
Quantum Statistics in Quantum Information



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Contents

- Two problems in QIT where state estimation plays a prominent role:
 - 1. Estimating the spectrum of a density matrix [Keyl-Werner]
 - 2. Upper bounds on (PPT) distillability [Rains]



Part 1

Estimating the spectrum of a density matrix



Representation Theory

- A *representation* of a group G assigns a matrix $R(g)$ to every group element g :

$$R(g_1g_2) = R(g_1)R(g_2).$$

- Most representations decompose into direct sums of smaller representations
- Irreducible representations (**irreps**) are the basic building blocks
- Questions: Characterise irreps? How do representations decompose?
- We will focus on representations of the **symmetric group** S_n (all permutations) and of the **full linear group** $GL(d, \mathbb{C})$ (all matrices).



Representations of S_n and $GL(d, \mathbb{C})$

- Consider a Hilbert space $\mathcal{H} = \mathbb{C}^d$ and its n -fold tensor product $\mathcal{H}^{\otimes n}$.
- S_n acts on $\mathcal{H}^{\otimes n}$ by a permutation π of the tensor factors.

This is represented by a matrix P_π , the **index permutation matrix**:

$$\psi_1 \otimes \dots \otimes \psi_n \mapsto \psi_{\pi(1)} \otimes \dots \otimes \psi_{\pi(n)} = P_\pi (\psi_1 \otimes \dots \otimes \psi_n).$$

- $GL(d, \mathbb{C})$ acts on $\mathcal{H}^{\otimes n}$ by left-multiplying every factor by the same matrix A . This is represented by the **n -fold tensor power** $A^{\otimes n}$:

$$\psi_1 \otimes \dots \otimes \psi_n \mapsto (A\psi_1) \otimes \dots \otimes (A\psi_n) = A^{\otimes n} (\psi_1 \otimes \dots \otimes \psi_n).$$



Partitions

- To describe the irreps of S_n and $GL(d, \mathbb{C})$, we need the concept of **partition**.
- A partition λ of the integer n is a non-increasing sequence of non-negative integers λ_i adding up to n :

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0, \quad \sum_i \lambda_i = n.$$

- Notation: $\lambda \vdash n$.
- Example: Partitions of $n = 5$: $(5), (41), (32), (311), (221), (2111), (11111)$.
- Graphical representation: Young frames.



Wedderburn Decomposition

- There exist a basis, independent of π and A , in which we have the simultaneous decompositions

$$P_\pi = \bigoplus_{\lambda \vdash n} \mathbf{I}_{r(\lambda)} \otimes S_\lambda(\pi)$$
$$A^{\otimes n} = \bigoplus_{\lambda \vdash n} R_\lambda(A) \otimes \mathbf{I}_{s(\lambda)}$$

- Irreps $S_\lambda(\pi)$ and $R_\lambda(A)$ labelled by partitions of n .
- Irrep $R_\lambda(A)$ has dimension $r(\lambda)$ and occurs with multiplicity $s(\lambda)$.
- Irrep $S_\lambda(\pi)$ has dimension $s(\lambda)$ and occurs with multiplicity $r(\lambda)$.
- P_π and $A^{\otimes n}$ commute.



Group Characters of S_n

- The *group characters* are the traces of the irreps.
- $\text{Tr } S_\lambda(\pi)$ is denoted $\chi^\lambda(\pi)$.
- Special Case: dimension $s(\lambda)$ of S_n irreps:

$$\chi^\lambda(1) = s(\lambda) =: f^\lambda = n! \frac{\Delta(\nu_1, \dots, \nu_r)}{\nu_1! \dots \nu_r!}$$

with

$$\Delta(x_1, x_2, \dots, x_r) := \prod_{i < j} (x_i - x_j), \quad \nu_i(\lambda) := \lambda_i + r - i.$$

- f^λ = number of standard Young tableaux of shape λ .



Group Characters of $\mathrm{GL}(d, \mathbb{C})$

- $\mathrm{Tr} R_\lambda(A)$ is a symmetric polynomial in the eigenvalues of A ,
- called the **Schur polynomial**

$$s_\lambda(a_1, \dots, a_k) := \frac{\det(a_i^{\lambda_j + k - j})_{i,j=1}^k}{\det(a_i^{k-j})_{i,j=1}^k}$$

- For $k = 2$: $s_{(p,q)}(x, y) = \sum_{j=q}^p x^j y^{p+q-j}$.
- Special case: dimension $r(\lambda)$ of $\mathrm{GL}(d, \mathbb{C})$ irreps:

$$s_\lambda(1^{\times d}) = r(\lambda) = \frac{\Delta(\lambda_1 + d - 1, \lambda_2 + d - 2, \dots, \lambda_d)}{\Delta(d - 1, d - 2, \dots, 0)}.$$

- $s_\lambda(1^{\times d})$ = number of semistandard Young tableaux of shape λ and with d objects.



Invariant Subspaces

- Accordingly, the tensor space $\mathcal{H}^{\otimes n}$ splits up into invariant subspaces.

$$P_\pi = \bigoplus_{\lambda \vdash n} \mathbf{I}_{r(\lambda)} \otimes \mathcal{S}_\lambda(\pi)$$

$$A^{\otimes n} = \bigoplus_{\lambda \vdash n} R_\lambda(A) \otimes \mathbf{I}_{s(\lambda)}$$

$$\mathcal{H}^{\otimes n} = \bigoplus_{\lambda \vdash n} \mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$$

- Subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$, $\lambda \vdash n$, are invariant under all $A^{\otimes n}$ and all P_π .
- They are called the *symmetry classes* of the tensor space.
- Examples: totally symmetric and totally antisymmetric subspaces.



Projectors on Invariant Subspaces

- We denote the projector on symmetry class λ by P^λ .
- Relation to index permutations:

$$P^\lambda = (n!)^{-1} f^\lambda \sum_{\pi \in S_n} \chi^\lambda(\pi) P_\pi.$$

- Example: totally symmetric projector:

$$P^{(n)} = (n!)^{-1} \sum_{\pi \in S_n} P_\pi.$$

- Relation to Schur polynomials (Schur):

$$\text{Tr}[P^\lambda A^{\otimes n}] = f^\lambda s_\lambda(a_1, \dots, a_d).$$



Projectors on Invariant Subspaces

- The projectors P^λ form an orthogonal set and add up to \mathbf{I} on the full tensor space:

$$P^\lambda P^{\lambda'} = \delta_{\lambda\lambda'} P^\lambda$$
$$\sum_{\lambda \vdash n} P^\lambda = \mathbf{I}.$$

- Hence they constitute a set of von Neumann projective measurement operators, applicable on n copies of a state ρ .
- The outcome is one of the allowed partitions λ .
- The probability of outcome λ is

$$p_\lambda = \text{Tr}[P^\lambda \rho^{\otimes n}] = f^\lambda s_\lambda(r),$$

where r is the spectrum of ρ .



Asymptotic behaviour of $f^{\lambda} s_{\lambda}(r)$

- **Keyl-Werner Theorem:** for large n , p_{λ} tends to a Gaussian centered at $\lambda \approx n r^{\downarrow}$ with decreasing variance.
- Hence, for large n , we can measure the spectrum of ρ !
- Example: (Borrowed from Keyl and Werner without permission)
 $r^{\downarrow} = (0.6, 0.3, 0.1)$.
- The allowed values of λ/n form a grid on the triangle defined by the points

$$A = (1 \ 0 \ 0)$$

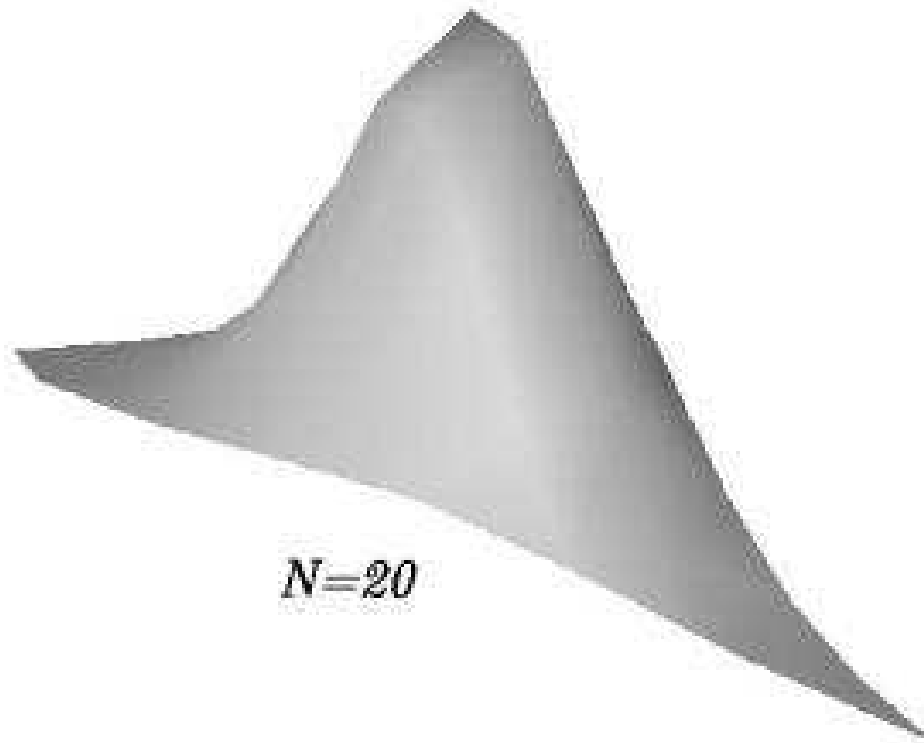
$$B = (1/2 \ 1/2 \ 0)$$

$$C = (1/3 \ 1/3 \ 1/3)$$

- Display p_{λ} at every grid point



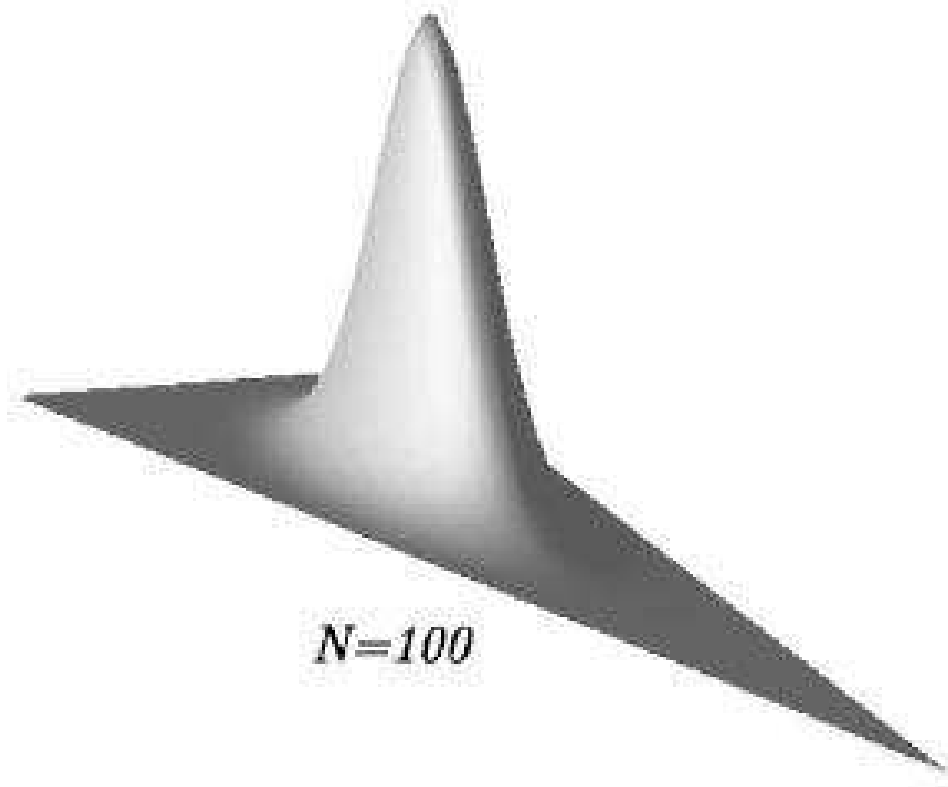
Examples



$N=20$

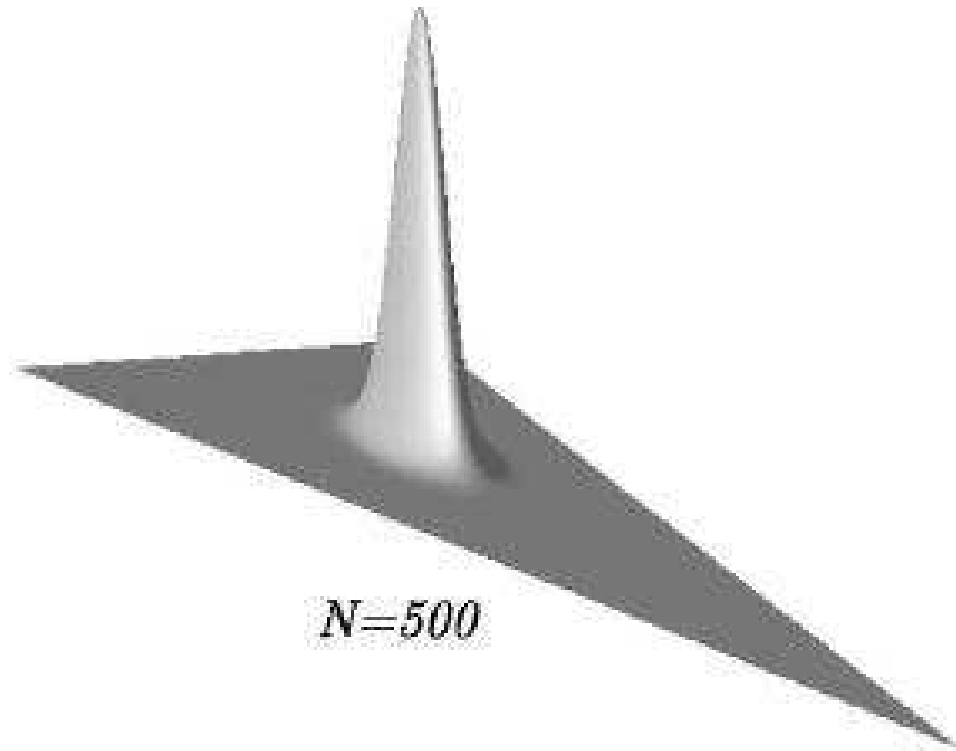


Examples





Examples





History of the Theorem

- 1906 (?): Schur's PhD thesis: $\text{Tr}[P^\lambda A^{\otimes n}] = f^\lambda s_\lambda(a_1, \dots, a_d)$;
- 1988: Alicki, Rudnicki and Sadowski: rediscovered Schur's formula + found asymptotic behaviour;
- 2001: Keyl and Werner: recalled Schur's formula + rediscovered asymptotic behaviour;
- 2002: Hayashi and Matsumoto: easier proof, with quantum-statistical flavour.



Sketch of Hayashi-Matsumoto Proof

- Schur:

$$\text{Tr}[P^\lambda \rho^{\otimes n}] = f^\lambda s_\lambda(r).$$

- Bound on $s_\lambda(r)$:

$$s_\lambda(r) \leq s_\lambda(1^{\times d}) r_1^{\lambda_1} \dots r_d^{\lambda_d}.$$

- Bound on f^λ :

$$f^\lambda \leq \frac{n!}{\lambda_1! \dots \lambda_d!}.$$

- From Stirling's formula:

$$\frac{n!}{\lambda_1! \dots \lambda_d!} \approx e^{nS(\lambda/n)}.$$



Sketch of Hayashi-Matsumoto Proof

- Thus (Hayashi-Matsumoto):

$$\mathrm{Tr}[P^\lambda \rho^{\otimes n}] \leq s_\lambda(1^{\times d}) e^{-nS(\lambda/n||r^\downarrow)}.$$

- From this one can prove: for a sequence $\lambda(n)$ such that $\lambda(n)/n$ tends to s^\downarrow , and for a state with ordered spectrum r^\downarrow , the convergence rate is

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathrm{Tr}[P^{\lambda(n)} \rho^{\otimes n}] = S(s^\downarrow||r^\downarrow).$$

- ...and also (Keyl-Werner): for a continuous function g on the set of ordered spectra,

$$g(r^\downarrow) = \lim_{n \rightarrow \infty} \sum_{\lambda \vdash n} g(\lambda/n) \mathrm{Tr}[P^\lambda \rho^{\otimes n}].$$



Part 2

Rains' bound on distillable entanglement



Distillable Entanglement

- Distillable entanglement = given a supply of a bipartite state ρ , what is the rate at which one can “distill” maximally entangled pairs from it using an LOCC protocol?
- If, from n copies of ρ we make m entangled pairs, the rate, $D(\rho)$, is m/n .
- Problem: membership of LOCC class is hard to decide
- Simplification: restrict to protocols that preserve PPT-ness (PPT= positive partial transpose ρ^Γ)
- PPT-distillability D_Γ



Distillable Entanglement

- Characterisation of D_Γ by Rains
- Let ρ be a bipartite state on $\mathcal{H}_A \otimes \mathcal{H}_B$.
- Let Φ_K be a $K \times K$ -dimensional maximally entangled state.
- The **fidelity of K -dim PPT distillation** of a state ρ is

$$F(\rho; K) = \max_{\Omega} \{ \text{Tr}[\Phi_K \Omega(\rho)] : \Omega \text{ PPT op. from } \mathcal{H}_A \otimes \mathcal{H}_B \text{ to } \mathbb{C}^K \otimes \mathbb{C}^K \}$$

- The **PPT distillable entanglement**:

$$D_\Gamma(\rho) = \sup_r \{ r : \lim_{n \rightarrow \infty} F(\rho^{\otimes n}; K = \lfloor 2^{rn} \rfloor) = 1 \}$$



Distillable Entanglement

- Fidelity of PPT distillation can be expressed as a semidefinite program (SDP).
- **Primal program**: maximisation over positive operators G

$$F(\rho; K) = \max_G \{ \text{Tr}[G\rho] : 0 \leq G \leq \mathbf{I}, |G^\Gamma| \leq 1/K \}.$$

- Any allowed choice of G gives a **lower bound** (not quite useful).
- **Dual program**: minimisation over operators D

$$F(\rho; K) = \min_D \text{Tr}(\rho - D)^+ + \text{Tr} |D^\Gamma|/K.$$

- Any D gives an **upper bound** (potentially very useful).
- The dual program yields bounds based on the relative entropy.



Relative Entropy Bounds

- Celebrated **Rains bound**:

$$D_{\Gamma}(\rho) \leq \min_{\sigma} S(\rho||\sigma) + \log \text{Tr} |\sigma^{\Gamma}|$$

- Proof proceeds via Stein's Lemma and related results
- Lemma 1 [Hiai and Petz]: defining

$$\gamma_n(\epsilon) := -\frac{1}{n} \log \min_P \{ \text{Tr}[P\sigma^{\otimes n}] : \text{Tr}[P\rho^{\otimes n}] \geq 1 - \epsilon \},$$

$$S(\rho||\sigma) \leq \liminf_{n \rightarrow \infty} \gamma_n(\epsilon) \leq \limsup_{n \rightarrow \infty} \gamma_n(\epsilon) \leq \frac{1}{1 - \epsilon} S(\rho||\sigma).$$

- Lemma 2: Whenever $y > S(\rho||\sigma)$

$$\limsup_{n \rightarrow \infty} \text{Tr}(\rho^{\otimes n} - 2^{yn} \sigma^{\otimes n})^+ < 1.$$