see Chapter 18) can be obtained in a similar way. Similarly, consider the virtual link diagram as a plane graph. As the graph is plane, there is a cyclic order of the half-edges at each vertex. Writing the cyclic order of half-edges as $(abcd)$, we call a vertex state *crossing* if pairs ac and bd . With this, Equations (13.5) and (13.6) hold when D is a virtual link diagram, provided that the sum is restricted to graph states in which each virtual crossing has a crossing vertex state.

13.3.5 The Transition polynomial

All the above skein polynomials are weighted sums over graph states and have a very similar form. Jaeger in [651] defined a polynomial of 4-regular graphs, called the *transition polynomial*, which contains each of these polynomials as specializations. Subsequently, more general forms of the transition polynomial, for higher degree graphs, graphs with weight systems, embedded graphs, and even non-Eulerian graphs emerged, all also loosely referred to as transition polynomials, sometimes with descriptors such as *generalized transition polynomial* or *topological transition polynomials* (see [448, 456, 457, 458, 460]). These assimilate the Martin and circuit partition polynomials for arbitrary graphs as well as the various generalizations of the Penrose polynomial and Kauffman bracket to surface embeddings.

We begin with Jaeger's polynomial in the 4-regular setting. Let G be a 4-regular graph. As in Figure 13.1, there are three types of vertex state at each vertex. Suppose there is a way to distinguish among these vertex states (for example, by labelling half-edges, by using an underlying orientation, by a checkerboard coloring, etc.). Then we may use such distinctions to define weight systems and the transition polynomial, as follows.

A *weight system* W for G is function that assigns an element of a unitary ring $\mathfrak{R}$ to each vertex state of G. If s is a graph state of G and v a vertex, let $\omega(v,s)$ denote the element W assigns to the vertex state at v in s, and let $\omega(s) := \prod_{v\in V(F)} \omega(v,s)$ be the *state weight* of s.

Definition 13.15. The *transition polynomial* of G with weight system W is then

$$q(G;W,t) = \sum_s \omega(s)\, t^{c(s)}, \tag{13.7}$$

where the sum is over all graph states s of G.

Note that in some references, such as Jaeger's original paper [651], the exponent $c(s)-1$ is used in place of $c(s)$.

Comparing definitions immediately gives the following identities.

1. Let G be a 4-regular graph. Then

$$q(G;W_J,\lambda) = J(G;\lambda),$$

where the weight system W_J, takes the value 1 for all vertex states.

2. Let $\vec{G}$ be a 4-regular digraph, with underlying graph G. Then

$$q(G;W_j,\lambda) = j(\vec{G};\lambda),$$

where the weight system W_j, takes the value 1 when the vertex state corresponds to pairing incoming with outgoing edges in the underlying orientation of $\vec{G}$ and 0 otherwise.

3. Let G be an embedded graph. Then

$$q(G_m;W_p,\lambda) = P(G;\lambda),$$

where the weight system W_p, with respect to the canonical checkerboard coloring of G_m and conventions in Figure 13.2, takes the value 1 when the vertex state is a white smoothing, 0 when it is a gray smoothing, and -1 when it is a crossing.

4. When G is a plane graph,

$$q(G_m;W_t,t) = t^{k(G)}a^{n(G)}b^{r(G)}T\Big(G;\frac{at}{b}+1,\frac{bt}{a}+1\Big),$$

and for any embedded graph

$$q(G_m;W_t,t) = t^{k(G)}a^{n(G)}b^{r(G)}R\Big(G;\frac{at}{b}+1,\frac{bt}{a},\frac{1}{t},1\Big),$$

where W_t is the weight system of a canonically checkerboard colored medial graph that takes the value b when the vertex state is a white smoothing, a when it is a gray smoothing, and 0 when it is a crossing.

5. If D is a classical link diagram

$$q(G_m; W_b, d) = d[D](A, B, d),$$

where G_m is the underlying graph of D, W_b is the weight system that takes the value A on an A-splicing, B on an B-splicing, and 0 otherwise.

6. If D is a virtual link diagram

$$q(G_m; W_{vb}, d) = d[D](A, B, d),$$

where G_m is the underlying graph of D, W_{vb} is the weight system that on a vertex corresponding to a classical crossing takes the value A on an A-splicing, B on an B-splicing, and 0 otherwise; and on a vertex corresponding to a virtual crossing takes the value 1 on a crossing state and 0 otherwise.

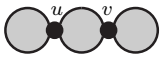
The requirement that G is 4-regular is not essential to the definition of the transition polynomial; in fact G need only be Eulerian. For higher valencies, a vertex state is just a partition into pairs of its incident edges, and then the definitions of graph states and weight systems extend in the obvious way. With this, Equation (13.7) defines the *generalized transition polynomial*, $q(G; W, t)$, introduced in [460]. Using the generalized transition polynomial, Items 1 and 2 above extend to the full circuit partition polynomials for arbitrary Eulerian graphs and digraphs.

A special weight system, called the *medial weight system*, can be used to encode topological information about embedded graph. If G is an embedded graph, then its canonically checkerboard colored medial graph G_m is 4-regular and the checkerboard coloring distinguishes the states at a vertex as either a white smoothing, gray smoothing or crossing as in Figure 13.2. This weight system gives rise to the *topological transition polynomial* introduced in [456].

Definition 13.16. If G is an embedded graph with canonically checkerboard colored medial graph G_m, then each vertex may be assigned state weights via an ordered triple $(\alpha_v,\ \beta_v,\ \gamma_v)$, that indicates the weights of the white smoothing, gray smoothing, and crossing state, in that order. The resulting *medial weight system*, $W_m(G_m)$, is denoted by the resulting set of ordered triples, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, indexed equivalently either by the vertices of G_m or by the edges of G. The *topological transition polynomial* of G is then:

$$Q(G; (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), t) := q(G_m; W_m, t).$$

Example 13.17. If $G =$ 

, then $G_m =$ 

and so

$$\begin{aligned} Q(G; (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), t) = \ & \alpha_u \alpha_v\ Q\Big(\Big) + \alpha_u \beta_v\ Q\Big(\Big) \\ & + \alpha_u \gamma_v\ Q\Big(\Big) + \cdots \\ = \ & \alpha_u \alpha_v t + \alpha_u \beta_v t^2 + \alpha_u \gamma_v t + \cdots \end{aligned}$$

The definition of the topological transition polynomial requires passing through medial graphs. This can be, and in the literature often is, avoided by describing embedded graphs as ribbon graphs (see Chapter 27 for a definition of ribbon graphs). For an edge e of a ribbon graph G, if $G\backslash e$, G/e, and $G^{\tau(e)}$ denote ribbon graph deletion, contraction, and partial Petriality (see Subsection 1.2.16, and full definitions for them may be found in, for example, [456, 457]), then

$$Q(G;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) = \sum_{(A,B,C)} \Big(\prod_{e\in A}\alpha_e\Big)\Big(\prod_{e\in B}\beta_e\Big)\Big(\prod_{e\in C}\gamma_e\Big)t^{\mathrm{bc}(G^{\tau(C)}\backslash B)},$$

where the sum is over ordered partitions (A,B,C) of $E(G)$, and where $\mathrm{bc}(G)$ denotes the number of boundary components of G (see Chapter 27). Furthermore, $Q(G)$ can be defined by the recursion relation

$$Q(G) = \alpha_e Q(G/e) + \beta_e Q(G\backslash e) + \gamma_e Q(G^{\tau(e)}/e)$$

together with its value of $t^{\mathrm{bc}(G)}$ on edgeless ribbon graphs. Duality and twisted duality relations allow permutations of the vertex state weights in the weight system so that, for example, from [462] we have $Q(G;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) = Q(G^*;(\boldsymbol{\beta},\boldsymbol{\alpha},\boldsymbol{\gamma}),t)$, and from [456] we have $Q(G;(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}),t) = Q(G^\Gamma,(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})^\Gamma,t)$, where G^Γ is a twisted dual (see Subsection 1.2.16).

Brijder and Hoogeboom defined a transition polynomial for vf-safe delta-matroids in [206]. If the delta-matroid corresponds to an embedded graph, then this definition agrees with the topological transition polynomial up to a factor of $t^{k(G)}$ (see [318]).

We also note that the space of Eulerian graphs has a Hopf algebra structure with multiplication given by disjoint union and and comultiplication given by summing over ordered partitions into edge disjoint Eulerian subgraphs. With this, the generalized transition polynomial of is a Hopf algebra map from the Hopf algebra of Eulerian graphs to the binomial bialgebra. Full details may be found in [460], including extensions to non-Eulerian graphs.

13.4 Evaluations of the Tutte polynomial along $x = y$

In addition to the elementary interpretations of the Tutte polynomial along $x = y$ given in Chapter 3 (e.g. at $(1,1)$ and $(2,2)$), several other interpretations of the Tutte polynomial at $(-1,-1)$ and $(3,3)$ are known. Moreover, the relationship between the Martin and Tutte polynomials for plane graphs via medial graphs in Theorem 13.7 leads to combinatorial interpretations of the Tutte polynomial along the line $x = y$. See also Section 21.3.2 for some of these results from a Hollant perspective.

1. Let G be a connected planar graph and G_m be the checkerboard colored medial graph of any plane embedding of G. The *all-crossing state* of G_m is the state that results from choosing the crossing vertex state at each vertex (see Figure 13.2). The components of an all-crossing state are called the *anti-circuits* or *crossing circuits* of G_m (or equivalently of $\vec{G}_m$, where they result from following edges in alternating direction, forward-backward-forward-etc., to form circuits). Martin [825, 826] found that

$$T(G; -1, -1) = (-1)^{v(G)}(-2)^{\mu(G_m)-1},$$

where $\mu(G_m)$ is the number *anti-circuits* of G_m.

Note that this interpretation of $T(G; -1, -1)$ can also be deducted through knot theory. The Jones polynomial (see Chapter 18) evaluated at $t = 1$ is known to equal $(-2)^{c(L)-1}$ where $c(L)$ is the number of components of a link. An application of Theorem 18.18 then gives the result.

2. If M is a binary matroid, then $T(M; -1, -1) = (-1)^{r(M)}(-2)^d$, where $d = \dim V_2(M) \cap V_2^{\perp}(M)$ and $V_2(M)$ is the vector space over $GF(2)$ generated by the circuits of M. See Section 12.4 for details. This is due to Rosenstiehl and Read [970].

3. Let G be a connected planar graph and G_m be the medial graph of any plane embedding of G. Say that v is a *saddle vertex* if the directions on its incident edges alternate in direction as in-out-in-out in their cyclic ordering about v. Then

$$T(G; 3, 3) = \sum_{k \geq 0} 2^{k-1} e_k(G_m),$$

where $e_k(G_m)$ is the number of Eulerian orientations of G_m with exactly k saddle vertices. This interpretation is due to Las Vergnas, [752].

4. A *T-tetromino* is an arrangement of four unit squares into a 'T' shape. Korn and Pak [706] showed that if G is an $m \times n$ grid, then $T(G; 3, 3)$ is $1/2$ the number of ways to tile a $4m \times 4n$ rectangle with T-tetrominoes. (In fact this was shown for a larger class of graphs.)

5. A *claw covering* of a graph G is a spanning subgraph of G where every component isomorphic to $K_{1,3}$. Korn and Pak [705] showed that if G, is a connected plane graph, then $T(G; 3, 3)$ equals $1/2$ the number of claw coverings of a graph associated with G.

6. Let G be a planar graph and $\vec{G}_m$ be the directed medial graph of any plane embedding of G. Then Ellis-Monaghan [451] gave the following interpretation for a positive integer n.

$$T(G; 1 + n, 1 + n) = \left(\frac{1}{n}\right)^{k(G)} \sum_{\phi} 2^{\mu(\phi)},$$

where the sum is over all edge colorings ϕ of $\vec{G}_m$ with n colors so that each (possibly empty) set of monochromatic edges forms an Eulerian digraph, and where $\mu(\phi)$ is the number of monochromatic vertices (those whose incident edges are all the same color) in the coloring ϕ. See also Theorems 21.9 and 21.10.

7. Let G be a planar graph and $\vec{G}_m$ be the directed medial graph of any plane embedding of G. Let n be a positive integer. Then

$$T(G; 1-n, 1-n) = \left(\frac{-1}{n}\right)^{k(G)} \sum_{D_n(\vec{G}_m)} (-1)^{\sum_{i=1}^{n} k(D_i)}.$$

where $D_n(\vec{G}_m)$ is the set of all ordered partitions $(D_1, \ldots, D_n)$ of $E(\vec{G}_m)$ such that $\vec{G}_m$ restricted to each D_i is 2-regular and consistently oriented (i.e., each vertex has in-degree one and out-degree one). This interpretation is due to Ellis-Monaghan [451].

13.4.1 Derivatives of the Tutte polynomial

Results from [450] give interpretations for derivatives of the Tutte polynomial along the line $x = y$. See Chapter 28 for more on the derivatives of the Tutte polynomial.

For the following, let $P_n(\vec{G})$ be the set of ordered n-tuples $\overline{p} := (p_1, \ldots, p_n)$ where the p_i are consistently oriented edge-disjoint closed trails in $\vec{G}$. We denote by $\overline{p}^c$ the edges of $\vec{G}$ that are not in any trail of $\overline{p}$. Furthermore we write $m(\overline{p})$ for the number of vertices of $\vec{G}$ not belonging to any of the trails of $\overline{p}$. If G is a connected plane graph, then for all nonnegative integers n,

$$\frac{d^n}{dx^n} T(G; x, x)\Big|_{x=2} = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \sum_{\overline{p} \in P_k(\vec{G}_m)} 2^{m(\overline{p})}.$$

Recall that if G has more than one edge, then the coefficients of x and of y in the Tutte polynomial are equal, and this joint value is the β-invariant, $\beta(G)$. Noting that for a plane graph G the derivative $\frac{d}{dx} T(G; x, x)\Big|_{x=0} = 2\beta = -j'(G; -1)$ yields the following interpretation of the β-invariant:

$$\beta = \frac{1}{2} \sum (-1)^{k(P^c)+1}.$$

Here the sum is over all closed trails P in $\vec{G}_m$ which visit all vertices at least once, and P^c is the set of edges not in P.

13.5 Open problems

In [752], Las Vergnas conjectured that if M is a binary matroid, then $T^{(n)}(G; -1, -1) = K2^{d-n}$ where K is an integer, and $n \in \{0, \ldots d\}$. This long-standing question is still open. See also Conjectures 28.40 and 28.41.

The results given here for the Tutte polynomial along $x = y$ are all for integer values, and in fact nearly all that is known about combinatorial interpretations of the Tutte polynomial involve integer evaluations. This begs the question of what might be encoded at rational, irrational, or even complex values. The line $x = y$, like the hyperbola family $(x - 1)(y - 1) = q$ that plays such an important role in the computational complexity of the Tutte polynomial, may offer a somewhat more accessible setting in which to begin address this question than just generic points (x, y).

The Tutte polynomial has been generalized or adapted to a wide range of combinatorial objects including matroids, matroid perspectives, embedded graphs, delta-matroids, etc. This expansion has only just begun for the generalized transition polynomial for example with adaptations for embedded graphs (see [448, 456, 457, 458, 460]) and vf-safe delta-matroids in [206]. Given how such extensions have deepened the theory of the Tutte polynomial, it is likely they would be similarly fruitful for generalized transition polynomials.

Little is known about zeros of the circuit partition polynomials, and hence about zeros of the Tutte polynomial along the line $y = x$. Zeros on other curves have been heavily studied, e.g. $y = 0$ in the context of the chromatic polynomial, $x = 0$ for tensions and flows, and the hyperbolas $(x-1)(y-1) = q$ particularly for the Potts model (see Chapter 25). However, behavior along the line $y = x$ is ripe for exploration.