# Quantum Theory
# and Global Optimisation

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July 5, 2004

# The Trouble with QIT

- Quantum Information Theory (QIT) deals with bipartite and multipartite states.
- These are essentially higher-order tensors: one pair of indices (row, column) for every subsystem.
- Life would be easier if higher-order generalisations of eigenvalue and singular value decompositions existed.
- They do not, so we get a lot of difficult problems in QIT.
- We need all the help we can get, which is why I focus on the mathematical tools.
- In this talk, I will consider the so-called additivity problems in QIT.

# Entanglement Cost

- One of the Big Open Problems of QIT is calculating the entanglement cost $E_C$.
- $E_C$ defined in an operational way, nearly impossible to calculate
- Hayden, Horodecki and Terhal: $E_C$ is equal to the *regularisation* of the entanglement of formation $E_F$:

$$E_C(\rho) = \lim_{n\to\infty} E_F(\rho^{\otimes n})/n.$$

- $E_C$ would be equal to $E_F$ if $E_F$ were additive.

$$E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)\,?$$

- This is Additivity Problem #1.

# Classical Capacity of Quantum Channels

- Noisy communication channels modelled as completely positive trace-preserving (CPTP) maps between operator algebras.
- Another fundamental question in QIT: determination of *classical capacity of a quantum channel* i.e. the capacity of quantum channels to transmit classical information.
- Much more difficult than its purely classical counterpart due to the existence of *entanglement*.
- Optimal quantum channel *decoder* uses *entangled measurements* over the channel output states; i.e. the measurement alternatives are entangled states

# Additivity of Holevo Capacity

- Is entanglement also necessary to obtain an optimal *encoder*?
- Widely believed not to be the case, i.e. no benefit is expected in having entanglement between the (single-letter) states sent over the channel.
- To prove this, it is necessary to show that the single-letter classical capacity (a.k.a. the Holevo capacity $\chi$) of a quantum channel is additive.
- The Holevo capacity of a channel $\Phi$ is:

$$\chi(\Phi) = \sup_{\{(\pi_i,\rho_i)\}} S(\sum \pi_i \Phi(\rho_i)) - \sum \pi_i S(\Phi(\rho_i)).$$

- Is this additive? $\chi(\Phi_1 \otimes \Phi_2) = \chi(\Phi_1) + \chi(\Phi_2)$?
- This is Additivity Problem #2.

# Maximal Output Purity of Channels

- Holevo capacity is a complicated quantity. A simpler channel property, called the minimal output entropy (MOE), has been introduced.
- In general, if a pure state is sent through a channel it will become noisy, and turn into a mixed state.
- Consider the state that is least affected and measure its noisiness...
- ... using the Schatten $q$-norm $||X||_q = (\mathrm{Tr}\, X^q)^{1/q}$, giving the *maximal output $q$-purity* (MOP), $\nu_q$, of the channel:

$$\nu_q(\Omega) = \max_\rho ||\Omega(\rho)||_q$$

- ... or using the entropy $S$, giving the *minimal output entropy* (MOE), $\nu_S$:

$$\nu_S(\Omega) = \min_\rho S(\Omega(\rho)).$$

# Additivity Problem #3

- Is MOE additive? $\nu_S(\Phi_1 \otimes \Phi_2) = \nu_S(\Phi_1) + \nu_S(\Phi_2)$?
- Could shed some light on the additivity problem for the Holevo capacity.
- Additivity of MOE would follow from multiplicativity of MOP for "small" $q$ (Amosov, Holevo and Werner): $\nu_q(\Phi \otimes \Omega) = \nu_q(\Phi)\nu_q(\Omega)$.
- Multiplicativity of MOP proven by Chris King for entanglement breaking channels, unital qubit maps and depolarising channels.
- Refuted for values of $q > 4.79$ (Holevo and Werner)
- Nevertheless, it could still hold for $q \downarrow 1$ (Hope springs eternal...)
- If we could prove it for $q = 2$ this would already be uplifting...

# Local and Global Optimisation

- All QIT quantities mentioned above are expressed as maximisations
- There are lots of techniques for finding the maximum of a general function
- Most methods can get stuck in local maxima
- Global optimisation is about finding the biggest local maximum
- No method exists that guaranteedly finds that
- So we are very happy when we have a problem for which there is only one local optimum
- Convex problems are in that league.
- Convex problems have the additional benefit that a *duality theory* exists.

# Relation between EoF and MOP

- Using the duality theory of convex optimisation, Audenaert and Braunstein proved that the additivity problem for the EoF is equivalent to an additivity problem for the Legendre transform of the pure state entanglement function.
- Using the Lie-Trotter relation, entropic quantities can be converted to power-law quantities, and A&B also proved:
  If $\nu_q$ is multiplicative for $q \downarrow 1$ and for all completely positive maps, then the EoF is strongly superadditive.
- Strong superadditivity is a conjectured property for EoF, stronger than additivity; former implies latter.
- So two additivity problems are closely related!

# Equivalence of all Additivity Problems

- And then Peter Shor proved equivalence of four additivity problems!
  - additivity of EoF,
  - strong superadditivity of EoF,
  - additivity of classical capacity of a quantum channel,
  - additivity of the MOE: $\nu_S(\Phi \otimes \Omega) = \nu_S(\Phi) + \nu_S(\Omega)$.
- Latter seems easiest to deal with
- The stakes for proving multiplicativity of MOP have been raised.
- So can we prove multiplicativity of MOP?

# The way forward...

- It is my belief that convexity theory will not help us further anymore
- MOP is a *maximisation* of a convex function over a convex set

$$\nu_q(\Omega) = \max_\rho ||\Omega(\rho)||_q$$

- This is *not* a convex problem!
- Only thing convexity tells us is that the maximum will be achieved in an extreme point: a pure state.
- Possible way forward: characterise MOP in an asymptotic, “pseudo-linear” way.
- Two approaches: one based on the Quantum de Finetti Theorem and one based on representing a maximisation of a positive function by its $l_\infty$-norm.

# Some Notation

- Consider order-$n$ tensors $x$ in $\mathcal{H}^{\otimes n}$.
- Let $P_\pi$ be an index permutation matrix, permuting indices according to $\pi \in S_n$:

$$(P_\pi x)_{(i)} = x_{\pi(i)}.$$

- An operator $A$ on $\mathcal{H}^{\otimes n}$ is *symmetric* if and only if $\forall \pi \in S_n$, $P_\pi^\dagger A P_\pi = A$.
- The linear map $\mathsf{P}_n$ that projects all operators to the symmetric subspace of operators is

$$\mathsf{P}_n(A) = \frac{1}{n!} \sum_{\pi \in S_n} P_\pi^\dagger A P_\pi.$$

We call $\mathsf{P}_n(A)$ the *symmetric part* of $A$.

# Maximal Output Purity

- Consider the MOP $\nu_q$, but now for *integer* $q$; then

$$\nu_q^q(\Phi) = \max_{\rho \in \mathcal{S}(\mathcal{H})} \mathrm{Tr}[(\Phi(\rho))^q],$$

- Note that $\mathrm{Tr}[(\Phi(\rho))^q] = \mathrm{Tr}[\Phi(\rho)\Phi(\rho)\ldots\Phi(\rho)]$, with $q$ factors.
- We can write

$$\mathrm{Tr}[(\Phi(\rho))^q] = \mathrm{Tr}[A\,\rho^{\otimes q}],$$

with

$$A_{(i),(j)} = \mathrm{Tr}[\mathbf{\Phi}_{i_1,j_1}\ldots\mathbf{\Phi}_{i_q,j_q}],$$

where $\mathbf{\Phi}$ is the *Choi matrix* of the map $\Phi$.

# Representation 1

- For any $q, n \in \mathbb{N}$, and for any operator $A$ over $\mathcal{H}^{\otimes q}$ with Hermitian symmetric part, define

$$\mu_n(A) := \lambda_{\max}(\mathsf{P}_{q+n}(A \otimes \mathbf{I}^{\otimes n})).$$

- The sequence $(\mu_n(A))_n$ is non-increasing and converges to

$$\lim_{n\to\infty} \mu_n(A) = \max_{\rho} \mathrm{Tr}[A\rho^{\otimes q}].$$

- Hence, the MOP for integer $q$ can be expressed in this way.
- Proof is based on the Quantum de Finetti theorem (Hudson and Moody)

# What about it?

- Algorithmic issues: with some more symmetry theory, we get a modestly efficient algorithm that calculates guaranteed upper bounds (we already have lower bounds) and has no problems at all with local maxima!
- See quant-ph/0402076 for more details than you'd care for.
- Theoretical Issues: reduces a difficult optimisation problem to an eigenvalue problem
- The appearance of this $\lambda_{\max}$ still makes applying this characterisation quite difficult; so let's try something even simpler...

# Maximisation as infinity-norm

- The maximum of a continuous positive real function $g$ over a closed set $S$ can be written as

$$\max_{x\in S} g(x) = l_\infty(g) = \lim_{n\to\infty} l_n(g) = \lim_{n\to\infty} \left(\int_S g(x)^n d\mu(x)\right)^{1/n},$$

  where $d\mu(x)$ is a positive measure on $S$.

- In Polya and Szegö's famous book, Problem 198 of Part II, this is proven for $S$ a 1-dimensional closed interval, but the proof holds without modification for arbitrary closed sets.

- Here, we take as $S$ the $d$-dimensional complex sphere of norm-1 vectors,

- and as $d\mu(x)$ the unitarily invariant measure on the complex sphere.

With integer $q$ and $A$ an operator on $\mathcal{H}^{\otimes q}$, let $g(\phi)$ be a $q$-Hermitian form on a Hilbert space $\mathcal{H}$

$$g(\phi) = \mathrm{Tr}[A\,|\phi\rangle\langle\phi|^{\otimes q}],$$

$$g(\phi)^n = \mathrm{Tr}[A^{\otimes n}\,|\phi\rangle\langle\phi|^{\otimes qn}],$$

$$\begin{aligned}\int_S g(\phi)^n d\mu(\phi) &= \mathrm{Tr}[A^{\otimes n}\int_S d\mu(\phi)|\phi\rangle\langle\phi|^{\otimes qn}] \\ &= \mathrm{Tr}[A^{\otimes n}\frac{1}{C_{qn+d-1}^{d-1}(qn)!}\sum_{\pi\in S_{qn}} P_\pi].\end{aligned}$$

# Representation 2

- We thus get

$$\max_{\phi} g(\phi) = \lim_{n\to\infty} \left(\frac{1}{(qn)!} \sum_{\pi\in S_{qn}} \mathrm{Tr}[P_\pi A^{\otimes n}]\right)^{1/n}.$$

- We can apply this to the problem at hand:

$$\nu_q^q(\Phi) = \max_{\psi} \mathrm{Tr}[(\Phi(|\psi\rangle\langle\psi|))^q] = \max_{\psi} \mathrm{Tr}[A\,(|\psi\rangle\langle\psi|)^{\otimes q}] = \max_{\psi} g(\psi).$$

- Not an eigenvalue problem, but simply a trace!
- For $q = 1$, this reduces to the power method for calculating maximal eigenvalues.

# A Result

- Using Representation 2 we can prove for *integer* $q$:
- If $\nu_q$ is multiplicative for $\Phi_1 \otimes \Phi_1^T$ and for $\Phi_2 \otimes \Phi_2^T$, then it is also multiplicative for $\Phi_1 \otimes \Phi_2$.
  - Here, $\Phi^T$ denotes the *transpose* of channel $\Phi$: $\Phi^T := T \circ \Phi \circ T$
  - The Choi matrix of the transposed channel is simply the transposed of the channel's Choi matrix, which is its complex conjugate.
- In Representation 2, $\nu_q(\Phi_1 \otimes \Phi_2)$ is essentially an inner product of two functions over the symmetric group, one related to $\Phi_1$, the other to $\Phi_2^T$.
- Using the Schwarzine quality then yields the proof.
- Work in progress: a proof (?) for non-integer $q$...

# To be continued...