Understanding the MSW correspondence between Holevo capacity and Entanglement of Formation

Koenraad M.R. Audenaert*
University of Wales Bangor, School of Informatics,
Dean Street, Bangor, Gwynedd LL57 1UT, United Kingdom

I. INTRODUCTION

II. HOLEVO CAPACITY

I shall denote an ensemble by a Fraktur-E, $\mathfrak{E} = \{p_i, \rho_i\}$. The **Holevo Mutual Information** of an ensemble is defined as

$$I(\mathfrak{E}) = S(\sum_i p_i\rho_i) - \sum_i p_i S(\rho_i).$$

From a more mathematical point of view, one could call this the **concavity** of the von Neumann entropy over the ensemble. This is my own word for it, but I think it's well chosen because it measures the amount of concavity of a function. For concave functions, the concavity is positive, for convex functions it is negative, and for affine functions it is zero.

The **Holevo Capacity** of a channel T is

$$C(T) = \sup_{\mathfrak{E}}\{I(T(\mathfrak{E})) : \mathfrak{E} \text{ is an ensemble of pure states}\}.$$

By $T(\mathfrak{E})$ I mean the channel output ensemble $\{p_i, T(\rho_i)\}$ if $\mathfrak{E} = \{p_i, \rho_i\}$ is the input ensemble.

III. STINESPRING FORM OF A CHANNEL

By Lindblad's theorem, given a completely positive trace-preserving (CPTP) map T from input space $\mathcal{H}_{\text{in}}$ to $\mathcal{H}_{\text{out}}$, one can find an ancilla system $\mathcal{H}_{\text{aux}}$, a state ν supported on $\mathcal{H}_{\text{aux}}$, and a unitary map U on $\mathcal{H}_{\text{out}}\otimes\mathcal{H}_{\text{aux}}$ such that

$$T(\rho) = \text{Tr}_{\text{aux}}(U\,(\rho\otimes\nu)\,U^\dagger).$$

Typically, the so-called reference state ν is taken to be a pure state.

In order to find this representation starting from the Kraus representation of the map, one could proceed as follows [1]. For ease of exposition I shall assume that $\mathcal{H}_{\text{out}}$ and $\mathcal{H}_{\text{in}}$ are identical, and I will denote them by $\mathcal{H}$. Suppose T has a K-element Kraus representation

$$T(\rho) = \sum_{k=1}^K V_k\rho V_k^\dagger,$$

where the condition for trace preservation reads

$$\sum_k V_k^\dagger V_k = \mathbb{1}.$$

Take for $\mathcal{H}_{\text{aux}}$ a K-dimensional Hilbert space with basis vectors denoted by $|k\rangle$, $k = 1, 2, \ldots, K$. Fix a normalised pure state $\nu = |\nu\rangle\langle\nu|$ on $\mathcal{H}_{\text{aux}}$. Define the matrix U by its action on the pure states of $\mathcal{H}$ tensored with the reference state,

$$\begin{aligned} U(|\psi\rangle\otimes|\nu\rangle) &= \sum_k (V_k|\psi\rangle)\otimes|k\rangle \\ &= \bigoplus_k V_k|\psi\rangle. \end{aligned}$$

Defined as such, U is an isometry from $\mathcal{H}\otimes\nu$ to $\mathcal{H}\otimes\mathcal{H}_{\text{aux}}$. It is an isometry because it doesn't change the norm:

$$\begin{aligned} ||U(|\psi\rangle\otimes|\nu\rangle)||_2^2 &= ||\bigoplus_k V_k|\psi\rangle||_2^2 \\ &= \sum_k ||V_k|\psi\rangle||_2^2 \\ &= \sum_k \langle\psi|V_k^\dagger V_k|\psi\rangle \\ &= \langle\psi|\mathbb{1}|\psi\rangle \\ &= ||\psi||_2^2. \end{aligned}$$

Here I have used the condition for trace preservation of the map.

To finish off the definition of U, we have to describe how it acts on all other states in $\mathcal{H}\otimes\mathcal{H}_{\text{aux}}$. If we just let it map these other states to 0, then U is called a partial isometry. Partial isometries have singular values 0 and 1 only. If we then replace each of the 0 singular values by 1, U turns into a unitary matrix (a square matrix with singular values all 1). In effect, U is now a unitary on $\mathcal{H}\otimes\mathcal{H}_{\text{aux}}$, as required.

To see that this indeed reproduces the original CPTP map T, calculate the action on pure states $\Psi = |\psi\rangle\langle\psi|$:

$$\begin{aligned} &\text{Tr}_{\text{aux}}(U\,(\Psi\otimes\nu)\,U^\dagger) \\ &= \text{Tr}_{\text{aux}}\left(\sum_{k,l}(V_k|\psi\rangle)\otimes|k\rangle\,.\,(\langle\psi|V_l^\dagger)\otimes\langle l|\right) \\ &= \sum_{k,l} V_k|\psi\rangle\langle\psi|V_l^\dagger\ \text{Tr}_{\text{aux}}(|k\rangle\langle l|) \\ &= \sum_k V_k\Psi V_k^\dagger \\ &= T(\Psi). \end{aligned}$$

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*Electronic address: `kauden@informatics.bangor.ac.uk`

By linearity of T, the same holds for mixed states.

To establish contact with the MSW paper [2] and Shor's paper [3], the following table can be helpful:

My	MSW	Shor
$\mathcal{H}_{\text{in}}$	$\mathcal{H}$	$\mathcal{H}_{\text{IN}}$
$\mathcal{H}_{\text{out}}$	$\mathcal{H}_2$	$\mathcal{H}_A$
$\mathcal{H}_{\text{aux}}$	$\mathcal{H}_1$	$\mathcal{H}_B$

Furthermore, in the MSW paper, the image of $\mathcal{H}\otimes\nu$ under U is denoted by $\mathcal{K}$. This is a subspace of $\mathcal{H}\otimes\mathcal{H}_{\text{aux}}$.

IV. HOLEVO CAPACITY IN THE MSW SETTING

For the calculation of the Holevo capacity $C(T)$ of T, we have to let T act on the whole space $\mathcal{H}_{\text{in}}$. However, by the above, T can be split up as an isometry U acting on $\mathcal{H}_{\text{in}}\otimes\nu$, followed by the partial trace over $\mathcal{H}_{\text{aux}}$. Thus, for the purposes of calculating $C(T)$, T is equivalent to the partial trace $\operatorname{Tr}_{\text{aux}}$ acting on $\mathcal{K}$ (the image of $\mathcal{H}_{\text{in}}\otimes\nu$ under U). As U is an isometry, pure states remain pure states, so that we have

$$C(T)=\sup_{\mathfrak{E}:\mathcal{K}}\{I(\operatorname{Tr}_{\text{aux}}(\mathfrak{E}):\mathfrak{E}\text{ pure }\}.$$

I will be using the notation $\rho:\mathcal{K}$ to mean that ρ is supported on $\mathcal{K}$. Analogously, $\mathfrak{E}:\mathcal{K}$ means that all states in the ensemble are supported on $\mathcal{K}$.

Now, the average state $\rho=\sum_i p_i\Psi_i := \langle\mathfrak{E}\rangle$ of an ensemble $\mathfrak{E}$ with state vectors $\psi_i\in\mathcal{K}$ is a state with support on $\mathcal{K}$, and conversely, every realising ensemble of a state ρ with support on $\mathcal{K}$ must consist of states with support on $\mathcal{K}$ too. Thus:

Proposition 1 *For any ensemble $\mathfrak{E}$:*

$$\mathfrak{E}:\mathcal{K} \iff \langle\mathfrak{E}\rangle:\mathcal{K}.$$

Proof. We have $\mathcal{H}_{\text{in}}\otimes\mathcal{H}_{\text{aux}}=\mathcal{K}\oplus\mathcal{K}^\perp$. Choose a basis for $\mathcal{H}_{\text{in}}\otimes\mathcal{H}_{\text{aux}}$ consisting of vectors from $\mathcal{K}$ and vectors from the orthogonal complement $\mathcal{K}^\perp$. In this basis any state supported on $\mathcal{K}$ can be written as a block matrix

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$$

Suppose we have a state ρ with a realising ensemble $\{p_i,\rho_i\}$ such that some ρ_i are not supported on $\mathcal{K}$ only. That means they have some strictly positive diagonal elements on the $\mathcal{K}^\perp$ part. Since states are positive semidefinite, their diagonal elements are always ≥ 0. But then, since the p_i are strictly positive, ρ itself will also have some strictly positive diagonal elements on the $\mathcal{K}^\perp$ part (everything must be positive or zero, so there are no cancellations possible). Thus ρ is not supported on $\mathcal{K}$ only.

Conversely, if all ρ_i are supported on $\mathcal{K}$, then they all have the block structure

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$$

and, obviously, so has ρ. Hence, ρ is supported on $\mathcal{K}$ too. □

We can now combine all that to show the MSW correspondence

$$C(T)=\sup_{\rho:\mathcal{K}} S(\operatorname{Tr}_{\text{aux}}\rho)-E_F(\rho),$$

where E_F is the entanglement of formation with respect to the $\mathcal{H}_{\text{in}}$ vs. $\mathcal{H}_{\text{aux}}$ subdivision.

Proof. Indeed, denoting $\mathfrak{E}=\{p_i,\Psi_i\}$ (a pure state ensemble), the right-hand side is

$$\begin{aligned}
\text{RHS} &= \sup_{\rho:\mathcal{K}}\Big(S(\operatorname{Tr}_{\text{aux}}\rho) \\
&\qquad -\inf_{\mathfrak{E}}\Big\{\sum_i p_i S(\operatorname{Tr}_{\text{aux}}\Psi_i):\sum_i p_i\Psi_i=\rho\Big\}\Big) \\
&= \sup_{\mathfrak{E}:\mathcal{K}} S(\operatorname{Tr}_{\text{aux}}(\sum_i p_i\Psi_i))-\sum_i p_i S(\operatorname{Tr}_{\text{aux}}\Psi_i) \\
&= \sup_{\mathfrak{E}:\mathcal{K}} I(\operatorname{Tr}_{\text{aux}}\mathfrak{E}) \\
&= C(T).
\end{aligned}$$

In the second line I have used Proposition 1. □

[1] M.B. Ruskai, quant-ph/0205064.

[2] Matsumono, Shimono and Winter, quant-ph/0206148.

[3] P.W. Shor, quant-ph/0305035. (1991).