Applications of Representation Theory in Quantum Information

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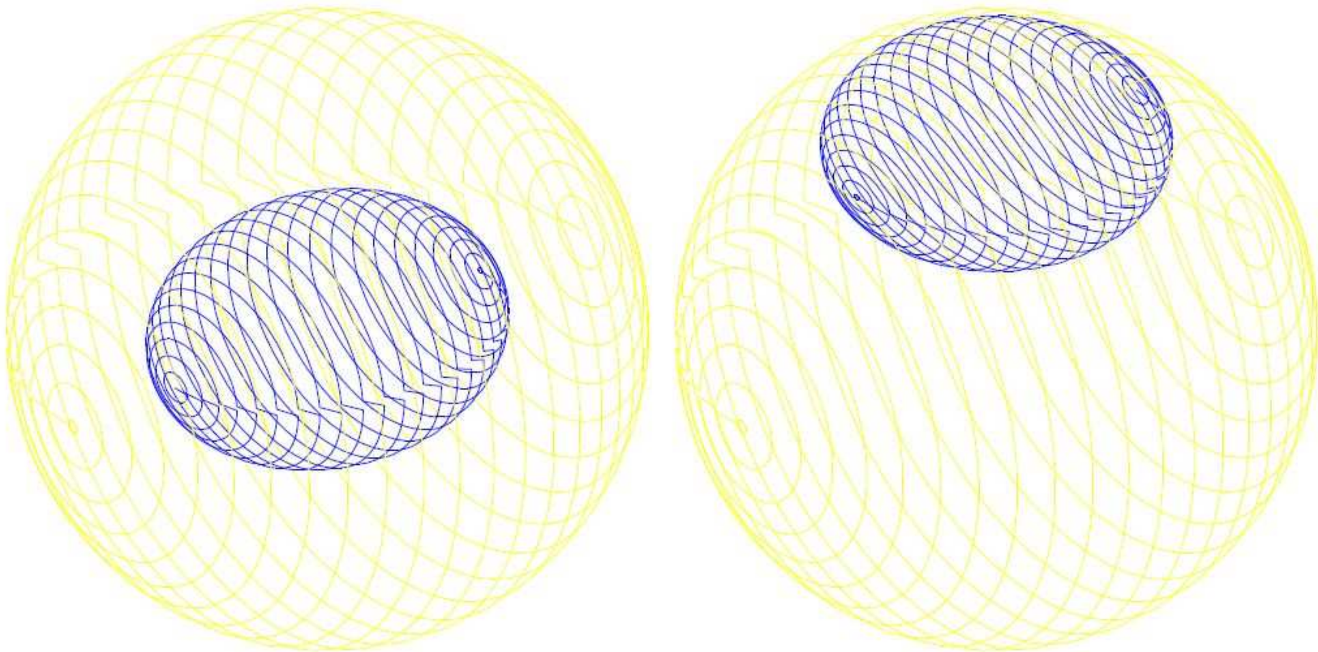
The Trouble with QIT

- Quantum Information Theory (QIT) deals with bipartite and multipartite states.
- These are essentially higher-order tensors: one pair of indices (row, column) for every subsystem.
- Life would be easier if higher-order generalisations of eigenvalue and singular value decompositions existed.
- They do not, so we get a lot of difficult problems in QIT:
  - Characterising separability
  - Are there NPT bound-entangled states?
  - Additivity questions
  - *** Add your own favourite problem ***

A Property of Channels

- In general, a pure state sent through a channel will become mixed (noisy)



- Consider the least affected state and measure its purity after the channel.

Maximal Output Purity (MOP)

- We can measure purity in a variety of ways
- One way uses the Schatten q-norm. For positive X:

$$||X||_q = (\mathrm{Tr}\, X^q)^{1/q}$$

- This norm is equal to the ℓ_q norm of the eigenvalues of X:

$$||X||_q = \left(\sum_i \lambda_i(X)^q \right)^{1/q} .$$

- We thus get the *maximal output q-purity* (MOP), ν_q of the channel:

$$\nu_q(\Omega) = \max_\rho ||\Omega(\rho)||_q$$

A Big Open Problem (BOP)

- Given that Schatten norms are multiplicative w.r.t. tensor product,

$$||A \otimes B||_q = ||A||_q \, ||B||_q$$

- Is the corresponding MOP also multiplicative?

$$\nu_q(\Phi \otimes \Omega) = \nu_q(\Phi)\nu_q(\Omega)?$$

- Can you get a higher purity by sending an entangled state through $\Phi \otimes \Omega$?

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$$\nu_q(\Phi \otimes \Omega) = \nu_q(\Phi)\nu_q(\Omega)?$$

- Can you get a higher purity by sending an entangled state through $\Phi \otimes \Omega$?
  - YES, for a whole zoo of channels: Entanglement Breaking Channels, Unital Qubit Maps, Depolarising Channels, Sky One and CNN Europe.
  - NO, for Holevo-Werner channel, when $q > 4.79$.
  - Nevertheless, it could still hold for q close to 1.

Why do we care about this?

- If MOP is multiplicative for $q \in [1, 1+\epsilon]$, then one can prove:
  - Strong superadditivity of Entanglement of Formation of states
  - Additivity of the Holevo Capacity of channels
- That would mean the following are impossible:
  - Getting "wholesale discounts" on LOCC creation of entanglement
  - Increase capacity by encoding classical info in entangled states
- On the other hand, it would allow us to calculate:
  - The entanglement cost of bipartite mixed states
  - The classical capacity of quantum channels

So how can we prove this?

- MOP is a *maximisation* of a convex function over a convex set

$$\nu_q(\Omega) = \max_{\rho} ||\Omega(\rho)||_q$$

- This is *not* a convex problem!
- The maximum will be achieved in an extreme point: a pure state.
- New approaches needed: Lifting = expressing a difficult problem as a simpler problem in higher dimensions
  - Hierarchies of Semidefinite Programs (Eisert, Doherty, Woerdeman)
  - "Pseudo-Linearisation"

Contents

Contents

On transforming
an unsolvable problem
to another unsolvable problem

MOP for $q = 2$

- Consider as an example the MOP ν_q, for $q = 2$:

$$\nu_2^2(\Phi) = \max_{\psi \in \mathcal{H}} \mathrm{Tr}[(\Phi(|\psi\rangle\langle\psi|))^2],$$

- Note that $\mathrm{Tr}[A^2] = \mathrm{Tr}[A.A]$.
- Using the flip operator F, $F.A \otimes B.F = B \otimes A$, we can write

$$\mathrm{Tr}[A^2] = \mathrm{Tr}[F\, A^{\otimes 2}].$$

(Audenaert '0402076; Giovannetti, Lloyd and Ruskai '0408103)

- So we need to find

$$\nu_2^2(\Phi) \;=\; \max_{\psi \in \mathcal{H}} \mathrm{Tr}[F.\Phi^{\otimes 2}(|\psi\rangle\langle\psi|^{\otimes 2})].$$

Maximum of a function is its ℓ_∞ norm

- The maximum of a continuous, positive function g over a closed set S can be expressed as

$$\begin{aligned}\max_{x\in S} g(x) &= l_\infty(g)\\ &= \lim_{n\to\infty} l_n(g)\\ &= \lim_{n\to\infty}\left(\int_S g(x)^n d\mu(x)\right)^{1/n},\end{aligned}$$

where $d\mu(x)$ is some strictly positive measure on S (independent of n).

Maximum of a function is its ℓ_∞ norm

- Consider the function $g(x) = 4x(1-x)$ over the interval $S = [0, 1]$.
- We know its maximum is 1 (at $x = 1/2$).
- Integral of g^n:

$$\int_0^1 dx\, [4x(1-x)]^n = \frac{\sqrt{\pi}\;\Gamma(n+1)}{2\;\Gamma(n+3/2)}$$

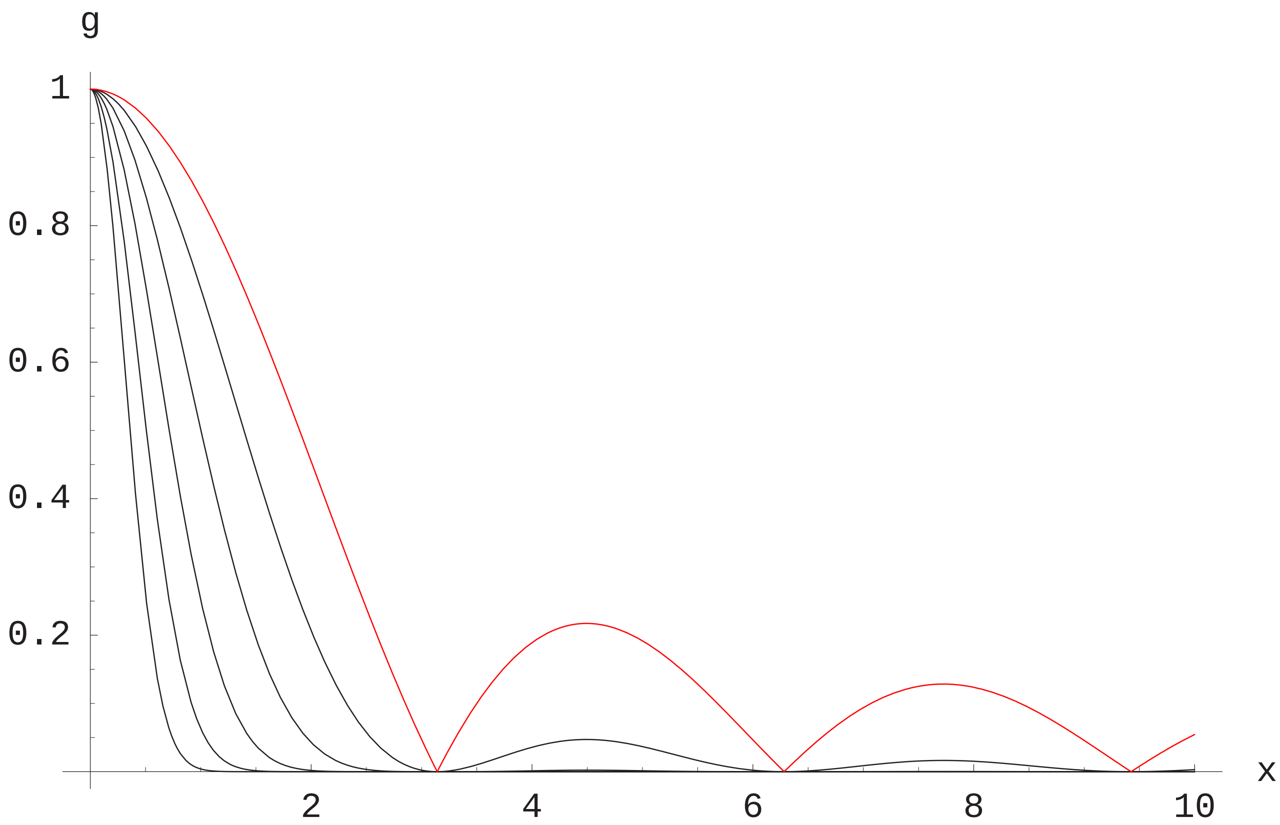
- As this lies in the interval

$$\frac{\sqrt{\pi}}{2}\,[\frac{1}{n+1}, 1]$$

the $1/n$-th power tends to 1, which is indeed the maximum of $g(x)$.

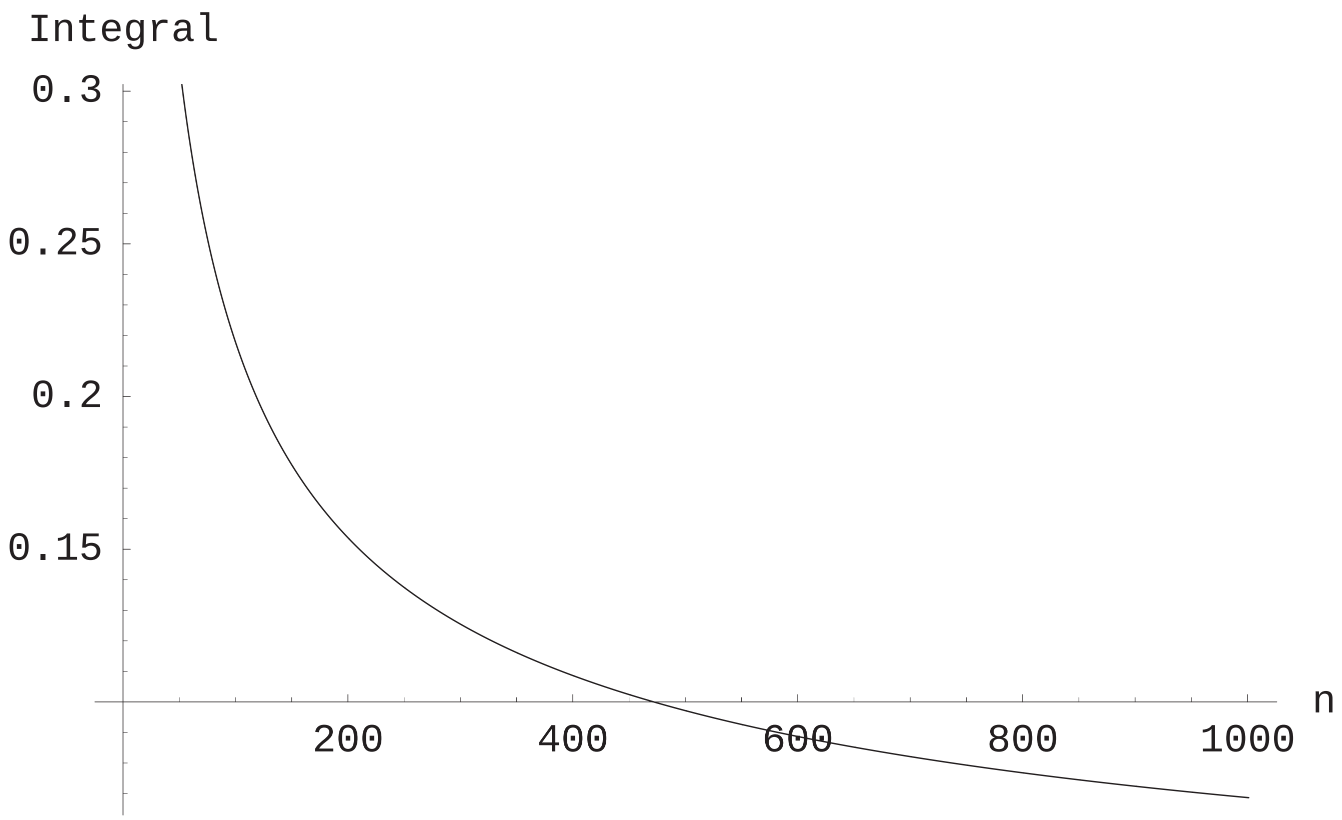
Maximum of a function is its ℓ_∞ norm

- This still works for more complicated g, even with local maxima.
- Plot of successive powers of $g(x) = |\sin(x)/x|$ over $x \in [0, 10]$:



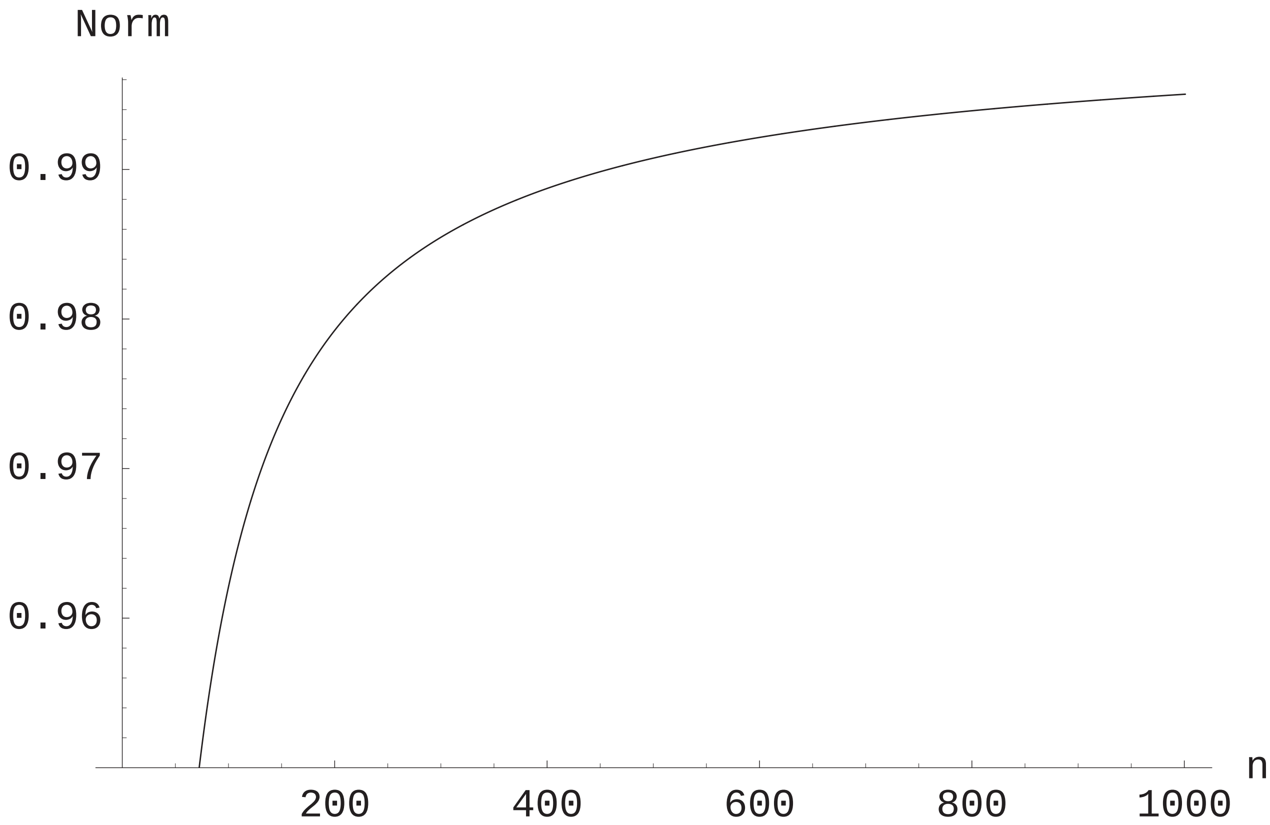
Maximum of a function is its ℓ_∞ norm

- Integral of $g(x)^n$ over $x \in [0, 10]$ versus n:



Maximum of a function is its ℓ_∞ norm

- $1/n$-th power of integral versus n:



Maximum of a function is its ℓ_∞ norm

- Is this trick/method actually useful?
  - Analytical? Only in very simple cases.
  - Numerical? Need to integrate, n must be *very* large.
- In our problem, the integration can be done analytically!

Pseudo-linearisation

- Take for S the d-dimensional complex sphere of norm-1 vectors,
- and for $d\mu(x)$ the unitarily invariant measure on the complex sphere.
- Consider an operator A on $\mathcal{H}^{\otimes 2}$, and $g(x) = g(\psi)$ the bi-Hermitian form on $\mathcal{H}$:

$$g(\psi) = \mathrm{Tr}[A\,|\psi\rangle\langle\psi|^{\otimes 2}].$$

- Then:

$$g(\psi)^n = \mathrm{Tr}[A^{\otimes n}\,|\psi\rangle\langle\psi|^{\otimes 2n}],$$

- Now this is linear in $|\psi\rangle\langle\psi|^{\otimes 2n}$, hence:

$$\int_S d\mu(\psi)\,g(\psi)^n = \mathrm{Tr}\left[A^{\otimes n}\int_S d\mu(\psi)\,|\psi\rangle\langle\psi|^{\otimes 2n}\right].$$

Pseudo-linearisation

- The integral over S can be calculated, thanks to the symmetry of S:

$$\int_S d\mu(\psi)\, |\psi\rangle\langle\psi|^{\otimes 2n} = \rho_{(2n)},$$

  the totally symmetric state of $\mathcal{H}^{\otimes 2n}$.

- Thus

$$\int_S d\mu(\psi)\, g(\psi)^n = \mathrm{Tr}[A^{\otimes n}\, \rho_{(2n)}].$$

- In particular:

$$\nu_2^2(\Phi) \;=\; \lim_{n\to\infty} \left(\mathrm{Tr}[F^{\otimes n}\;\Phi^{\otimes 2n}(\rho_{(2n)})\,]\right)^{1/n}.$$

Pseudo-linearisation

- Recall the basic ideas involved to get rid of the maximisation:
  - Express maximand as linear form in tensor powers of the argument;
  - Express maximisation as a limit of an ℓ_n norm;
  - Exploit the symmetry of the set over which to maximise.
- Questions:
  - Can we perform the remaining calculations involving $P^{(2n)}$?
  - What about non-integer q?
    The trick based on $A^q = A.A.\cdots.A$ no longer works...
- Representation theory will provide the key!

Representation Theory

- A *representation* of a group G assigns a matrix $R(g)$ to every group element g:

$$R(g_1 g_2) = R(g_1) R(g_2).$$

- Example: spatial rotations are represented by rotation matrices
- Example: matrices can be represented by their own tensor powers

$$(A_1 A_2)^{\otimes n} = A_1^{\otimes n} A_2^{\otimes n}$$

- Most representations decompose into direct sums of smaller representations
- Irreducible representations (irreps) are the basic building blocks
- Questions: Characterise irreps? How do representations decompose?

Decomposing tensor powers

- Consider a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Taking the second tensor power gives

$$A^{\otimes 2} = \begin{pmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{pmatrix}$$

- Note the multiple occurrences of some elements.
- There exist a basis in which $A^{\otimes 2}$ allows a more compact description: it decomposes as a direct sum.

Decomposing tensor powers

- Using the basis transformation

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}.$$

one gets

$$U\, A^{\otimes 2}\, U^{\dagger} = \begin{pmatrix} a^2 & \sqrt{2}\,ab & b^2 & 0 \\ \sqrt{2}\,ac & ad+bc & \sqrt{2}\,bd & 0 \\ c^2 & \sqrt{2}\,cd & d^2 & 0 \\ 0 & 0 & 0 & ad-bc \end{pmatrix}.$$

- For higher powers and higher dimensions, more blocks occur, of different sizes.

Representations of S_n and GL$(d, \mathbb{C})$

- We will focus on representations of the symmetric group S_n (all permutations) and of the full linear group $\mathrm{GL}(d, \mathbb{C})$ (all matrices).
- Consider a Hilbert space $\mathcal{H} = \mathbb{C}^d$ and its n-fold tensor product $\mathcal{H}^{\otimes n}$.
- S_n acts on $\mathcal{H}^{\otimes n}$ by a permutation π of the tensor factors. This is represented by a matrix P_π, the index permutation matrix:

$$\psi_1 \otimes \ldots \otimes \psi_n \;\mapsto\; \psi_{\pi(1)} \otimes \ldots \otimes \psi_{\pi(n)} = P_\pi \left(\psi_1 \otimes \ldots \otimes \psi_n\right).$$

- $\mathrm{GL}(d, \mathbb{C})$ acts on $\mathcal{H}^{\otimes n}$ by left-multiplying every factor by the same matrix A. This is represented by the n-fold tensor power $A^{\otimes n}$:

$$\psi_1 \otimes \ldots \otimes \psi_n \;\mapsto\; (A\psi_1) \otimes \ldots \otimes (A\psi_n) = A^{\otimes n} \left(\psi_1 \otimes \ldots \otimes \psi_n\right).$$

Partitions

- To describe the irreps of S_n and $\mathrm{GL}(d, \mathbb{C})$, we need the concept of partition.
- A partition λ of the integer n is:
  - a non-increasing sequence of non-negative integers λ_i

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0,$$

  - adding up to n

$$\sum_i \lambda_i = n.$$

- Notation: $\lambda \vdash n$.
- Example: Partitions of $n = 5$: $(5), (4\,1), (3\,2), (3\,1\,1), (2\,2\,1), (2\,1\,1\,1), (1\,1\,1\,1\,1)$.

Wedderburn Decomposition

- There exist a basis in which we have the simultaneous decompositions

$$P_\pi = \bigoplus_{\lambda \vdash n} \mathbf{I}_{r(\lambda)} \otimes S_\lambda(\pi)$$

$$A^{\otimes n} = \bigoplus_{\lambda \vdash n} R_\lambda(A) \otimes \mathbf{I}_{s(\lambda)}$$

- Irreps $S_\lambda(\pi)$ and $R_\lambda(A)$ labelled by partitions of n.
- Irrep $R_\lambda(A)$ has dimension $r(\lambda)$ and occurs with multiplicity $s(\lambda)$.
- Irrep $S_\lambda(\pi)$ has dimension $s(\lambda)$ and occurs with multiplicity $r(\lambda)$.
- P_π and $A^{\otimes n}$ commute

Group Characters of S_n

- The *group characters* are the traces of the irreps.
- $\mathrm{Tr}\, S_\lambda(\pi)$ is denoted $\chi^\lambda(\pi)$.
- Satisfies orthogonality relations, such as:

$$(n!)^{-1} \sum_{\pi \in S_n} \chi^\lambda(\pi)\chi^{\lambda'}(\pi) = \delta_{\lambda\lambda'}.$$

- Special Case:

$$\chi^\lambda(1) = s(\lambda) =: f^\lambda = n!\, \frac{\Delta(\nu_1, \ldots, \nu_r)}{\nu_1! \ldots \nu_r!}$$

with

$$\Delta(x_1, x_2, \ldots, x_r) := \prod_{i<j}(x_i - x_j), \qquad \nu_i(\lambda) := \lambda_i + r - i.$$

Group Characters of GL$(d, \mathbb{C})$

- $\mathrm{Tr}\, R_\lambda(A)$ is a symmetric polynomial in the eigenvalues of A,
- called the Schur polynomial

$$s_\lambda(a_1, \ldots, a_k) := \frac{\det(a_i^{\lambda_j + k - j})_{i,j=1}^k}{\det(a_i^{k-j})_{i,j=1}^k}$$

- For $k = 2$: $s_{(p,q)}(x, y) = \sum_{j=q}^p x^j y^{p+q-j}$.
- Special case:

$$s_\lambda(1^{\times d}) = r(\lambda) = \frac{\Delta(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_d)}{\Delta(d-1, d-2, \ldots, 0)}.$$

Invariant Subspaces

- Accordingly, the tensor space $\mathcal{H}^{\otimes n}$ splits up into invariant subspaces.

$$
\begin{aligned}
P_\pi &= \bigoplus_{\lambda \vdash n} \quad \mathbf{I}_{r(\lambda)} \otimes S_\lambda(\pi) \\
A^{\otimes n} &= \bigoplus_{\lambda \vdash n} R_\lambda(A) \otimes \mathbf{I}_{s(\lambda)} \\
\mathcal{H}^{\otimes n} &= \bigoplus_{\lambda \vdash n} \quad \mathcal{R}_\lambda \otimes \mathcal{S}_\lambda
\end{aligned}
$$

- Subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$, $\lambda \vdash n$, are invariant under all $A^{\otimes n}$ and all P_π.
- They are called the *symmetry classes* of the tensor space.
- We will denote the projectors on them by P^λ.

Examples

- The totally symmetric "boson" subspace $P^{(n)}$:
  - Multiplicity $f^{(n)} = 1$
  - Dimension $\binom{n+d-1}{d-1}$
- The totally antisymmetric "fermion" subspace $P^{(1^n)}$:
  - Multiplicity $f^{(1^n)} = 1$
  - Dimension $\binom{d}{n}$, if $d \geq n$, and 0 otherwise.

Projectors on Invariant Subspaces

- Relation to index permutations:

$$P^\lambda = (n!)^{-1} f^\lambda \sum_{\pi \in S_n} \chi^\lambda(\pi) P_\pi.$$

- Example:

$$P^{(n)} = (n!)^{-1} \sum_{\pi \in S_n} P_\pi.$$

- Relation to Schur polynomials:

$$\mathrm{Tr}[P^\lambda A^{\otimes n}] = f^\lambda s_\lambda(a_1, \ldots, a_d).$$

- Rank:

$$\mathrm{Tr}\, P^\lambda = f^\lambda s_\lambda(1^{\times d}).$$

Interlude: Multi-particle Twirls

- (Collins-Sniady) Let σ be a state on $(\mathbb{C}^d)^{\otimes n}$. The n-particle twirl of σ,

$$T_n(\sigma) := \int_{U(d)} d\mu(U)\, U^{\otimes n} \sigma U^{\dagger,\otimes n},$$

is equal to

$$T_n(\sigma) = \left(\frac{1}{n!} \sum_{\pi \in S_n} \mathrm{Tr}[\sigma P_\pi]\, P_\pi^T \right) \left(\sum_{\lambda \vdash n} \frac{f^\lambda}{s_\lambda(1^{\times d})} P^\lambda \right).$$

- In the particular case $\sigma = \rho^{\otimes n}$ one gets *generalised Werner states*:

$$T_n(\rho^{\otimes n}) = \sum_{\lambda \vdash n} \frac{\mathrm{Tr}[P^\lambda\, \rho^{\otimes n}]}{\mathrm{Tr}[P^\lambda]}\, P^\lambda = \sum_{\lambda \vdash n} \frac{s_\lambda(r)}{s_\lambda(1^{\times d})}\, P^\lambda,$$

where r is the spectrum of ρ.

Projectors on Invariant Subspaces

- The projectors P^λ form an orthogonal set and add up to **I** on the full tensor space:

$$\begin{aligned} P^\lambda P^{\lambda'} &= \delta_{\lambda\lambda'} P^\lambda \\ \sum_{\lambda\vdash n} P^\lambda &= \mathbf{I} \end{aligned}$$

- Hence they constitute a set of von Neumann projective measurement operators, applied on n copies of a state ρ.
- The outcome is one of the allowed partitions λ
- The probability of outcome λ is

$$p_\lambda = \mathrm{Tr}[P^\lambda \rho^{\otimes n}] = f^\lambda s_\lambda(r),$$

where r is the spectrum of ρ.

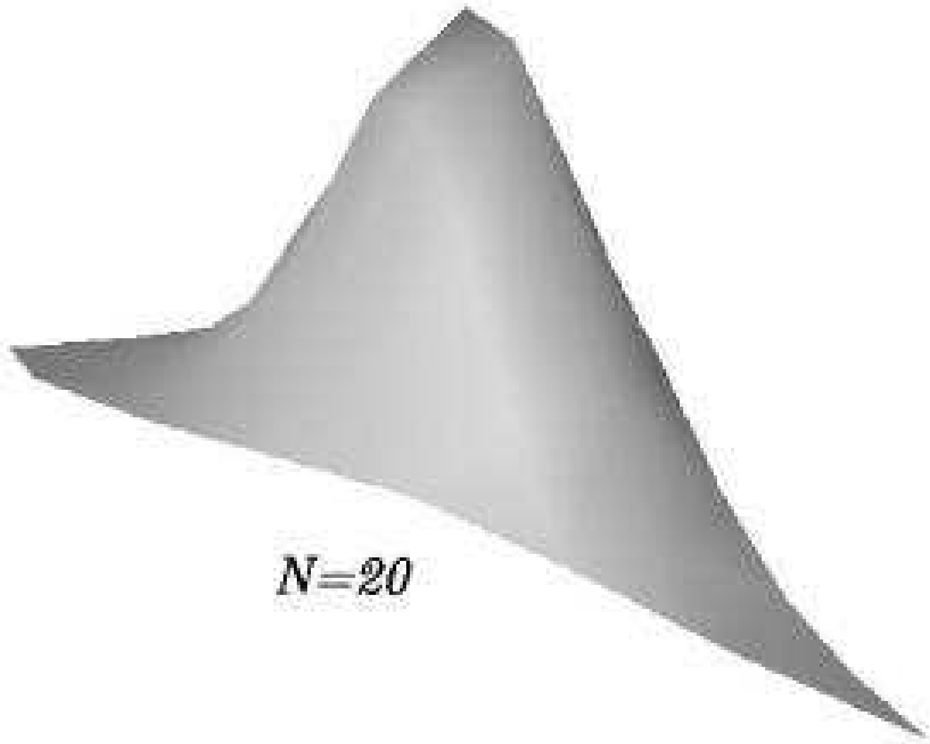
Asymptotic behaviour of $f^{\lambda} s_{\lambda}(r)$

- Keyl-Werner Theorem: for large n, p_λ tends to a Gaussian centered at $\lambda \approx n\, r^{\downarrow}$ with decreasing variance.
- Hence, for large n, we can measure the spectrum of ρ!
- Example: (Borrowed from Keyl and Werner without permission)
  $r^{\downarrow} = (0.6, 0.3, 0.1)$.
- The allowed values of λ/n form a grid on the triangle defined by the points

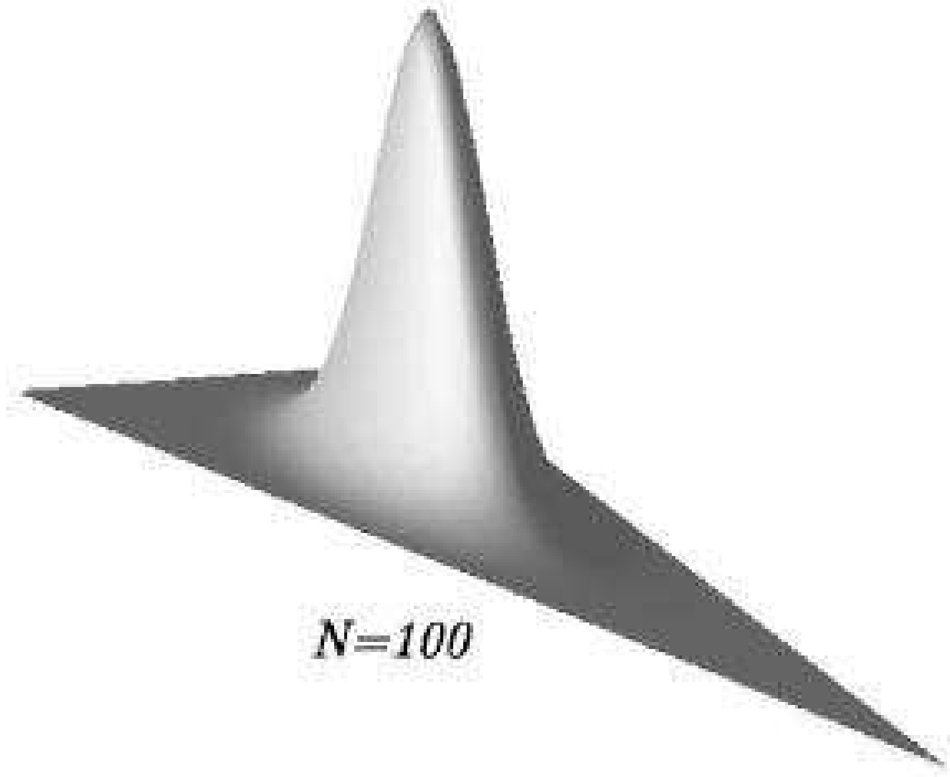
$$\begin{aligned} A &= (1\ 0\ 0) \\ B &= (1/2\ 1/2\ 0) \\ C &= (1/3\ 1/3\ 1/3) \end{aligned}$$

- Display p_λ at every grid point

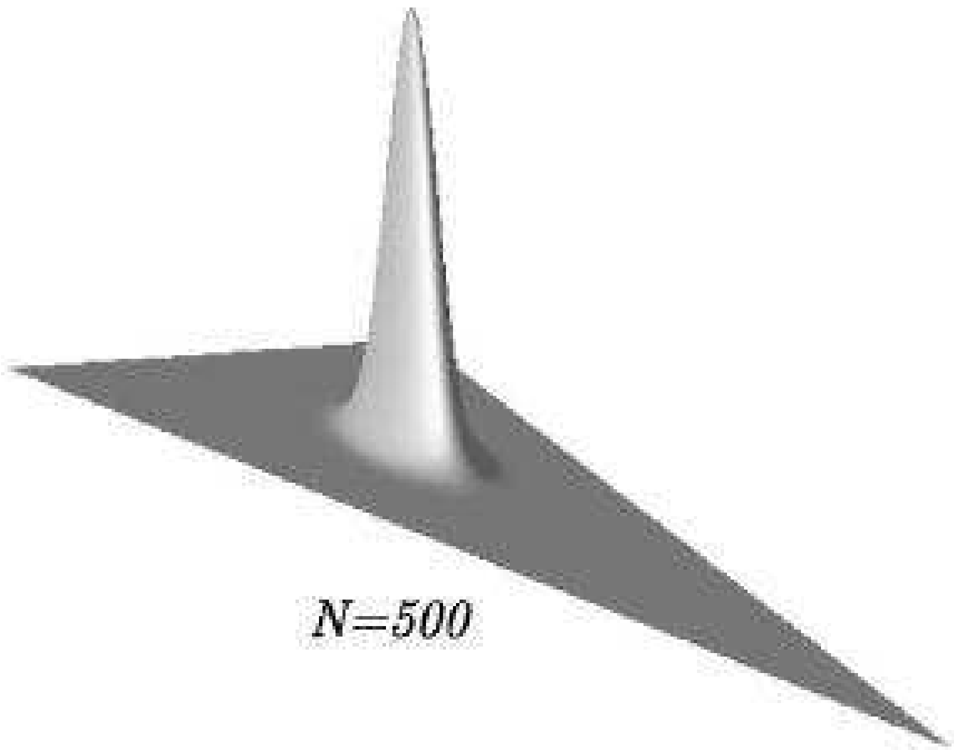
Examples



Examples



Examples



Maximal Output Purity, Again

- Goal: express MOP for any q as a pseudo-linearisation.
- For any unitarily invariant matrix norm $||\rho||$,

$$||\rho|| = \lim_{n\to\infty} \sum_{\lambda\vdash n} ||\lambda/n|| \, \mathrm{Tr}[P^\lambda \, \rho^{\otimes n}].$$

- Use this norm to measure output purity of a channel Φ:

$$\nu_{||.||}(\Phi) = \max_\psi ||\Phi(|\psi\rangle\langle\psi|)||$$

- This can be calculated as the limit

$$\nu_{||.||}(\Phi) = \lim_{m\to\infty} \left(\int d\psi \; ||\,\Phi(|\psi\rangle\langle\psi|)\,||^m \right)^{1/m}.$$

Maximal Output Purity, Again

- Applying Keyl-Werner on $||\Phi(|\psi\rangle\langle\psi|)||^m$ then gives for the integral

$$\lim_{n\to\infty}\int d\psi \sum_{\lambda\vdash n} ||\lambda/n||^m \;\mathrm{Tr}[P^\lambda\,\Phi(|\psi\rangle\langle\psi|)^{\otimes n}].$$

- This is then equal to

$$\lim_{n\to\infty}\mathrm{Tr}[\sum_{\lambda\vdash n} ||\lambda/n||^m P^\lambda\;\Phi^{\otimes n}(P^{(n)})].$$

- For the Schatten q-norm we get

$$\nu_q(\Phi) = \lim_{m\to\infty}\lim_{n\to\infty}\left(\sum_{\lambda\vdash n} ||\lambda/n||_q^m \;\mathrm{Tr}\left[P^\lambda\,\Phi^{\otimes n}(P^{(n)})\right]\right)^{\frac{1}{m}}.$$

Maximal Output Purity, Again

- The most important quantity (MIQ) in

$$\nu_q(\Phi) = \lim_{m\to\infty} \lim_{n\to\infty} \left(\sum_{\lambda\vdash n} ||\lambda/n||_q^m \ \mathrm{Tr}\left[P^\lambda \, \Phi^{\otimes n}(P^{(n)}) \right] \right)^{\frac{1}{m}} .$$

is

$$\mathrm{MIQ}_\lambda := \mathrm{Tr}\left[P^\lambda \, \Phi^{\otimes n}(P^{(n)}) \right] .$$

- It can be written in terms of index permutations as

$$(n!)^{-2} f^\lambda \sum_{\pi\in S_n} \sum_{\pi'\in S_n} \chi^\lambda(\pi) \, \mathrm{Tr}\left[P_\pi \, \Phi^{\otimes n}(P_{\pi'}) \right] .$$

- So the even more important quantity (EMIQ) is

$$\mathrm{EMIQ}_{\pi,\pi'} := \mathrm{Tr}\left[P_\pi \, \Phi^{\otimes n}(P_{\pi'}) \right] .$$

Multiplicativity of MOP?

- Multiplicativity of MOP means that for all channels Φ, Ω

$$\nu_q(\Phi \otimes \Omega) = \nu_q(\Phi)\nu_q(\Omega).$$

- Pseudo-linearisation yields an expression for $\nu_q(\Phi \otimes \Omega)$ based on

$$\text{EMIQ}_{\pi,\pi'} = \mathrm{Tr}\left[P_\pi\,(\Phi \otimes \Omega)^{\otimes n}(P_{\pi'})\right].$$

- Here, P_π and $P_{\pi'}$ are representations on the space $\mathcal{H}_I \otimes \mathcal{H}_{II}$, where Φ operates on $\mathcal{H}_I$ and Ω on $\mathcal{H}_{II}$.
- One can easily show $P_\pi(\mathcal{H}_I \otimes \mathcal{H}_{II}) = P_\pi(\mathcal{H}_I) \otimes P_\pi(\mathcal{H}_{II})$.

Multiplicativity of MOP?

- This yields, for whatever number K of channel copies:

$$\nu_q(\Phi^{\otimes K})$$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} \left(\sum_{\lambda \vdash n} ||\lambda/n||_q^m \ (n!)^{-2} f^\lambda \sum_{\substack{\pi\in S_n \\ \pi'\in S_n}} \chi^\lambda(\pi) \left(\mathrm{Tr}\left[P_\pi \, \Phi^{\otimes n}(P_{\pi'})\right]\right)^K \right)^{\frac{1}{m}}$$

$$= \lim_{\substack{m\to\infty \\ n\to\infty}} \left(\frac{1}{n!} \sum_{\pi\in S_n} \left(\sum_{\lambda\vdash n} ||\lambda/n||_q^m \ f^\lambda \chi^\lambda(\pi) \right) \ \frac{1}{n!} \sum_{\pi'\in S_n} \left(\mathrm{Tr}\left[P_\pi \, \Phi^{\otimes n}(P_{\pi'})\right]\right)^K \right)^{\frac{1}{m}}.$$

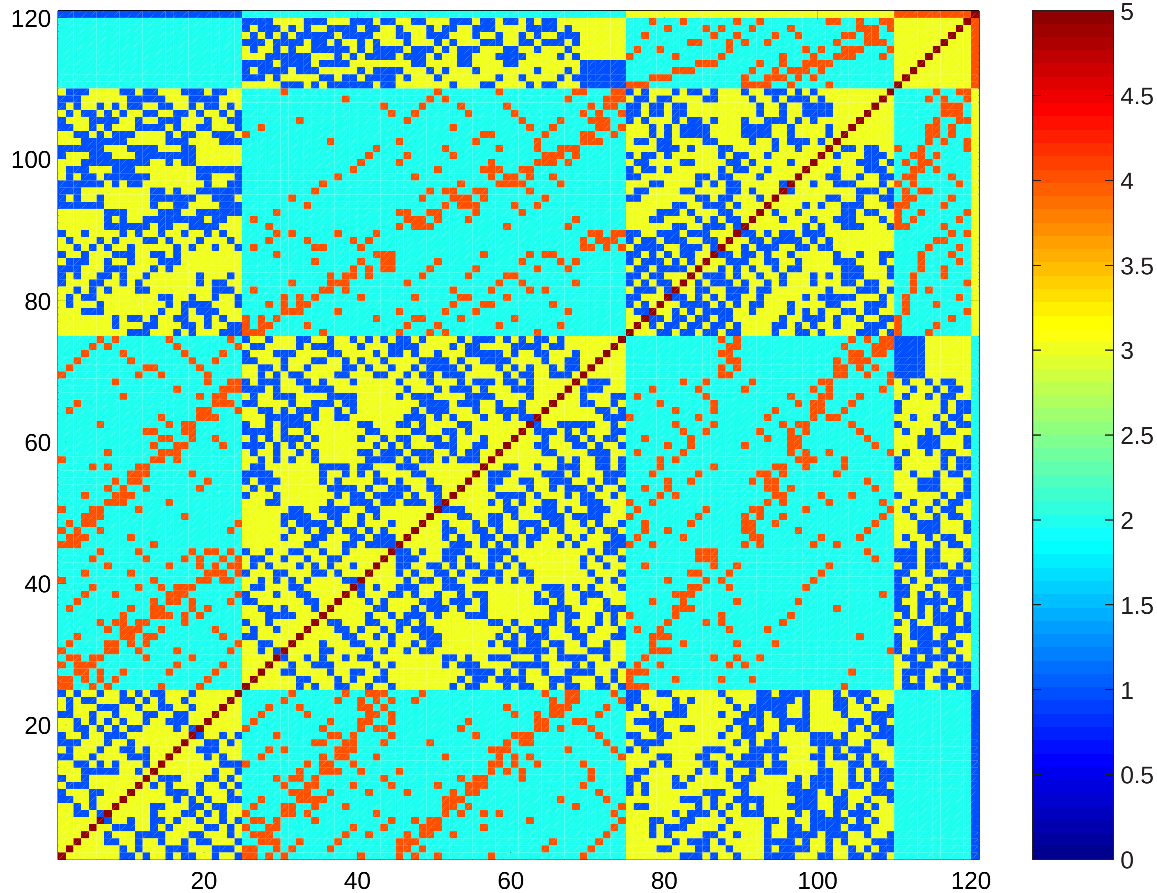
Enters some Combinatorics

- Basic quantity to be calculated is

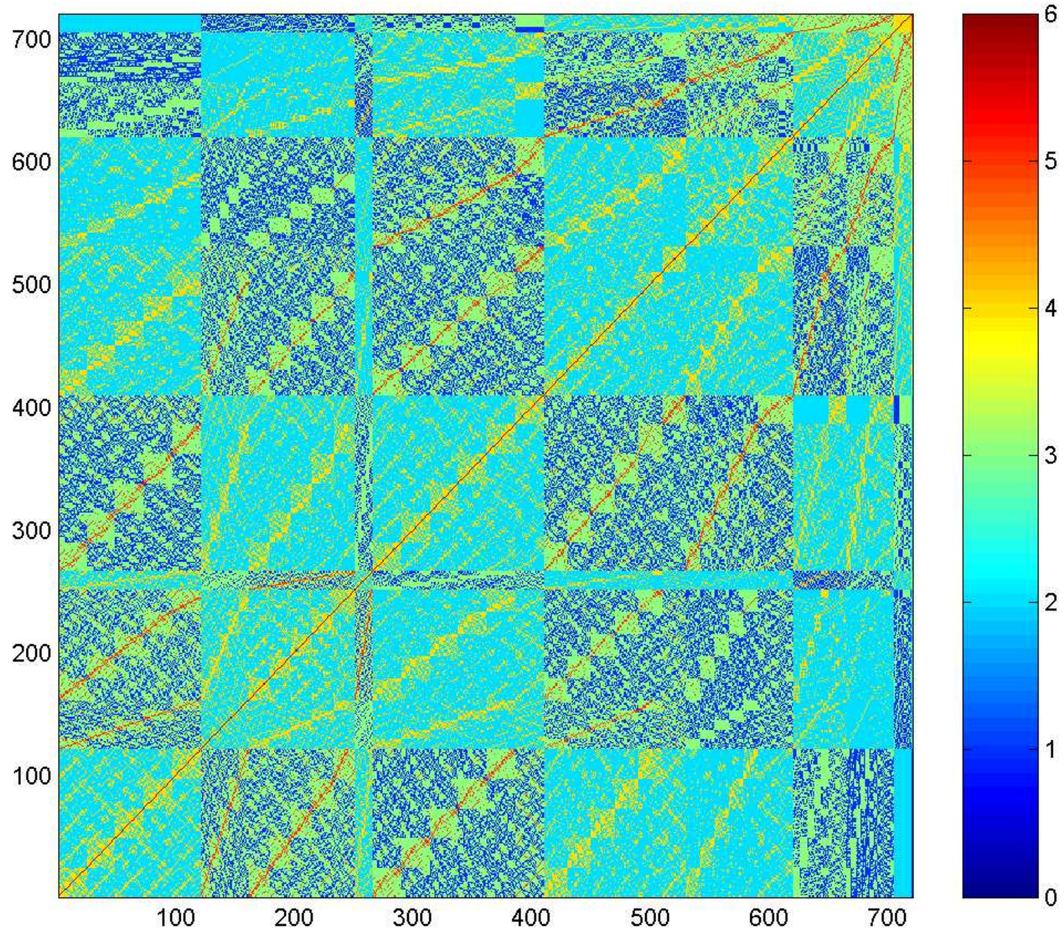
$$\text{EMIQ}_{\pi,\pi'} = \text{Tr}\left[P_\pi\, \Phi^{\otimes n}(P_{\pi'})\right].$$

- Even for very simple channels this leads to combinatorial questions.
- For example, let Φ be the identity channel: $\Phi(\rho) = \rho$.
- In that case, we need to consider $\text{Tr}[P_\pi P_{\pi'}^T]$ for any pair $\pi, \pi' \in S_n$.
- This is equal to d^ℓ, where ℓ is the number of cycles in $\pi \circ \pi'^{-1}$.
- Plotting ℓ in terms of π and π' produces nice wallpaper designs.

Example: S_5

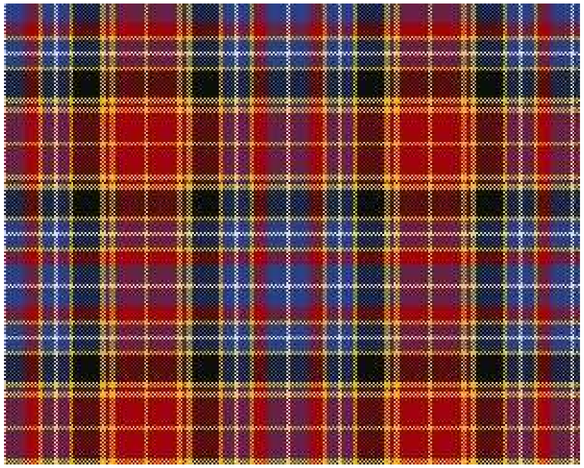


Example: S_6

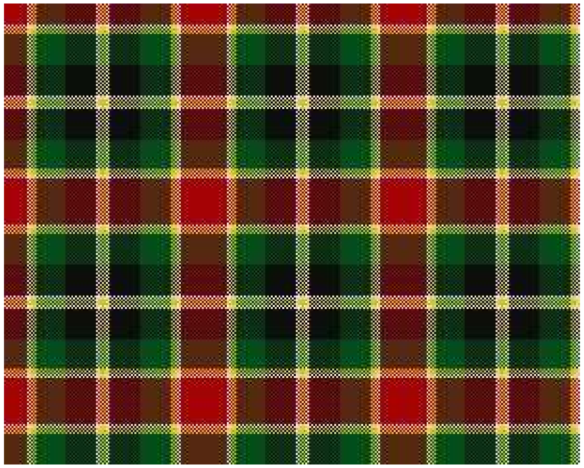


Example: Subgroups of S_n

Cameron



Dalrymple



MacLachlan

A Simplification

- Fortunately, we don't need $\mathrm{Tr}\left[P_\pi\,\Phi^{\otimes n}(P_{\pi'})\right]$ *per se*!
- To calculate $\nu_q(\Phi \otimes \Phi)$, we need the sums

$$\Sigma_K(\pi) := \sum_{\pi' \in S_n} \left(\mathrm{Tr}\left[P_\pi\,\Phi^{\otimes n}(P_{\pi'})\right]\right)^K$$

- It is enough to know the histogram of the trace: the number $n(\alpha)$ of permutations π' for which the trace assumes the value α.
- Then $\Sigma_K(\pi)$ is the K-th order moment of $n(\alpha;\pi)$:

$$\Sigma_K(\pi) = \int d\alpha\ n(\alpha;\pi)\alpha^K.$$

Example: Φ identity map.

- We want to calculate

$$\Sigma_K(\pi) := \sum_{\pi' \in S_n} \left(\mathrm{Tr}\left[P_\pi \, P_{\pi'}\right]\right)^K$$

- Histogram of trace independent of π: π can be absorbed in π';
- and given by absolute value of Stirling numbers of first kind:

$$n(\alpha) = |s(n,k)|, \qquad \alpha = d^k.$$

- Sums of powers follow from generating function of Stirling numbers:

$$\Sigma_K = \sum_{k=0}^{n} |s(n,k)| \, (d^k)^K = d^K \, (d^K + 1) \ldots (d^K + n - 1).$$

Conclusion

- We have built a bridge from (multi)linear algebra to combinatorics
- Known results in 1 might lead to new results in 2
- Fighting chance for solving BOP of MOP
- Easter Egg Wish List:
  - Toolset for calculations with P^λ and friends
  - Asymptotic formulas for $\lambda \vdash n \to \infty$ *a la* Keyl-Werner
- For some notes on these and related topics, see:

`http://www.qols.ph.imperial.ac.uk/~kauden/QITnotes`