Convergence of Sequences of Functions and Dini's Theorem

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1. Pointwise and Uniform Convergence

In this note I am considering sequences of real-valued functions and their convergence properties. The topics I have chosen relate to what I needed for a certain proof in [2]. As customary, I denote such a sequence by $\{f_n\}$, with $n = 1, 2, \ldots$, and every function f_n is defined on a given set S.

It is said that $\{f_n\}$ converges **pointwise** to a limit function f on S if for all $x \in S$, the sequence of values $\{f_n(x)\}$ converges to the value $f(x)$. Convergence of a sequence of real numbers is defined as:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |f_n(x) - f(x)| < \epsilon.$$

Formally this gives for pointwise convergence:

$$\forall x \in S, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |f_n(x) - f(x)| < \epsilon. \tag{1}$$

The important thing to note is that the value of N will depend on ϵ but also on x.

Uniform convergence is a stronger notion of convergence, such that this value of N will not depend on x. Instead we have:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in S : |f_n(x) - f(x)| < \epsilon. \tag{2}$$

Note the interchange of the quantifier on x.

An example of pointwise but non uniform convergence: $S = [0, 1]$, $f_n = x^n$, converges pointwise to f,

$$f(x) = \begin{cases} 0, x < 1 \\ 1, x = 1 \end{cases}$$

Notice in this example that the f_n are continuous, but converge (pointwise) to a discontinuous function f. With uniform convergence, such a phenomenon does not occur, as witnessed by the following theorem (no proof given)

Theorem 1. *If a sequence $\{f_n\}$ of continuous functions converges uniformly to f, then f is continuous.*

Uniform convergence is also needed, for example, to show that the supremum of f_n over S converges to the supremum of f over S (at least for positive functions). Typically, this is formulated in terms of the sup-norm. The **sup-norm** of a function f over S is

$$||f||_S = \sup_{x\in S} |f(x)|.$$

Uniform convergence is *equivalent* with **convergence in the sup-norm**. Indeed, convergence in the sup-norm means that $||f_n - f||_S$ converges to 0, that is

$$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : ||f_n - f||_S < \epsilon,$$

and the statement $||f_n - f||_S < \epsilon$ is equivalent with $\forall x \in S : |f_n(x) - f(x)| < \epsilon$. Substituting this yields the formulation of uniform convergence.

By the triangle inequality for norms,

$$||f_n - f||_S \geq \big|\, ||f_n||_S - ||f||_S \,\big|.$$

Therefore, uniform convergence also implies that $||f_n||_S$ converges to $||f||_S$.

If the convergence of $\{f_n\}$ to f is only pointwise, then convergence in sup-norm is not guaranteed. Examples of this will be exhibited in the next section.

2. Dini's Theorem

There are several ways to test whether sequences of functions converge uniformly, and Dini's Theorem is one of them [1].

Theorem 2 (Dini). *If $\{f_n\}$ is a sequence of* continuous *functions converging pointwise to a* continuous *function f over a* compact *set S, such that $\forall x \in S, \forall n$: $f_n(x) \geq f_{n+1}(x)$ (*monotonous *convergence), then the convergence is* uniform.

Note that we could equally well have put $f_n(x) \leq f_{n+1}$ for monotonicity. The proof then goes through simply by considering $-f_n$ instead of f_n.

Proof. Define $g_n = f_n - f$. Both f_n and f are continuous, so g_n is continuous as well, and g_n converges pointwise and monotonically to zero. Formally,

$$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall x \in S : 0 \leq g_N(x) < \epsilon.$$

Actually, pointwise convergence states that $g_n(X) < \epsilon$ for all $n \geq N$, but we can't use this in the proof. It turns out we need something more powerful.

By continuity of the g_n, we can find a finite neighbourhood $B(x)$ around x within which g_N stays smaller than 2ϵ (while keeping the same N):

$$\forall\epsilon, \forall x, \exists N, \exists B(x) : \forall y \in B(x) : g_N(y) < 2\epsilon.$$

By the monotonicity of g_n we also have that $g_n(y) < 2\epsilon$ for every $n > N$ and (!) for every $y \in B(x)$. This is the more powerful statement I just referred to.

Since S is compact, for every value of ϵ it can be covered by the union of neighbourhoods $B(x_i)$ of a finite number of points x_i. In each of the x_i we might need a different value N_i of N. Let's take the maximum of these values: $M = \max_i N_i$. Due to the monotonicity of g_n, $g_n(y) < 2\epsilon$ will hold for every value of $n > M$ in every neighbourhood $B(x_i)$, i.e. in the whole of S. Gluing these facts together we get exactly the statement of uniform convergence. □

The compactness condition for S is crucial, as can be seen by considering the $f_n(x) = x^n$ example again, but now on the half-open interval $S = [0, 1[$. We have pointwise convergence of f_n to zero everywhere in S, but the sup-norm of f_n is 1 for every n.

The monotonicity condition cannot be removed either. To see this, consider the non-monotonic sequence of triangular functions $\{f_n\}$ of width $2/n$ over the closed interval $S = [0, 2]$:

$$f_n(x) = \begin{cases} nx, & 0 \leq x \leq 1/n \\ 1 - n(x - 1/n), & 1/n \leq x \leq 2/n \\ 0, & 2/n \leq x \leq 2 \end{cases}$$

For $x = 0$, $f_n(x) = 0$ for every value of n, while for $x > 0$, $f_n(x) = 0$ as soon as $n \geq 2/x$. Thus, f_n converges pointwise to zero everywhere in $[0, 2]$. In contrast, f_n does not converge uniformly to zero, because the sup-norm of f_n is 1 for whatever value of n.

References

1. T.M. Apostol, *Mathematical Analysis*, Addison-Wesley, 1974.
2. K.M.R. Audenaert and S.L. Braunstein, On Strong Superadditivity of the Entanglement of Formation, preprint quant-ph/0303045, available online at http://ArXiv.org (2003).