Discrete symmetries in statistical image analysis and beyond

Alexey Koloydenko

Department of Mathematics
Royal Holloway University of London

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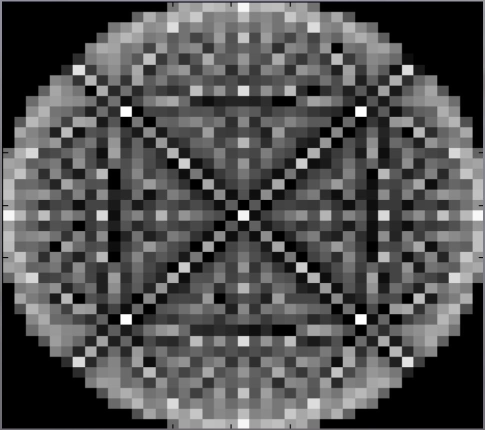
A natural image, collection of van Hateren (J. H. van Hateren, A. van der Schaaf 1998)

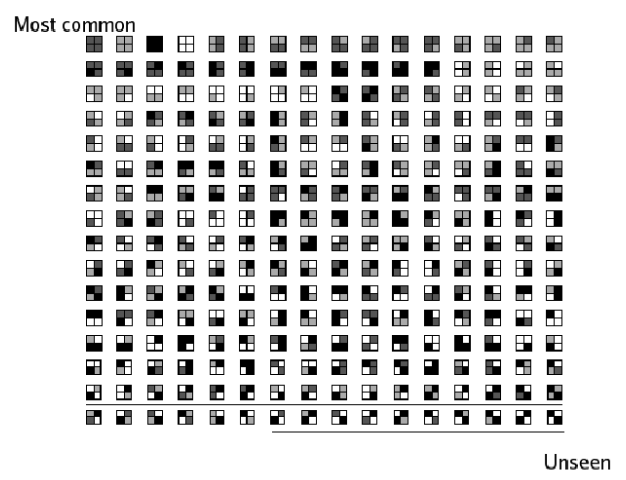
A joint histogram of contrast-normalized differences

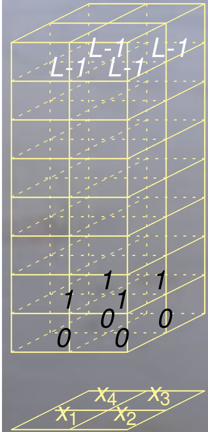
$$(x_4 + x_3 - x_1 - x_2, x_4 + x_1 - x_3 - x_2)\,/\,c(\mathbf{x})$$

within microimage $\mathbf{x} = \begin{matrix} x_4 & x_3 \\ x_1 & x_2 \end{matrix}$

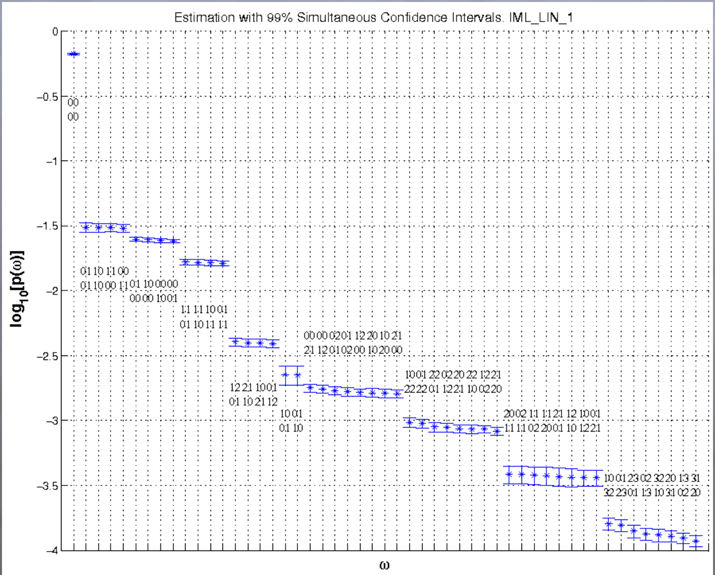


Modes: $\left\{ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}, \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\}$; $\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}, \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$; $\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}, \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix}, \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}, \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}$





Micropatterns $\mathbf{x} \in \Omega = \mathrm{Mat}_{2\times 2}([L])$, $L = 4$, ranked by frequency. Symmetry group of square cuboid: $G = D_{4h} \cong D_8 \times C_2$ (Geman, Koloydenko 1998; Koloydenko, Geman 2006)



Estimation with 99% Simultaneous Confidence Intervals. IML_LIN_1
0
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
log10[p(ω)]
ω
00
00
01 10 11 00
01 10 00 11
01 10 00 00
00 00 10 01
11 11 10 01
01 10 11 11
00 00 02 01 12 20 10 21
21 12 01 02 00 10 20 00
12 21 10 01
01 10 21 12
10 01
01 10
10 01 22 02 20 22 12 21
22 22 01 12 21 10 02 20
20 02 11 11 21 12 10 01
11 11 02 20 01 10 12 21
10 01 23 02 32 20 13 31
32 23 01 13 10 31 02 20

Full Group G of Microimage Symmetries

$$p(\mathbf{x}) = p(g\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \ \forall g \in G$$

$$r = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad i = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Presentation of
$G = \langle r, s, i | r^4 = s^2 = i^2 = 1, si = is, ri = ir, rs = sr^3 \rangle$.

Note: Ω is translated down to be centered around $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ in order for i to act as $-1\times$

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- G-action on $\mathbb{R}[x_1, x_2, x_3, x_4]$ is defined as follows:

$$rx_1 = x_2; \quad rx_2 = x_3; \quad rx_3 = x_4; \quad rx_4 = x_1;$$
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$$ix_k = -x_k, \; k = 1, 2, 3, 4$$

- *Theory of polynomial invariants of finite groups* (D. Hilbert 1890, E. Noether 1916):
- $\mathbb{R}[x_1, x_2, \ldots, x_m]^G$ has a finite set of generators $f_1(x_1, x_2, \ldots, x_m), f_2(x_1, x_2, \ldots, x_m), \ldots, f_N(x_1, x_2, \ldots, x_m)$

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Theorem

$$\begin{aligned} f_1(x) &= (x_1 + x_3)(x_2 + x_4), \\ f_2(x) &= x_1 x_3 + x_2 x_4, \\ f_3(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ f_4(x) &= x_1 x_2 x_3 x_4, \\ f_5(x) &= (x_1^2 + x_3^2)(x_2^2 + x_4^2). \end{aligned}$$

Also,

$$\mathbb{R}[x_1, x_2, x_3, x_4]^G \overset{(f_1,\dots,f_5)}{\cong} \mathbb{R}[w_1, w_2, w_3, w_4, w_5]/J_F, \text{ where}$$

$$J_F = \{h \in \mathbb{R}[w_1, w_2, w_3, w_4, w_5] : h(f_1, f_2, f_3, f_4, f_5) = 0 \in \mathbb{R}[x_1, x_2, x_3, x_4]\} = \langle q \rangle, \text{ and}$$

$$q(w_1, w_2, w_3, w_4, w_5) = 4w_1^2 w_3 + 8w_1 w_2 w_5 + 2w_1 w_3 w_5 - 2w_1 w_4^2 w_5 + \\ +16w_2^2 - 8w_2 w_3 - 8w_2 w_4^2 + 4w_2 w_5^2 + w_3^2 - 2w_3 w_4^2 + w_4^4.$$

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General measure: ordinary mixed moments		G-invariant measure: *mixed G-invariant moments*
$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m}$	$\leftrightarrow$	$f_1^{\beta_1} f_2^{\beta_2} \dots f_N^{\beta_N}$

Symbolic algebra tools (e.g. Gap, INVAR, Macaulay2, Magma) are readily available to produce generators $f_1, f_2, \dots, f_N$ (and J_F, relations on them).

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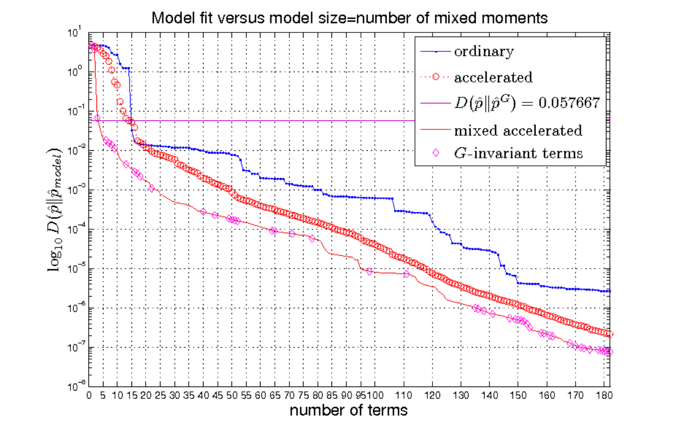

Log-linear models with increasing number of ordinary and *G*-invariant terms are fit to natural microimage data. The maximal *G*-invariant model has 30 free parameters but its four parameter simplification with three nonconstant *G*-invariant terms (f_3, f_1, and f_3^2) nearly achieves the best *G*-invariant fit. These together with *G*-invariant terms ($f_1 f_3, f_4, f_3^3$) are also included by greedy ("accelerated") constructor in the best-fitting ten parameter model.

Implications for statistical modelling

1. Equality constraints $\leftrightarrow$ structure of Ω
2. Ordered levels $[L] = \{1, 2, \ldots, L\} \Rightarrow \Omega \subset \mathbb{R}^m$
3. Structure: $\{\mathcal{O}\} \cong \Omega / G$, G - a subgroup of the full symmetry group of Ω
4. Multiple data sets of common origin (common constraints) but variable resolution (quantisation) $\Rightarrow$ multiple Ω's identified with common solid
5. Statistical models should be compatible (e.g. admit appropriate extensions (P. McCullagh 2002) with progressive refining of Ω)

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Summary

1. Generalisation of the multivariate *Problem of moments*
2. A new connection with computational algebraic geometry
3. Incremental model construction in the presence of G-invariance
4. Reduction of computations for estimation

Extensions

G-invariant polynomials are "all" patch features invariant to G. Hence, any (possibly non-linear) regression on G-invariant features, or predictors, should be carried out in terms of $f_1, f_2, \ldots, f_N$. Besides modeling the microimage distributions p (possibly conditional on membership of the patch in some class of interest), the response might naturally be probability of membership of the patch in some class of interest.

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 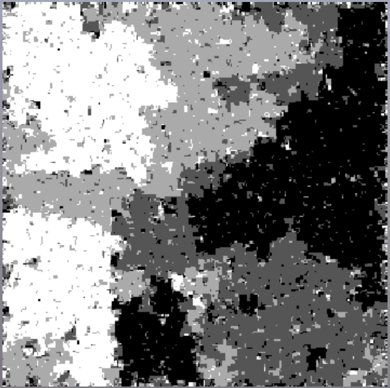

Left: The natural image with $L = 4$ grey levels. Right: A realization of the Maximum Entropy extension of the G-invariant estimates of the natural 2×2 microimage distribution (courtesy of Prof. L. Younes, the Johns Hopkins University)

Reference

"Symmetric Measures via Moments" *Bernoulli* Volume 14, Number 2, pp. 362–390, 2008