1 Inverse functions

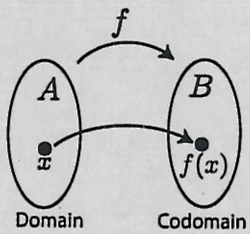
READING	Read Section 1.5 of Rogawski

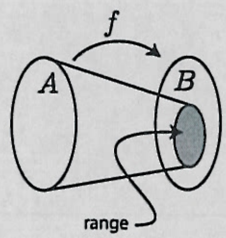
Reading

1.1 A very fast review of functions

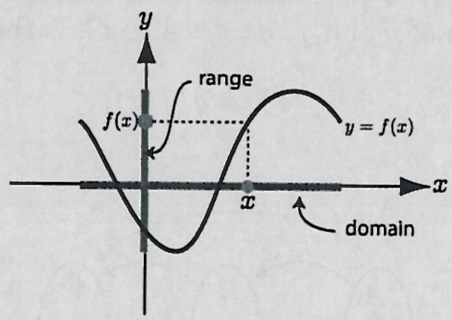
A **function**, $f : A \to B$, from a set A to a set B is a rule that assigns *exactly one* element $f(a)$ in B to each element a in A.

A is called the **domain** of f, B is called the **codomain** of f, and the **range** of f is the set $\{f(a) \mid a \in B\}$. The range is the set of all elements in the codomain, B, that are an image of an element in the domain, A.





In terms of graphs:



Recall that the graph of a function passes the **vertical line test**: every vertical line $x = a$ intersects the graph in *at most* one point.

Example 1.1. Let $f(x) = \sin(x)$ be a function from the real numbers to the numbers (this is denoted by $f : \mathbb{R} \to \mathbb{R}$). Write down the domain, codomain and the range of f. Suppose that $g(x) = \sin(x)$ with domain $[0, \pi]$, what is the range of $g(x)$?

Solution.

For f: domain $\mathbb{R}$, codomain $\mathbb{R}$, range $[-1,1]$

Example 1.2. What is the largest possible domain of the function $f(x) = \frac{x^7+9x^2-2}{\sin(3x)}$?

Solution.

Defined as long as $\sin(3x) \neq 0$

$\sin(t) = 0$ when $t = \pm k\pi$, for $k \in \mathbb{Z}$[1]

So $\sin(3x) = 0$ when $t = \pm \frac{k\pi}{3}$

Example 1.3. What is the largest possible (real) domain of the function $h(x) = \frac{|x|}{x}$? With this domain, if the codomain is $\mathbb{R}$, what is the range of $h(x)$?

Solution. $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

range is $\{-1, +1\}$

Question 1.4. Why the vertical line test hold? *(Hint: what would happen if it did not hold?)*

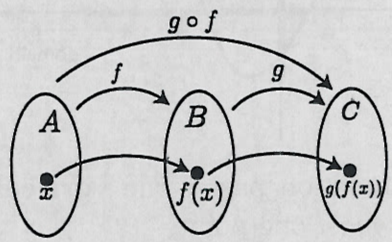
Answer. Suppose $x = a$ intersected the graph at (a, b) and (a, c)

then $f(a) = b$ and $f(a) = c$

But a function assigns one element to a, so $b = c$.

Let $f : A \to B$ and $g : B \to C$ be functions such that the range of f is a subset of the domain of g. The **composite** of g and f, $g \circ f : A \to C$, is the function defined by

$$g \circ f(x) := g(f(x)).$$



Example 1.5. Let $f(x) = \frac{x^2+7}{2x}$ and $g(x) = \sin(x+2)$. Find $f \circ g(x)$ and $g \circ f(x)$.

Solution.

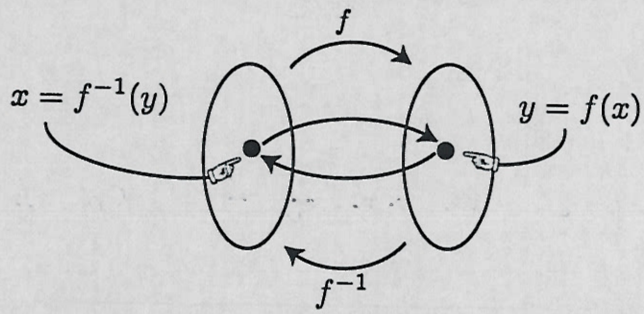
$g \circ f(x) = g(f(x)) = g\left(\frac{x^2+7}{2x}\right) = \sin\left(\frac{x^2+7}{2x} + 2\right) = \sin\left(\frac{x^2+4x+7}{2x}\right)$

$f \circ g(x) = f(g(x)) = f(\sin(x+2)) = \frac{\sin^2(x+2)+7}{2\sin(x+2)}$

1.2 Inverse functions

1.2.1 The definition of an inverse

The inverse, $f^{-1}(x)$, of a function $f(x)$, is a function that "undoes the effect of $f(x)$". For example, the inverse of the function $f(x) = x^3$ is the function $f(x) = \sqrt[3]{x}$.



Definition 1.6. Let $f(x)$ be a function with domain D and range R. The **inverse** $f^{-1}(x)$ (if it exists) is the function with domain R such that

$$f^{-1}(f(x)) = x, \quad \text{for each } x \text{ in } D,$$

and

$$f\left(f^{-1}(x)\right) = x, \quad \text{for each } x \text{ in } R.$$

If such a function f^{-1} exists, f is said to be **invertible**.

Note that not all functions have inverses. Also note that, in general, $f^{-1}(x) \neq 1/f(x)$.

Example 1.7. Verify that $g(x) = \frac{1-3x}{x-1}$ is the inverse of $f(x) = \frac{x+1}{x+3}$.

Solution. We need to verify that $g(f(x)) = x$ and that $f(g(x)) = x$.

$$g(f(x)) = g\left(\frac{x+1}{x+3}\right) = \frac{1 - 3\left(\frac{x+1}{x+3}\right)}{\left(\frac{x+1}{x+3}\right) - 1} = \frac{\left(\frac{(x+3)-3(x+1)}{x+3}\right)}{\left(\frac{(x+1)-(x+3)}{(x+3)}\right)} = \frac{-2x}{-2} = x.$$

Similarly,

$$f(g(x)) = f\left(\frac{1-3x}{x-1}\right) = \frac{\left(\frac{1-3x}{x-1} + 1\right)}{\left(\frac{1-3x}{x-1} + 3\right)} = \frac{\left(\frac{1-3x+x-1}{x-1}\right)}{\left(\frac{1-3x+3x-3}{x-1}\right)} = \left(\frac{-2x}{x-1}\right)\left(\frac{x-1}{-2}\right) = x$$

Example 1.8. Give an example of (a) a function that does not have an inverse; (b) a function such that $f^{-1}(x) = 1/f(x)$; and (c) a function such that $f^{-1}(x) \neq 1/f(x)$

Solution. (a) $f(x) = x^2$

(b) $f(x) = 2$

(c) $f(x) = x^3$

1.2.2 Finding the inverse

To find $f^{-1}(x)$ *(if it exists)*	**Step 1.** Set $y = f(x)$. **Step 2.** Solve for x in terms of y. **Step 3.** Switch the variables x and y.

Example 1.9. Find the inverse of $f(x) = \frac{2x-1}{x+2}$.

Solution. $y = \frac{2x-1}{x+2}$

Solve for y : $y(x+2) = 2x-1 \Leftrightarrow yx + 2y = 2x-1 \Leftrightarrow (yx-2x) = -1-2y$

$\Leftrightarrow (y-2)x = -1-2y \Leftrightarrow x = \frac{-1-2y}{y-1}$ So $f^{-1}(x) = \frac{-1-2x}{x-1}$

HOMEWORK	Rowgowski Section 1.5: Q 1, 4, 5, 6

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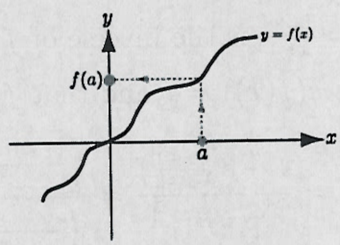
1.2.3 The graph of an invertible function

It is easy to decide whether or not a function has an inverse by looking at its graph.

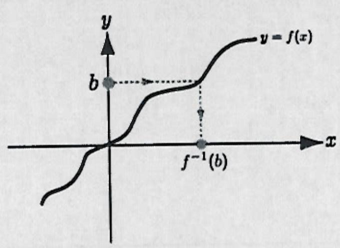
Consider the graph of $y = f(x)$ To find $f(a)$:

(1) travel vertically from the point a on the x-axis to the graph.
(2) travel horizontally from that point of the graph to the y-axis. This point on the y-axis gives the value of $f(a)$.



If f is invertible, then we can find the value of $f^{-1}(b)$ from the graph of $f(x)$ as follows:

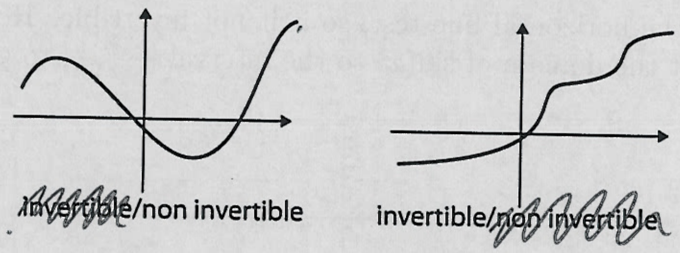
(1) travel horizontally from the point b on the y-axis to the graph.
(2) travel vertically from that point of the graph to the x-axis. This point on the x-axis gives the value of $f^{-1}(b)$.



Notice that this process works if and only if when we travel horizontally from b we can only meet one point (otherwise we can't decide which at point on the graph we start travelling vertically).

These considerations give rise to the **horizontal line test**: f is invertible if and only if every horizontal line meets its graph at most once.

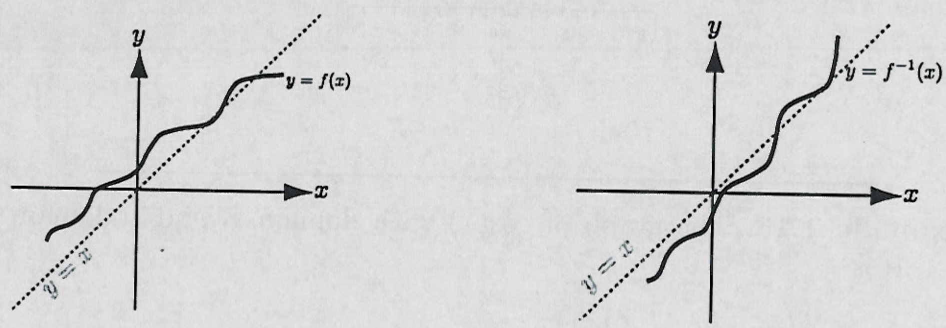
Example 1.10.



The above considerations also tell us how to find the graph of f^{-1} from the graph of f: simply "switch the x-axis and the y-axis". More formally:

To find the graph of $f^{-1}(x)$	Reflect the graph of f^{-1} in the line $y = x$.

Example 1.11.

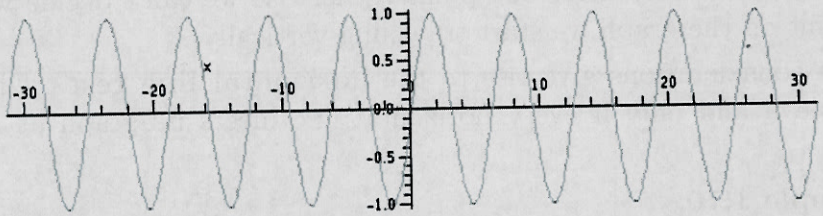


1.2.4 Restricting the domain

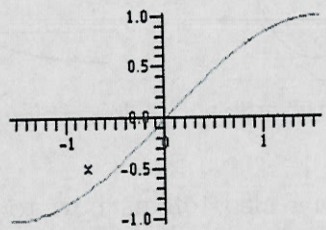
There are many functions that are not invertible, even though we would like them to be. We can **restrict the domain** of a non-invertible function to obtain an invertible function. Intuitively, we take a "vertical slice" of the graph of a function in such a way that the new graph passes the horizontal line test. This is illustrated in the following example.

Example 1.12. The graph of $\sin(x)$ with domain $\mathbb{R}$ and codomain

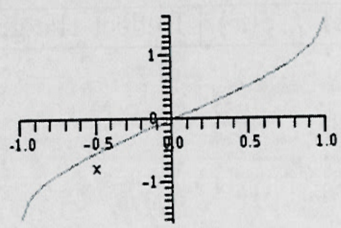
$[-1, 1]$ is:



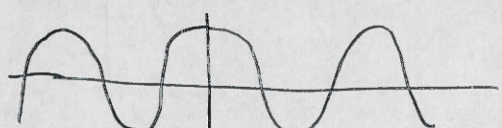
This fails the horizontal line test, so it it not invertible. However, we can restrict the domain of $\sin(x)$ to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to get:



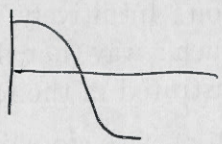
This is an invertible function (as it passes the horizontal line test). The graph of its inverse, $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, is



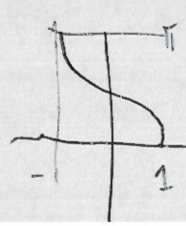
Example 1.13. The graph of $\cos(x)$ with domain $\mathbb{R}$ and codomain $[-1, 1]$ is:



This fails the horizontal line test, so it it not invertible. However, we can restrict the domain of $\cos(x)$ to the interval $[0, \pi]$ to get the graph:



This is an invertible function (as it passes the horizontal line test). The graph of its inverse, $\cos^{-1} : [-1, 1] \to [0, \pi]$, is



HOMEWORK	Rowgowski Section 1.5: Q 16, 17

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2 Exponentials and logarithms

READING	Read Section 1.6 of Rogawski

Reading

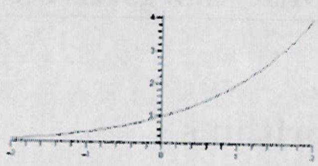
2.1 Exponential functions

Let b be a real number such that $b > 0$ and $b \neq 1$. Then the **exponential function** to the base b is the function

$$f(x) = b^x.$$

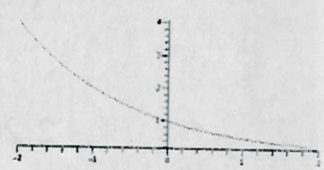
We have:

- $b^x > 0$ for all numbers x.
- $f(x) = b^x$ is an increasing function if $b > 1$; a decreasing function if $b < 1$.
- The range of $f(x) = b^x$ is the interval $(0, \infty)$.
- For each b, the graph of $f(x) = b^x$ will pass through the point $(0, 1)$.
- The function $f(x) = e^x$, where $e \approx 2.71828183\cdots$, is very important (we will see why later in the course).
- Typical graphs of $y = b^x$ are:



when $b > 1$

and



when $b < 1$.

Exponentials satisfy various laws:

$$b^x = b^y \iff x = y$$
$$b^x \cdot b^y = b^{x+y}$$
$$b^0 = 1$$

$$\frac{b^x}{b^y} = b^{x-y}$$
$$b^{-x} = \frac{1}{b^x}$$
$$(b^x)^y = b^{xy}$$
$$b^{1/n} = \sqrt[n]{b}.$$

Example 2.1. Simplify $(ab)^2(a^{-2}+b^{-2})$.

Solution. $= (ab)^2 a^{-2} + (ab)^2 b^{-2} = a^0 b^2 + a^2 b^0 = b^2 + a^2$

Example 2.2. Solve the equation $(a^2)^{y+1} = a^{-6}$ for a.

Solution.

$(a^2)^{y+1} = a^{-6} \Leftrightarrow a^{2y+2} = a^{-6}$

$\Leftrightarrow 2y+2 = -6 \Leftrightarrow y+1 = -3 \Leftrightarrow y = -4$

Example 2.3. Solve the equation $b^4 = 10^{12}$ for $b > 0$.

Solution.

$$b^4 = 10^{12} \iff (b^4)^{1/4} = (10^{12})^{1/4} \iff b^{4/4} = 10^{12/4}$$

Since $b > 0$ this happens if and only if $b = 10^3$.

Warning: remember $(a+b)^x \neq a^x + b^x$. Also notice that when b is a number $b^{-1} = 1/b$, however, when f is a function, f^{-1} denotes the inverse function of f.

HOMEWORK	Rowgowski Section 1.6: Q 1, 3, 5, 7, 9

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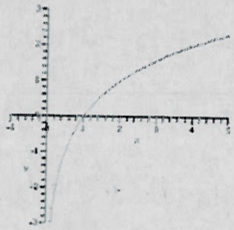
2.2 Logarithms

Since the graph of $y = b^x$ satisfies the horizontal line test, it is invertible. The inverse of $f(x) = b^x$ is $f^{-1}(x) = \log_b(x)$.

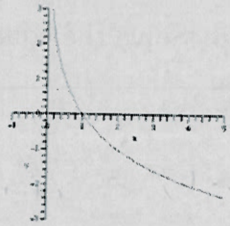
- $\log_b(x)$ is the **logarithm** to the base b.
- $\log_e(x)$ is important and is denoted by $\ln(x)$. This is called the **natural logarithm**.

- Typical graphs of $y = \log_b(x)$ are:



when $b > 1$

and



when $b < 1$.

By definition we have

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$$b^{\log_b(x)} = x \quad \text{and} \quad \log_b(b^x) = x.$$

In addition,

$$\log_b(x) = \log_b(y) \quad \Longleftrightarrow \quad x = y.$$

Logarithms satisfy the following very useful laws:

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$\log_b(1) = 0$	$\log_b(b) = 1$
$\log_b(xy) = \log_b(x) + \log_b(y)$	$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
$\log_b(\frac{1}{x}) = -\log_b(x)$	$\log_b(x^n) = n\log_b(x)$

Example 2.4. Evaluate $\ln\left(\sqrt{e}\cdot e^{7/5}\right)$.

Solution. $= \ln\left(e^{\frac{1}{2}}e^{7/5}\right) = \ln\left(e^{\frac{1}{2}+\frac{7}{5}}\right) = \ln\left(e^{\left(\frac{5}{10}+\frac{14}{10}\right)}\right)$

$= \ln\left(e^{19/10}\right) = \frac{19}{10}$

Example 2.5. Solve the equation $6e^{-4t} = 2$.

Solution.

$6e^{-4t} = 2 \iff e^{-4t} = \frac{1}{3} \iff -4t = \ln\left(\frac{1}{3}\right)$

$\iff t = -\frac{1}{4}\ln\left(\frac{1}{3}\right) = \frac{1}{4}\ln\left(\left(\frac{1}{3}\right)^{-1}\right) = \frac{1}{4}\ln(3)$

$= \ln\left(3^{\frac{1}{4}}\right) = \ln\left(\sqrt[4]{3}\right)$

Example 2.6. Solve the equation $\ln(x^2+1) - 3\ln(x) = \ln(2)$.

Solution. Note $\ln(a) = \ln(b) \iff a = b$

$$\ln(x^2+1) - 3\ln(x) = \ln(2) \iff \ln(x^2+1) - \ln(x^3) = \ln(2)$$

$$\iff \ln\left(\frac{x^2+1}{x^3}\right) = \ln(2) \iff \frac{x^2+1}{x^3} = 2 \iff 2x^3 - x^2 - 1 = 0$$

by inspection $x=1$ is a root so

$$(2x^3 - x^2 - 1) = (x-1)(\underbrace{2}_{\text{must be } 2}x^2 + x + \underbrace{1}_{\text{must be } 1})$$

solving the quadratic $x = \dfrac{-1 \pm \sqrt{1^2 - 8}}{4}$ no real solution

so $x=1$ is the only solution.

HOMEWORK	Rowgowski Section 1.6: Q 11, 13, 21, 23, 24, 25, 27, 29, 43, 44

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