Convex Analysis in QIT

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Convex Analysis

- Convex Sets
- Convex Hulls
- Convex Functions
- Convex Closures
- An Application in QIT

Convex Sets and Convex Hulls

- A set S is **convex** if and only if

$$\forall p, q \in S : \overline{pq} \in S$$

  where $\overline{pq}$ is the line segment joining p and q

- Dually, a convex set S equals the intersection of all halfplanes containing S
- The **Convex Hull** is a kind of completion of sets in this sense
- The convex hull $\mathrm{Conv}(S)$ of a set S is the *union* of all line segments $\overline{pq}$ where $p, q \in S$
- Dually, the convex hull of S is the *intersection* of all halfplanes containing S

Convex Functions

- Whatever one can say about sets, one can say about functions
- The **Graph** of a function $y = f(x)$ is the set of points $\{(x, y) : y = f(x)\}$
- The **Epigraph** of a function $y = f(x)$ is the set of points

$$\mathrm{Epi}(f) := \{(x, y) : y \geq f(x)\}$$

- A function is convex if and only if its epigraph is a convex set
- A function f is **concave** if $-f$ is convex

Testing Convexity of Functions

- Functionally, checking convexity of f can be based on **convex combinations** $\sum_i a_i f(x_i)$, where $a_i \geq 0$ and $\sum_i a_i = 1$
- The function f is convex if and only if

$$\forall\{(a_i, x_i)\} : f(\sum_i a_i x_i) \leq \sum a_i f(x_i)$$

- Dually, f is convex if and only if
  f equals the pointwise supremum of all affine functions majorised by f

$$\forall x : f(x) = \sup_{a,b}\{a^T x + b : (\forall y : a^T y + b \leq f(y))\}$$

Convex Closures

- The convex closure $\hat{f}$ of a function f is defined by

$$\mathrm{Epi}(\hat{f}) = \mathrm{Conv}(\mathrm{Epi}(f))$$

- Functionally, the convex closure of f can also based on taking convex combinations

$$\hat{f}(x) = \min_{\{(a_i,x_i)\}} \{\sum_i a_i f(x_i) : \sum_i a_i x_i = x\}$$

- And, dually, the convex closure of f is
  the pointwise supremum of all affine functions majorised by f

$$\hat{f}(x) = \sup_{a,b} \{a^T x + b : (\forall y : a^T y + b \leq f(y))\}$$

Application to Entanglement Cost

- One of the Big Open Problems of QIT: how to calculate E_C?
- E_C defined in an operational way, nearly impossible to calculate
- First theoretical breakthrough (Hayden, Horodecki and Terhal):

  E_C is equal to the *regularisation* of E_F, the entanglement of formation (EoF):

$$E_C(\rho) = \lim_{n \to \infty} E_F(\rho^{\otimes n})/n.$$

- EoF defined in a mathematical and non-operational way (see below)
- Much more amenable to calculation than E_C
- For 2-qubit mixed states, a closed formula for E_F exists (Wootters).
- E_C still requires calculations over infinite-dimensional states.

Is EoF additive?

- The regularisation $\rho \mapsto \rho^{\otimes n}$ is about calculating potential “wholesale discounts”
- We would love it if there were no such discounts
- We want **Additivity**...

$$E_F(\rho_1 \otimes \rho_2) =? \, E_F(\rho_1) + E_F(\rho_2)$$

- ... because then $E_C = E_F$
- Additivity has been proven in specific instances.
- Some of these additivity results are sufficiently powerful to allow calculating E_C for certain classes of mixed states.
- The much sought-after general proof, however, remains elusive for the time being and, in fact, general additivity is still a conjecture.

Strong Superadditivity of EoF?

- It is very easy to show that the EoF is **subadditive**:

$$E_F(\rho_1 \otimes \rho_2) \leq E_F(\rho_1) + E_F(\rho_2).$$

- Additivity would then follow from **superadditivity**,

$$E_F(\rho_1 \otimes \rho_2) \geq ? \, E_F(\rho_1) + E_F(\rho_2).$$

- Vollbrecht and Werner conjectured a stronger property implying superadditivity: **Strong Superadditivity**

$$E_F(\rho) \geq ? \, E_F(\rho_I) + E_F(\rho_{II})$$

Here ρ is a general state over a duplicated Hilbert space and ρ_I and ρ_{II} are its reductions to the different copies of that space.

Entanglement of Formation Defined

- Any state ρ can be realised by an **ensemble** of pure states
- An ensemble is specified by a set of pairs $\{(p_i, \psi_i)\}_{i=1}^{N}$
  - of N state vectors ψ_i and statistical weights p_i
  - with $p_i \geq 0$ and $\sum_i p_i = 1$
- The entanglement of formation (EoF) of a bipartite state ρ (over the bi-partite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$), is

$$E_F(\rho) = \min_{\{(p_i,\psi_i)\}} \{ \sum_i p_i S(\mathrm{Tr}_A \, \Psi_i) : \sum_i p_i \Psi_i = \rho \}.$$

EoF is a Convex Closure

- **Observation 1**: The definition of the EoF really means that the EoF is the **convex closure** of the pure state entanglement function

$$E(\Psi) = S(\mathrm{Tr}_A \Psi)$$

- Consider bounded functions f whose domain is the set of states $\mathcal{S}(\mathcal{H})$
- We can apply real convex analysis because $\mathcal{B}(\mathcal{H})$ with $\langle A, B\rangle = \mathrm{Tr}[AB]$ is a real vector space
- **Definition 1**: The convex closure of f is

$$\hat{f}(\rho) = \min_{\{(p_i,\rho_i)\}} \{\sum_i p_i f(\rho_i) : \sum_i p_i \rho_i = \rho\}$$

EoF is a Convex Closure

- **Observation 2**: The second, dual, formulation of the convex closure looks easier
- **Definition 2**: The convex closure of a function f is
  the pointwise supremum of all affine functions on $\mathcal{S}(\mathcal{H})$ majorised by f

$$\hat{f}(\rho) = \sup_{X \in \mathcal{B}(\mathcal{H})} \{\mathrm{Tr}[\rho X] : (\forall \sigma \in \mathcal{S}(\mathcal{H}) : \mathrm{Tr}[\sigma X] \leq f(\sigma))\}$$

- Moreover, this definition can be pulled apart into two identical parts, based on the concept of the **conjugate function**

Conjugate function

- Define the **conjugate function** f^*:

$$f^*(X) = \max_{\rho \in \mathcal{S}(\mathcal{H})} \mathrm{Tr}[\rho X] - f(\rho)$$

- If f is continuous this is called the **Legendre transform** of f.
- The conjugate is convex in X: pointwise maximum of affine functions

- The conjugate and convex closure determine each other completely

$$f \quad \xrightarrow{\;*\;} \quad f^* \quad \xleftrightarrow{\;*\;} \quad \hat{f}$$

  - The convex closure of f is the conjugate of the conjugate of f: $\hat{f} = f^{**}$
  - The conjugate of the convex closure of f is the conjugate of f: $\hat{f}^* = f^*$

Central Lemma

Lemma 1 *For any bounded function* f *defined on* $\mathcal{S}(\mathcal{H})$ *strong superadditivity (SSA) of* $\hat{f}$

$$\hat{f}(\rho) \geq ? \, \hat{f}(\rho_I) + \hat{f}(\rho_{II}),$$

is equivalent to subadditivity (sA) of the conjugate f^* *with respect to the Kronecker sum:*

$$f^*(X_1 \otimes \mathbf{I} + \mathbf{I} \otimes X_2) \leq ? \, f^*(X_1) + f^*(X_2).$$

Proof of Lemma (I)

Set $Z = X_1 \otimes \mathbf{I} + \mathbf{I} \otimes X_2$. Assuming sA of f^* yields

$$\begin{aligned}
\hat{f}(\rho) &= \sup_X \mathrm{Tr}[\rho X] - f^*(X) \\
&\geq \sup_{X_1,X_2} \mathrm{Tr}[\rho Z] - f^*(Z) \\
&\geq \sup_{X_1,X_2} \mathrm{Tr}[\rho_I X_1 + \rho_{II} X_2] - f^*(X_1) - f^*(X_2) \\
&= \hat{f}(\rho_I) + \hat{f}(\rho_{II}),
\end{aligned}$$

which is SSA of $\hat{f}$.

Proof of Lemma (II)

Conversely, assuming SSA of $\hat{f}$ yields

$$\begin{aligned} f^*(Z) &= \max_\rho \mathrm{Tr}[\rho Z] - \hat{f}(\rho) \\ &\le \max_\rho \mathrm{Tr}[\rho_I X_1 + \rho_{II} X_2] - \hat{f}(\rho_I) - \hat{f}(\rho_{II}) \\ &= \max_{\rho_1,\rho_2} \mathrm{Tr}[\rho_1 X_1 + \rho_2 X_2] - \hat{f}(\rho_1) - \hat{f}(\rho_2) \\ &= f^*(X) + f^*(Y), \end{aligned}$$

which is sA of f^*. $\square$

Getting Rid of the Kronecker Sum

The appearance of the Kronecker sum in Lemma 1 suggests that the consideration of the function $f^* \circ \log$ is a more natural setting for studying additivity.

By setting $g = f^* \circ \log$ and $X_i = \log A_i$

sA of f^* turns into

$$g(A_1 \otimes A_2) \leq ?\, g(A_1) + g(A_2)$$

and the conjugate and convex closure turn into

$$\begin{aligned} g(A) &= \max_{\rho \in \mathcal{S}(\mathcal{H})} \mathrm{Tr}[\rho \log(A)] - f(\rho) \\ \hat{f}(\rho) &= \max_{A \in \mathcal{B}^+(\mathcal{H})} \mathrm{Tr}[\rho \log(A)] - g(A) \end{aligned}$$

Main Result

Theorem 1 *For any function* f *defined on* $\mathcal{S}(\mathcal{H})$*, and with* g *defined on* $\mathcal{B}^+(\mathcal{H})$ *by*

$$g(A) = \max_{\rho \in \mathcal{S}(\mathcal{H})} \mathrm{Tr}[\rho \log(A)] - f(\rho)$$

strong superadditivity of the convex closure $\hat{f}$*,*

$$\hat{f}(\rho) \geq ? \, \hat{f}(\rho_I) + \hat{f}(\rho_{II}),$$

is equivalent to subadditivity of g*,*

$$g(A_1 \otimes A_2) \leq ? \, g(A_1) + g(A_2).$$

Applied to the EoF

Since E is concave, the optimal ρ in the definition of E^* must be an extreme point of the feasible set, i.e. a pure state

Corollary 1 *With* g *defined on* $\mathcal{B}^+(\mathcal{H})$ *by*

$$g(A) = \max_{\psi \in \mathcal{H}} \mathrm{Tr}[\Psi \log(A)] - E(\Psi)$$

strong superadditivity of the EoF,

$$E_F(\rho) \geq ?\, E_F(\rho_I) + E_F(\rho_{II}),$$

is equivalent to subadditivity of g*,*

$$g(A_1 \otimes A_2) \leq ?\, g(A_1) + g(A_2).$$

Relation to quantum channels

Let $||.||_q$ be the Schatten q-norm

$$||X||_q = \mathrm{Tr}[|X|^q]^{1/q}$$

Define MOP = **Maximal Output Purity** (Amosov, Holevo and Werner) of a quantum channel Λ:

$$\nu_q(\Lambda) = \max_{\phi} ||\Lambda(\Phi)||_q$$

Theorem 2 *If* $\nu_q(\Lambda)$ *is multiplicative for* $q \downarrow 1$ *and for all non-trace-preserving channels, then the entanglement of formation is strongly superadditive.*

Multiplicativity of the MOP

- Multiplicativity of ν_q conjectured by AHW for trace preserving channels.
- Proven by King for
  - entanglement breaking channels
  - unital qubit maps
  - depolarising channels
- Refuted for values of $q > 4.79$ (Holevo and Werner)
- Nevertheless, hopefully it still holds for $q \downarrow 1$

Conclusion

- Thanks to a simple convex analytical trick, we have:
  - A dual, easier, formulation for the EoF
  - EoF is strongly superadditive if and only if $E^* \circ \log$ is subadditive
  - EoF is strongly superadditive if ν_q is multiplicative for $q \downarrow 1$
- This might convince you that convex analysis is a useful tool in QIT
- Read the preprint:

`http://arXiv.org : quant-ph/0303045`