HW2 Partial Solutions (The remaining questions were solved in class.)

A1 (i) $\displaystyle \lim_{x\to\infty} \frac{7x-9}{4x^2+3} = \lim_{x\to\infty} \frac{7x-9}{4x^2+3} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\left(\frac{7x}{x^2} - \frac{9}{x^2}\right)}{\left(\frac{4x^2}{x^2} + \frac{3}{x^2}\right)} = \frac{\lim_{x\to\infty}\left(\frac{7}{x}\right) - 9\left(\lim_{x\to\infty}\frac{1}{x}\right)^2}{\lim_{x\to\infty}(4) + 3\left(\lim_{x\to\infty}\frac{1}{x}\right)^2} = 0$

(ii) $\displaystyle \lim_{x\to\infty} \frac{3x^2+20x}{2x^2+9} = \lim_{x\to\infty} \frac{3x^2+20x}{2x^2+9} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\left(3+\frac{20}{x}\right)}{\left(2+\frac{9}{x^2}\right)} = \frac{\left(\lim_{x\to\infty} 3\right) + 20\lim_{x\to\infty}\left(\frac{1}{x}\right)}{\left(\lim_{x\to\infty} 2\right) + 9\left(\lim_{x\to\infty}\frac{1}{x}\right)^2} = \frac{3}{2}$

(iii) and (iv) are solved similarly. The limits are both zero

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B1

7) $\displaystyle \lim_{x\to 2}\left(\frac{x^2-3x+2}{x-2}\right) = \lim_{x\to 2}\left(\frac{(x-2)(x-1)}{x-2}\right) = \lim_{x\to 2}(x-1) = 1$

9) $\displaystyle \lim_{x\to 2}\left(\frac{x-2}{x^3-4x}\right) = \lim_{x\to 2}\left(\frac{x-2}{x(x-2)(x+2)}\right) = \lim_{x\to 2}\left(\frac{1}{x(x+2)}\right) = \frac{1}{8}$

11) $\displaystyle \lim_{h\to 0}\left(\frac{(1+h)^3-1}{h}\right) = \lim_{h\to 0}\left(\frac{1+3h+3h^2+h^3-1}{h}\right) = \lim_{h\to 0}(3+3h+h^2) = 3$

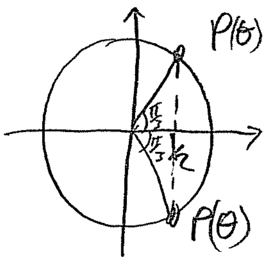
B2

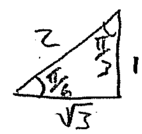
43) $\displaystyle \lim_{x\to 0}(2a+x) = 2\left(\lim_{x\to 0} a\right) + \left(\lim_{x\to 0} x\right) = 2a$

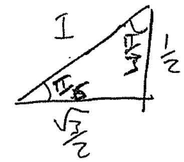
51) $\displaystyle \lim_{x\to 0}\left(\frac{(x+a)^3-a^3}{x}\right) = \lim_{x\to 0}\left(\frac{x^3+3x^2a+3xa^2+a^3-a^3}{x}\right) = \lim_{x\to 0}\left(x^2+3xa+3a^2\right) = 3a^2$

C2

(i)

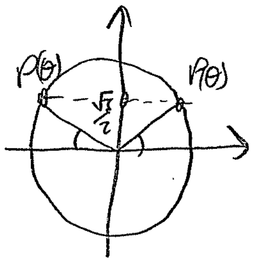


using 

which is similar to 

$\theta = \frac{\pi}{3}$ or $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

(ii)



$\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

(iii)

$\cos^2(\theta) + \sin(\theta)\cos(\theta) = 0 \iff \cos(\theta)\left[\cos\theta + \sin(\theta)\right] = 0$

$\iff \cos\theta = 0$ or $\cos(\theta) = -\sin(\theta)$

$\cos(\theta) = 0$ if $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\cos\theta = -\sin\theta$ $\qquad$ $P(\theta)$ is of form $(a, -a)$

$P(\theta) = (a, -a)$ is on unit circle so $a^2 + (-a)^2 = 1$ so $2a^2 = 1$

so $a = \frac{\pm 1}{\sqrt{2}}$

So $P(\theta) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ or $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

solving as usual gives $\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

So

$\cos^2\theta + \sin\theta\cos\theta = 0$ when $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}$

E1

i) $\lim_{x\to 0}\left(\frac{\sin x\cos x}{4x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\cos x}{4}\right) = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\cos x}{4}\right) = \frac{1}{4}$

ii) $\lim_{t\to 0}\left(\frac{\sin^2 t}{t}\right) = \lim_{t\to 0}\left(\frac{\sin t}{t}\cdot\sin t\right) = \left(\lim_{t\to 0}\frac{\sin t}{t}\right)\left(\lim_{t\to 0}\sin t\right) = 0$

iii) $\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right) = \lim_{x\to 0}\frac{1}{\left(\frac{\sin^2 x}{x^2}\right)} = \frac{\left(\lim_{x\to 0} 1\right)}{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)} = \frac{1}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2} = 1$

iv)
put $\theta = 6x$

$\lim_{x\to 0}\left(\frac{\sin 6x}{x}\right) = \lim_{\theta\to 0}\left(\frac{\sin\theta}{(\theta/6)}\right) = 6\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 6$

(v)
Similar to (iv), but use $\theta = 7t$

(vi) in class.

E2 in class.

B1 Q19

$$\lim_{h\to 0}\left(\frac{\sqrt{2+h}-2}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{2+h}-2}{h}\cdot\frac{\sqrt{2+h}+2}{\sqrt{2+h}+2}\right) = \lim_{h\to 0}\left(\frac{2+h-4}{h(\sqrt{2+h}+2)}\right)$$

$$= \lim_{h\to 0}\left(\frac{h-2}{h(\sqrt{2+h}+2)}\right)$$

You can convince yourself very easily that $\lim_{h\to 0_+}\left(\frac{h-2}{h(\sqrt{2+h}+2)}\right) = -\infty$ and $\lim_{h\to 0_-}\left(\frac{h-2}{h(\sqrt{2+h}+2)}\right) = \infty$, so the limit does not exist.

However, it is a pain to show this is true:

Case 1

Let $0<h<1$, then $(h-2) \leq -1$. Since $h(\sqrt{2+h}+2)$ is positive, we have

$$\frac{h-2}{h(\sqrt{2+h}+2)} \leq \frac{-1}{h(\sqrt{2+h}+2)} \leq \frac{-1}{\sqrt{2+h}+2} \leq \frac{-1}{(2+h)+2} \leq \frac{-1}{h+4} \to -\infty \text{ as } h\to 0_+$$

So $\lim_{h\to 0_+}\frac{h-2}{h(\sqrt{2+h}+2)} \to -\infty$

Case 2

$-1 \leq h < 0$

$$\frac{h-2}{h(\sqrt{2+h}+2)} \geq \frac{-2}{h(\sqrt{2+h}+2)} \to \infty \quad \text{as} \quad x\to 0_-$$

So $\lim_{h\to 0_-}\frac{h-2}{h(\sqrt{2+h}+2)} = \infty$