Example 5: Calculate the derivative of $-2\cos(x^2\sin(x))$ with respect to x.

Solution: $-2\cos(x^2\sin(x)) = f(g(x))$ when $f(x) = -2\cos(x)$ and $g(x) = x^2\sin x$

The chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Let $u = g(x) = x^2\sin(x)$

$y = -2\cos(u)$

$\frac{dy}{du} = 2\sin(u)$

To find $u'(x)$ we need to use the product rule.

$\frac{du}{dx} = \left(\frac{d}{dx}x^2\right)\sin(x) + x^2\left(\frac{d}{dx}\sin(x)\right) = 2x\sin(x) + x^2\cos(x)$

so

$\frac{d}{dx}(-2\cos(x^2\sin(x))) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2\sin(u)\left(2x\sin(x) + x^2\cos(x)\right)$

$= 2x\sin(x^2\sin(x))(2\sin(x) + x\cos(x))$

Example 6: Calculate the derivative of $\sqrt{\frac{x+1}{x-1}}$ with respect to x.

Solution:

$\sqrt{\frac{x+1}{x-1}} = f(g(x))$ when $f(x) = \sqrt{x}$ and $g(x) = \frac{x+1}{x-1}$.

The chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Let $u = g(x) = \frac{x+1}{x-1}$

$y = \sqrt{u}$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$

To find $u'(x)$ we need to use the quotient rule.

$\frac{du}{dx} = \frac{d}{dx}\frac{x+1}{x-1} = \frac{\left(\frac{d}{dx}(x+1)\right)(x-1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

so

$\sqrt{\frac{x+1}{x-1}} = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot \left(\frac{-2}{(x-1)^2}\right) = \frac{-1}{\sqrt{\frac{x+1}{x-1}}\,(x-1)^2} = \frac{-1}{(x-1)^{3/2}(x+1)^{1/2}}$

$= \frac{-1}{\sqrt{(x-1)^3(x+1)}}$

Example 3: Calculate the derivative of $(x + \sin(x))^4$.

Solution: $(x + \sin(x))^4 = f(g(x))$ when $f(x) = x^4$ and $g(x) = x + \sin(x)$

The chain rule is

$$(f(g(x))' = f'(g(x)) \cdot g'(x).$$

We have

$f'(x) = 4x^3$

$f'(g(x)) = 4(x+\sin(x))^3$

$g'(x) = 1 + \cos(x)$

so

$\frac{d}{dx}(x + \sin(x))^4 = (f(g(x))' = f'(g(x)) \cdot g'(x) = 4(x+\sin(x))^3(1+\cos x)$

Example 4: Calculate the derivative of e^{x^2}.

Solution: $e^{x^2} = f(g(x))$ when $f(x) = e^x$ and $g(x) = x^2$.

The chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Let $u = g(x) =$ then

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = 2x$

so

$\frac{d}{dx}e^{x^2} = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot 2x = 2xe^{x^2}$

Example 7: Calculate the derivative of $\sin(e^{x^2+x})$

Solution: $\sin(e^{x^2+x}) = f(g(x))$ when $f(x) = \sin(x)$ and $g(x) = e^{x^2+x}$. The chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Let $u = g(x) =$ ~~then~~ e^{x^2+x}

$y = \sin(u)$

and so

$\frac{dy}{du} = \cos(u)$

We also need to find

$\frac{du}{dx} = \frac{d}{dx}e^{x^2+x}$

To calculate this derivative we need to use the chain rule again:

$e^{x^2+x} = p(q(x))$ where $p(x) = e^x$ and $q(x) = x^2+x$

The chain rule is

$$\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx}$$

(notice the change in variables).

Let $t = q(x) =$ ~~then~~ x^2+x

$u = e^t$

$\frac{du}{dt} = e^t$

$\frac{dt}{dx} = 2x+1$

so

$\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = e^t(2x+1) = (2x+1)e^{x^2+x}$

Finally,

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos(u)\,(2x+1)e^{x^2+x} = (2x+1)e^{x^2+x}\cos(e^{x^2+x})$