Fractional powers of positive positive definite matrices

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We find counterexamples for one of our pithy conjectures that for positive positive definite matrices (no typo!) A the fractional power A^q is again positive. Here, 'positive' means that the elements are positive and 'positive definite' means that the eigenvalues are positive. The requirement for positive definiteness comes about because A^q would otherwise be ill-defined for fractional q. This could have complemented earlier work of FitzGerald and Horn, who studied positive definiteness of the fractional Hadamard power $A^{(q)}$ (C.H. FitzGerald and R.A. Horn, "On fractional Hadamard powers of positive definite matrices," J. Math. Anal. Appl. 61, 633–642 (1977)).

The statement is obviously true for integer p, as can be shown using ordinary multiplication: $A^p = A.A.\cdots.A$. If A consists of non-negative elements, then the product does too.

The fractional case can be written in terms of an integral. Consider the integral formula

$$A^p = \frac{\sin(|p|\pi)}{\pi}\int_0^\infty dt\ t^p(A+t\mathbb{1})^{-1},$$

which is valid for $-1 < p < 0$. Thus, for $k < q < k+1$,

$$A^q = \frac{\sin(|q-k-1|\pi)}{\pi}\int_0^\infty dt\ t^{q-k-1}A^{k+1}(A+t\mathbb{1})^{-1}.$$

Substituting $X(t) = A/t$ yields

$$A^q = \frac{\sin(|q-k-1|\pi)}{\pi}\int_0^\infty dt\ t^q X(t)^{k+1}(X(t)+\mathbb{1})^{-1}.$$

So we could have proven the conjecture if $X^{k+1}(X+\mathbb{1})^{-1}$ is non-negative for non-negative X and any integer k.

Using Matlab(TM), we found counterexamples to the latter statement for $k=1$, which also turned out to be counterexamples to the conjecture that positive positive definite matrices have positive power p, for $1 < p < 2$ (see below). That led us to weaken the conjecture to values of p, $p \geq d-2$, with d the dimension of the matrix (see the paper of FitzGerald and Horn cited above). But the above counterexamples can be used as a basis to find counterexamples for that too! (However, I lost the notes I have made about that, mea culpa...)

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>> a
a =
    0.0096523     0.026398   0.00012905   0.00059013
     0.026398      0.10053     0.030808   0.00029887
   0.00012905     0.030808      0.26252     0.032929
   0.00059013   0.00029887     0.032929    0.0052308
>> eig(a)
      0.27221
      0.10243
    0.0032829
  2.2429e-006
>> a^2*inv(a+eye(4))
   0.00071447    0.0026121   0.00062359 -8.1019e-007
    0.0026121     0.010375    0.0085961   0.00076993
   0.00062359    0.0085961     0.055805     0.006949
 -8.1019e-007   0.00076993     0.006949   0.00087846
>> a^1.5
    0.0025401    0.0087771    0.0010738 -6.8119e-005
    0.0087771     0.033883      0.01941    0.0013742
    0.0010738      0.01941      0.13636     0.017028
 -6.8119e-005    0.0013742     0.017028    0.0022174
```

And another example:

```
>> a
    0.0020154   0.00016458    0.0020483   0.00046405     0.012969
   0.00016458      0.21819    0.0078702     0.017057     0.040952
    0.0020483    0.0078702    0.0044392   0.00054496     0.020645
   0.00046405     0.017057   0.00054496    0.0022252  2.0703e-006
     0.012969     0.040952     0.020645  2.0703e-006      0.14002

>> eig(a)
      0.23791
      0.12608
  4.2513e-018
    0.0018437
    0.0010616

>> a^2*inv(a+eye(5))
   0.00015458   0.00042885   0.00024394 -1.6012e-006    0.0016373
   0.00042885     0.040303    0.0020405    0.0030685     0.011523
   0.00024394    0.0020405   0.00043521   0.00010347    0.0028382
 -1.6012e-006    0.0030685   0.00010347   0.00024347   0.00051605
    0.0016373     0.011523    0.0028382   0.00051605     0.018707

>> a^1.1
    0.0014306   0.00041716    0.0016582   0.00018033     0.010681
   0.00041716      0.18792    0.0071763     0.014648     0.037899
    0.0016582    0.0071763     0.003277   0.00040149     0.017176
   0.00018033     0.014648   0.00040149    0.0016334   0.00036639
     0.010681     0.037899     0.017176   0.00036639      0.11584

>> a^1.5
   0.00049966    0.0007439   0.00073736  -2.339e-005    0.0049208
    0.0007439      0.10373    0.0047172    0.0079804     0.026014
   0.00073736    0.0047172    0.0012968   0.00021733     0.008228
  -2.339e-005    0.0079804   0.00021733   0.00067409   0.00086794
    0.0049208     0.026014     0.008228   0.00086794     0.054726
```