Theorem $(A \cup B) \cap (A \cup B^c) = A$

proof

We first show $A \subseteq (A \cup B) \cap (A \cup B^c)$. Let $x \in A$. Then $x \in A \cup B$ and $x \in A \cup B^c$. It then follows that $x \in (A \cup B) \cap (A \cup B^c)$. Therefore $A \subseteq (A \cup B) \cap (A \cup B^c)$

To show that $A \supseteq (A \cup B) \cap (A \cup B^c)$, let $x \in (A \cup B) \cap (A \cup B^c)$. Then $x \in A \cup B$ and $x \in A \cup B^c$. Since $x \in A \cup B$, $x \in A$ or $x \in B$. We will show by contradiction that x must be in A. Suppose that $x \notin A$. Then since $x \in A \cup B$, $x \in B$. Also since $x \in A \cup B^c$, then $x \in B^c$. Therefore $x \in B$ and $x \in B^c$, this is a contradiction, so $x \in A$.

$\square$

Theorem $A \subseteq B \Rightarrow A \setminus C \subseteq B \setminus C$

proof Suppose that $A \subseteq B$. Let $x \in A \setminus C$. Then $x \in A$ and $x \notin C$. Since $A \subseteq B$, $x \in B$. So $x \in B$ and $x \notin C$. Thus $x \in B \setminus C$.

$\square$

Theorem $C \subseteq A$ and $D \subseteq B \Rightarrow C \cup D \subseteq A \cup B$.

proof Suppose that $C \subseteq A$ and $D \subseteq B$. Let $x \in C \cup D$. Then $x \in C$ and $x \in D$. Since $C \subseteq A$, $x \in A$. Since $D \subseteq B$, $x \in B$. So $x \in A$ and $x \in B$, and so $x \in A \cup B$.

$\square$

Using only the facts that $\frac{d}{dx}x=1$ and the product rule, prove that $\frac{d}{dx}x^n = nx^{n-1}$, for all $n \in \mathbb{N}$

Solution We prove this using induction.

Base case: If $n=1$ then $\frac{d}{dx}x^1 = 1 = 1x^0$

Inductive hypothesis: Assume the formula holds for $n=k$

Inductive step: Let $n=k+1$. Then

$$\frac{d}{dx}x^n = \frac{d}{dx}x(x^{n-1}) = x^{n-1}\left(\frac{dx}{dx}\right) + x\left(\frac{d}{dx}x^{n-1}\right), \text{ by the product rule}$$

$$= x^{n-1} + x(n-1\,x^{n-2}) \quad \text{by the inductive hyp. and the fact that } \frac{dx}{dx}=1$$

$$= x^{n-1} + (n-1)x^{n-1} = nx^{n-1},$$

as required.

Theorem $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$

Proof

Base case if $n=1$, then $\sum_{i=1}^{n} i\,i! = 1 \cdot 1! = 1 = (1+1)! - 1$

Inductive hyp Assume the formula holds for $n=k$.

Inductive step

$$\sum_{i=1}^{k+1} i \cdot i! = \left(\sum_{i=1}^{k} i \cdot i!\right) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)!\,(1+k+1) - 1 = (k+1)!\,(k+2) - 1 = (k+2)! - 1,$$

as required.