3 The intuitive idea of a limit

READING	Read Section 2.2 of Rogawski

Reading

3.1 Guessing limits

We begin with an example.

Example 1. Consider the function

$$f(x) = \frac{\sin^2(x)}{x}.$$

$f(x)$ is not defined when $x = 0$, but is defined for all real numbers. (So we can take the domain of f to be $\mathbb{R}\backslash\{0\}$.)

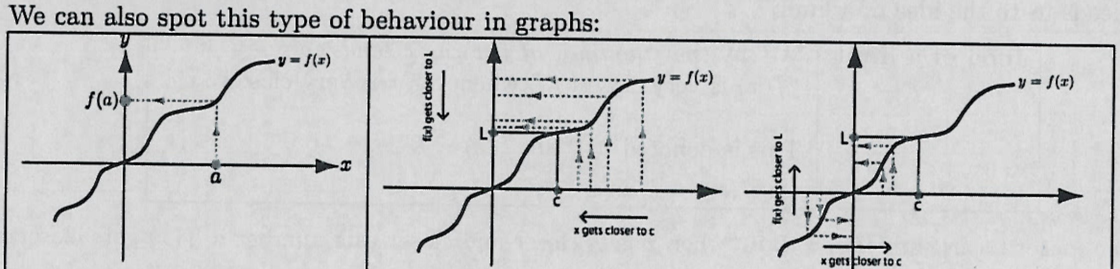
However, $f(x)$ can be calculated for values arbitrarily close to zero:

x	f(x)
3	0.006638285563
2	0.4134109052
1	0.7080734183
0.1	0.09966711080
0.01	0.009999666671
0.0001	0.00009999999966
0.000001	0.000001000000000
0.000000001	10^{-9}

x	f(x)
-0.5	-0.4596976942
-0.02	-0.01999733347
-0.01	-0.009999666671
-0.006	-0.005999928000
-0.003	-0.002999991000
-0.001	-0.0009999996666
-0.0007	-0.0006999998856
-0.000001	-0.000001000000000

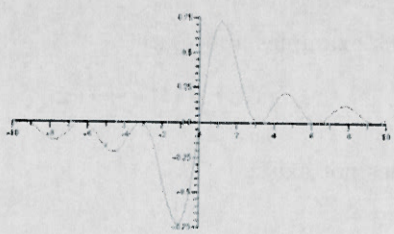
It appears that "when x is very close to 0, $f(x)$ is very close to 0."

We can also spot this type of behaviour in graphs:



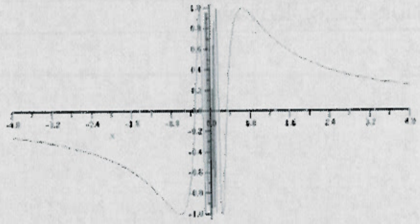
reading the value of $f(a)$ | Looking at what happens to $f(x)$ when x is close to c.

Example 2. The graph of $\frac{\sin^2(x)}{x}$:



From the graph,
It appears that "when x is very close to 0, $f(x)$ is very close to 0."
It appears that "when x is very close to $-\pi$, $f(x)$ is very close to 0."

Example 3. Let $f(x) = \sin(\frac{1}{x})$. The graph of this function is



Let's investigate numerically what happens to $f(x)$ as x gets close to 0.

x	$f(x)$
0.1	-0.5440211109
0.05	0.9129452507
0.01	0.8268795405
0.001	-0.3056143889
0.0001	0.03574879797
0.000001	-0.3499935022
0.0000000001	-0.4875060251
0.000000000000001	0.8582727932

x	$f(x)$
- 0.1	-0.5440211109
-0.05	-0.9129452507
-0.01	-0.8268795405
-0.001	0.3056143889
-0.0001	-0.03574879797
-0.000001	0.3499935022
-0.0000000001	0.4875060251
-0.000000000000001	-0.8582727932

When x is very close to zero, $f(x)$ DOES NOT GET CLOSE TO ANY NUMBER.

This leads us to the idea of a limit.

Idea of a limit	We say that *the limit of $f(x)$ as x tends to c is L* if "$f(x)$ is very close to L whenever x is very close to c". This is denoted by "$\lim_{x\to c} f(x) = L$", or "$f(x) \to L$ as $x \to c$".

In even looser terms, $\lim_{x\to c} f(x) = L$ if "when x gets closer and closer to a number c, $f(x)$ gets closer and closer to a number L".

Observe that limits do not always exist!

Example 4. The two limits from the examples above are:

$$\lim_{x\to 0} \frac{\sin^2(x)}{x} = 0 \qquad \lim_{x\to -\pi} \frac{\sin^2(x)}{x} = 0$$

Example 5. The following limit does not exist:

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \quad \text{DNE}$$

Example 6. Let $f(x) = \frac{x^2-4}{x-2}$. By calculating values, guess $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Solution.

x	f(x)
1.9	3.9
1.99	3.99
1.999	3.999
1.99999	3.99999
1.9999999	3.9999999

x	f(x)
2.1	4.1
2.01	4.01
2.0001	4.0001
2.00001	4.00001
2.0000001	4.0000001

From the values in the table, we guess that

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = 4$$

Example 7. Let $g(\theta) = \frac{\sin(\theta)}{\sin(\theta)}$. By calculating values, guess $\lim_{\theta\to\pi} g(\theta)$. *Remember to use x values both larger and smaller than π.)*

Solution.

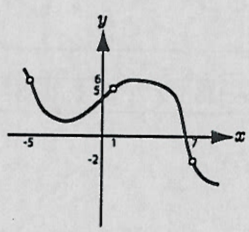
θ	$g(\theta)$
3.1	1
3.141	1
3.1415	1
3.1415922	1

θ	$g(\theta)$
3.15	1
3.142	1
3.1416	1
3.141563	1

From the values in the table, we guess that

$$\lim_{\theta\to\pi}\frac{\sin(\theta)}{\sin(\theta)} = 1$$

Example 8. Let $f(x)$ be the function with the following graph.



From the graph we can have

$$\lim_{x\to -5} f(x) = 6, \lim_{x\to 1} f(x) = 5, \lim_{x\to 7} f(x) = -2.$$

Example 9. Let $g(t) = \ln\left(\frac{\sin(t)}{t}\right)$. By calculating values, guess $\lim_{t\to 0}\ln\left(\frac{\sin(t)}{t}\right)$.

Solution.

t	g(t)
-1	-0.1726037463
-0.1	-0.001667222543
-0.01	-0.00001666673889
-0.0001	$-1.700000001 \times 10^{-9}$

t	g(t)
1	-0.1726037463
0.1	-0.001667222543
0.01	-0.00001666673889
0.0001	$-1.700000001 \times 10^{-9}$

From the values in the table, we guess that

$$\lim_{t\to 0} \ln\left(\frac{\sin(t)}{t}\right) = 0$$

Example 10. Let $h(t) = \ln\left(\frac{\sin(t)}{2t}\right)$. By calculating values, guess $\lim_{t\to 0} \ln\left(\frac{\sin(t)}{2t}\right)$.

Solution.

t	h(t)
-0.1	-0.6948144032
-0.05	-0.6935638820
- 0.01	-0.6931638473
- 0.0005	-0.6931472222
- 0.00007	-0.6931471814
-0.000001	-0.6931471806
-0.00000001	-0.6931471806

t	h(t)
0.1	-0.6948144032
0.05	-0.6935638820
0.01	-0.6931638473
0.0005	-0.6931472222
0.00007	-0.6931471814
0.000001	-0.6931471806
0.00000001	-0.6931471806

From the values in the table, we guess that

$$\lim_{t\to 0} \ln\left(\frac{\sin(t)}{2t}\right) = -0.693147806\ldots = -\ln(2)$$

Example 11. By comparing the last two examples, and considering the rules of logarithms, guess what the exact value of $\lim_{t\to 0} \ln\left(\frac{\sin(t)}{2t}\right)$ is.

$$\lim_{t\to 0} \ln\left(\frac{\sin(t)}{2t}\right) = \lim_{t\to 0}\left(\ln\left(\frac{\sin(t)}{t}\right) - \ln(2)\right) = \lim_{t\to 0}\left(\ln\left(\frac{\sin(t)}{t}\right)\right) - \lim_{t\to 0}(\ln(2))$$

$$= 0 - \ln(2) = -\ln(2)$$

HOMEWORK	Rowgowski Section 2.2: Q 1, 2, 3, 4, 7, 8, 21, 25, 29, 31

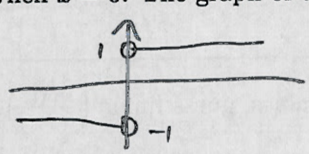
Homework

3.2 One-sided limits

Consider the function

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \end{cases}$$

This function is undefined when $x = 0$. The graph of this function is:



Observe that $\lim_{x\to 0} f(x)$ *does not exist.* This is since $f(x)$ is not close to any particular number when x is close to 0. (Depending on the value of x, $f(x)$ is close to either 1 or -1.)

However notice that:

- $f(x)$ is close to 1 when x is close to 0 and $x > 0$;
- $f(x)$ is close to -1 when x is close to 0 and $x < 0$.

This leads us to the idea of a one-sided limit.

Idea of one-sided limit	We say that the limit of $f(x)$ as x tends to c from the right is L if "$f(x)$ is very close to L whenever x is very close to c and $x > c$". This is denoted by "$\lim_{x \to c_+} f(x) = L$", We say that the limit of $f(x)$ as x tends to c from the left is L "$f(x)$ is very close to L whenever x is very close to c and $x < c$". This is denoted by "$\lim_{x \to c_-} f(x) = L$",

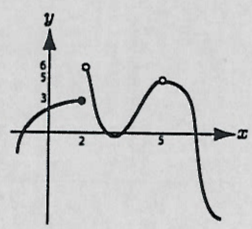
Example 12. From the above we guess

$$\lim_{x \to 0_+} \frac{x}{|x|} = 1 \qquad \lim_{x \to 0_-} \frac{x}{|x|} = -1$$

Note that

$$\boxed{\lim_{x \to c} f(x) = L \iff (\lim_{x \to c_+} f(x) = L \text{ and } \lim_{x \to c_-} f(x) = L)}$$

Example 13. Let $f(x)$ be given by the graph



Then

$$\lim_{x \to 2_-} f(x) = 2 \quad \lim_{x \to 2_+} f(x) = 6 \quad \lim_{x \to 5_-} f(x) = 5 \quad \lim_{x \to 5_+} f(x) = 5$$

Example 14. Investigate the one-sided limits $\lim_{x \to 0_\pm} |x|^{(\frac{1}{x})}$ numerically.

Solution.

x	$f(x)$
0.5	0.25
0.3	0.01807468966
0.2	0.00032
0.1	10^{-10}
0.05	$9.536743164 \times 10^{-27}$

x	$f(x)$
-0.5	4
-0.3	55.32598450
-0.2	3125.000000
-0.1	10^{10}
-0.05	$1.048576000 \times 10^{26}$

$\lim_{x \to 0_+} |x|^{(\frac{1}{x})} = 0$

$\lim_{x \to 0_-} |x|^{(\frac{1}{x})} = \text{DNE}$ (or ∞)

HOMEWORK	Rowgowski Section 2.2: Q 6, 37, 38, 39, 41, 43

Homework

4 Asymptotes

4.1 Guessing limits

We begin with an example.

Example 15. Consider the function

$$f(x) = \frac{1}{|x|}.$$

$f(x)$ is not defined when $x = 0$. Lets investigate what happens to $f(x)$ when x is close to 0.

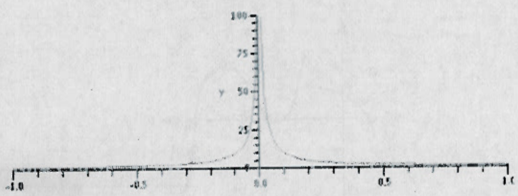
x	f(x)
0.1	10

x	f(x)
-0.1	10

We see that "when x is very close to 0, $f(x)$ is a very large positive number. In fact, we see we can make $f(x)$ as large as we like by taking x close enough to zero.

In such a situation, we say that "as x tends to 0, $f(x)$ tends to infinity".

We can also see the above behaviour in the graph of $1/|x|$:



Idea of an infinite limit	We say that *the limit of $f(x)$ as x tends to c is ∞* if "$f(x)$ becomes arbitrarily large and positive whenever x is close enough to c". This is denoted by "$\lim\limits_{x \to c} f(x) = \infty$", We say that *the limit of $f(x)$ as x tends to c is $-\infty$* if "$f(x)$ becomes arbitrarily large and negative whenever x is close enough to c" This is denoted by "$\lim\limits_{x \to c} f(x) = -\infty$", We say that $\lim\limits_{x \to c_-} f(x) = -\infty$ if "$f(x)$ becomes arbitrarily large and negative whenever x is close enough to c and $x < c$" and so on.

Example 16.

$$\lim_{x \to 0} \frac{1}{|x|} = \infty.$$

Example 17. Numerically investigate the one-sided limits of $f(x) = \frac{1}{x+2}$ as $x \to -2$.