Comment on “Permanents in Linear-Optics Networks” by S. Scheel

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Abstract: I comment on the paper [1] by S. Scheel. More precisely, I give a simpler derivation of its main result eq. (10). This also might be taken as evidence that there are benefits in leaving the trodden paths of traditional quantum optics methods.

Traditionally, bosonic states are described in the *occupation number* (ON) basis. In the context of optics networks, a photon can be in at most $N \in \mathbb{N}$ positions, where each position corresponds to a terminal of a network component. The basis vectors in this description are of the form $|n_1, n_2, \ldots, n_N\rangle$. Every n_i is a natural number describing the number of photons present at terminal i. It follows that the corresponding Hilbert space is $\mathcal{H}_N = (\mathbb{C}^\infty)^{\otimes N}$, to allow a countably infinite number of photons at every site. The benefit of this description is that symmetry, the distinctive property of bosonic systems, is implicit in it.

Describing the action of a linear-optics component, such as a beam-splitter, in the ON-basis can be quite complicated. Following the textbook [2] one starts with the transformation rule for amplitude operators:

$$\begin{aligned} \hat{a} &\mapsto \Lambda^+ \hat{a} \\ \hat{a}^\dagger &\mapsto \Lambda^T \hat{a}^\dagger, \end{aligned}$$

where Λ is a unitary matrix.

For example, in the case of a lossless beam-splitter, there are two input terminals and two output terminals, so there are two amplitude operators $\hat{a}_1$ and $\hat{a}_2$ corresponding to the left and right terminals, respectively. The matrix Λ is given by

$$\Lambda = \begin{pmatrix} T & R \\ -R^* & T^* \end{pmatrix},$$

with $R, T \in \mathbb{C}$ and $|R|^2 + |T|^2 = 1$.

The action of the component on the Fock states $|n_1, n_2, \ldots, n_N\rangle$ is then given by the unitary $\hat{U}$:

$$\hat{U}\,|n_1, n_2, \ldots, n_N\rangle = \prod_{i=1}^{N} \frac{1}{\sqrt{n_i!}} \left(\sum_{k_i=1}^{N} \Lambda_{k_i,i}\, \hat{a}^\dagger_{k_i} \right)^{n_i} |0, 0, \ldots, 0\rangle.$$

In [1] it is shown that the matrix elements of $\hat{U}$ can be expressed in terms of permanents of Λ:

$$\langle m_1, m_2, \ldots, m_N|\hat{U}\,|n_1, n_2, \ldots, n_N\rangle = \left(\prod_{i=1}^{N} \frac{1}{\sqrt{m_i! n_i!}} \right) \mathrm{Per}\,\Lambda[\Omega'|\Omega], \qquad (1)$$

with $\Omega = (1^{n_1}, 2^{n_2}, \ldots, N^{n_N})$ and $\Omega' = (1^{m_1}, 2^{m_2}, \ldots, N^{m_N})$.

In this Note I show that (1) can be derived easily from first principles, without having to take recourse to the formalism of amplitude operators. The basic idea is to add the indistinguishability of photons as an explicit feature to the calculations.

To this end I introduce the *photon position* (PP) basis, whose basis vectors are $|i\rangle$, $i = 1, 2, \ldots, N$ and correspond to each of the N sites. The state of a single photon can be expressed in this basis without further ado, giving rise to a Hilbert space $\mathbb{C}^N$.

An n-photon state can be similarly expressed in the tensor product basis $|i_1, i_2, \ldots, i_n\rangle$ corresponding to a Hilbert space $(\mathbb{C}^N)^{\otimes n}$. Boson statistics dictate that photon states should be symmetric under particle interchange, hence the only physically allowed states in $(\mathbb{C}^N)^{\otimes n}$ are the totally symmetric states. These states form a subspace in $(\mathbb{C}^N)^{\otimes n}$ of dimension C^{N-1}_{n+N-1}, which is customarily denoted by $(\mathbb{C}^N)^{\vee n}$ [3]. Following [3] I define the projection map

$$P_{\vee,n} : (\mathbb{C}^N)^{\otimes n} \mapsto (\mathbb{C}^N)^{\vee n}$$

and the inclusion map

$$Q_{\vee,n} : (\mathbb{C}^N)^{\vee n} \mapsto (\mathbb{C}^N)^{\otimes n}.$$

Obviously, $Q_{\vee,n} = P^\dagger_{\vee,n}$. Then $P_n := P_{\vee,n} Q_{\vee,n}$ is a projector operating within $(\mathbb{C}^N)^{\otimes n}$ and projecting onto the subspace $(\mathbb{C}^N)^{\vee n}$.

A general state, where the photon number n is not fixed, can then be expressed as a vector in the Hilbert space $\mathcal{H}$ consisting of the direct sum of all n-photon Hilbert spaces:

$$\mathcal{H} = \oplus_{n=0}^{\infty} (\mathbb{C}^N)^{\otimes n}.$$

Note that I am really using the full spaces $(\mathbb{C}^N)^{\otimes n}$ instead of the symmetric subspaces $(\mathbb{C}^N)^{\vee n}$, because I will explicitly use the projectors P_n in the calculations. The separate n-photon spaces are called *sectors*.

The mapping between ON and PP bases is very straightforward. First note that the basis vectors in the ON basis are ordered so as to accomodate product states. For example, in the $N = 2$ case, the ordering is

$$|00\rangle, |10\rangle, |20\rangle, \ldots, |01\rangle, |11\rangle, |21\rangle, \ldots$$

The first step in the conversion to the PP basis is sorting these vectors into sectors of constant total photon number $n = n_1 + n_2 + \ldots + n_N$. At the level of state vectors this is done simply using a permutation matrix $\mathcal{P}_N$. What we get as a result is the direct sum of the totally symmetric subspaces in each sector. The second step is injecting each of these totally symmetric subspaces to its respective sector. This is done using the combined injection

$$\oplus_{n=0}^{\infty} Q_{\vee,n}.$$

Hence, the total basis transformation matrix from the ON to the PP basis is

$$(\oplus_{n=0}^{\infty} Q_{\vee,n})\,\mathcal{P}_N.$$

The converse transformation, from PP to ON, is given by

$$\mathcal{P}_N^T\,(\oplus_{n=0}^{\infty} P_{\vee,n}).$$

I claim that it is worthwile to consider the PP basis approach because the description of the network components is much easier in that basis. Indeed, if Λ is the unitary used in the transformation rules for the amplitude operators, then that actually means in the PP basis that Λ is the unitary describing the action of the component on the $n = 1$ sector. What makes the PP basis so useful is that the action of the component on the sectors with higher total photon number n can be described by $\Lambda^{\otimes n}$, temporarily pretending that photons are distinguishable. Bosonic symmetry is then restored by going over to the ON basis, taking projections on the totally symmetric subspaces into account. The total action on $\mathcal{H} = \oplus_{n=0}^{\infty} (\mathbb{C}^N)^{\otimes n}$ is therefore, in the PP basis:

$$|\psi\rangle_{PP} \mapsto \hat{U}_{PP}|\psi\rangle_{PP} = \oplus_{n=0}^{\infty} \Lambda^{\otimes n}\,|\psi\rangle_{PP},$$

which becomes in the ON basis:

$$\begin{aligned}\psi \mapsto \hat{U}|\psi\rangle &= \mathcal{P}_N^T\,(\oplus_{n=0}^{\infty} P_{\vee,n})\left(\oplus_{n=0}^{\infty} \Lambda^{\otimes n}\right)(\oplus_{n=0}^{\infty} Q_{\vee,n})\,\mathcal{P}_N\,|\psi\rangle \\ &= \mathcal{P}_N^T\left(\oplus_{n=0}^{\infty} P_{\vee,n}\Lambda^{\otimes n}Q_{\vee,n}\right)\mathcal{P}_N\,|\psi\rangle \\ &= \mathcal{P}_N^T\,(\oplus_{n=0}^{\infty} \Lambda^{\vee n})\,\mathcal{P}_N\,|\psi\rangle.\end{aligned}$$

It is now a simple matter to calculate the matrix elements of $\hat{U}$. Obviously, $\hat{U}$ is block-diagonal w.r.t. the different sectors, since linear optical elements are photon number preserving. So in sector n we look at the matrix elements $\langle m_1, m_2, \ldots, m_N|\hat{U}|n_1, n_2, \ldots, n_N\rangle$, with $\sum_i m_i = \sum_i n_i = n$. These elements are given by

$$\begin{aligned}\langle m_1, m_2, \ldots, m_N|\hat{U}|n_1, n_2, \ldots, n_N\rangle &= \langle m_1, m_2, \ldots, m_N|\Lambda^{\vee n}|n_1, n_2, \ldots, n_N\rangle \\ &= \left(\prod_{i=1}^{N} \frac{1}{\sqrt{m_i! n_i!}}\right) \mathrm{Per}\,\Lambda[\Omega'|\Omega],\end{aligned}$$

with $\Omega = (1^{n_1}, 2^{n_2}, \ldots, N^{n_N})$ and $\Omega' = (1^{m_1}, 2^{m_2}, \ldots, N^{m_N})$. The last equality is taken *verbatim* from ([3], p.19). This is just the main result from [1].

References

1. S. Scheel, preprint quant-ph/0406127, available online at http://ArXiv.org (2004).
2. W. Vogel, D.-G. Welsch and S. Wallentowitz, *Quantum Optics, An Introduction*, Wiley-VCH, Berlin (2001).
3. R. Bhatia, *Matrix Analysis*, Springer (1996).