Norm Compression Inequalities for Block Partitioned Matrices

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Block Partitioned Matrices

- Matrix = representation of a linear mapping from one space V_1 to another, V_2
- Example: a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ can be represented by the matrix

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}.$$

- Suppose we single out orthogonal subspaces in V_1 and V_2 and partition V_1 and V_2 as direct sums

$$V_1 = \bigoplus_j V_{1j}$$
$$V_2 = \bigoplus_i V_{2i}$$

- Linear mappings between V_1 and V_2 can then be represented by Block Matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

or, for short,

$$A = [A_{ij}],$$

where A_{ij} are matrices themselves.

- A_{ij} represents a mapping from V_{1j} to V_{2i}
- When $V_1 = V_2$, so $V_{1i} = V_{2i}$, the partitioning of the block matrix is Symmetric: e.g.

$$A = \left(\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \hline a_3 & b_3 & c_3 \end{array}\right).$$

Matrix Trace Norms

- Among the zoo of norms that can be defined for matrices, the Schatten q-norms (a.k.a. Trace Norms) stand out:
- For $q \geq 1$, $||A||_q := (\mathrm{Tr}(|A|^q))^{1/q}$.
- The matrix absolute value $|A|$ is defined as $|A| = (A^* A)^{1/2}$.
- Non-commutative generalisation of ℓ_q-norms for vectors.
- Depend only on singular values of A: $||A||_q = (\sum_i \sigma_i(A)^q)^{1/q}$
- Special cases:
  - $q = 1$: Trace Norm $||A||_1 = \mathrm{Tr}\,|A|$
  - $q = 2$: Frobenius Norm $||A||_2 = (\mathrm{Tr}(A^* A))^{1/2} = \left(\sum_{ij} |A_{ij}|^2\right)^{1/2}$.
  - $q = \infty$: Operator Norm $||A||_\infty = \sigma_1^\downarrow(A)$.

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$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} ||A_{11}||_q & ||A_{12}||_q & \cdots \\ ||A_{21}||_q & ||A_{22}||_q & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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- Q: What can you then tell me about $||A||_q$?
- A: You could give me upper and lower bounds on $||A||_q$ using Norm Compression Inequalities (NCI's)!

Overview: NCI's for the Schatten norms

1. Previously known NCI's
2. An NCI for positive semidefinite matrices
3. Conjectures, with or without partial proofs

Learn more about this from math.FA/0505680 at `arxiv.org`

1. Previously known NCI’s

Pinching Inequality

- Let A be symmetrically partitioned: $A = [A_{ij}]$
- The Pinching of A is obtained by setting all off-diagonal blocks to 0:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} A_{11} & 0 & \dots \\ 0 & A_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \bigoplus_i A_{ii}.$$

- The Pinching Inequality says: for any unitarily invariant norm

$$|||A||| \geq |||\bigoplus_i A_{ii}|||.$$

- For Schatten q-norms, this is an NCI:

$$||A||_q \geq \left|\left|\bigoplus_i A_{ii}\right|\right|_q = \left(\sum_i ||A_{ii}||_q^q\right)^{1/q}$$

An Upper Bound

- There is a complementary NCI for *positive semidefinite* (PSD) symmetrically partitioned block matrices:

$$||A||_q \leq \sum_i ||A_{ii}||_q.$$

- Ref: Horn and Johnson's "Topic in Matrix Analysis", p. 217 *Problem* 22.
- When A is not PSD, there is no NCI upper bound in terms of only the diagonal block norms $||A_{ii}||_q$.

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- When A is not PSD, there is no NCI upper bound in terms of only the diagonal block norms $||A_{ii}||_q$.
- Reason: if one of the off-diagonal block norms goes off to ∞, so does $||A||_q$!
- That can't happen when A is PSD because then $||A_{ij}||_q^2 \le ||A_{ii}||_q \, ||A_{jj}||_q$.

NCI's exploiting all blocks

- For a general $d \times d$ partitioned matrix $A = [A_{ij}]$, with $1 \leq i, j \leq d$, one has, for $1 \leq q \leq 2$:

$$d^{2-q}||A||_q^q \geq \sum_{i,j} ||A_{ij}||_q^q \geq ||A||_q^q$$

- The reversed inequalities hold for $q \geq 2$.
- Ref: Bhatia and Kittaneh
- Note that the bound is stated in terms of the norm of the *vectorised* norm compression $(||A_{11}||_q, ||A_{12}||_q, \ldots)$
- The splitting up in two cases ($q < 2$, $q > 2$) is a common phenomenon; likely to happen when an NCI reduces to equality for $q = 2$.

An NCI for 2X2 PSD matrices

- Chris King found an NCI for PSD 2×2 block matrices:

$$\left|\left| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right|\right|_q \geq \left|\left| \begin{pmatrix} ||A_{11}||_q & ||A_{12}||_q \\ ||A_{21}||_q & ||A_{22}||_q \end{pmatrix} \right|\right|_q, \quad 1 \leq q \leq 2,$$

  while the reversed inequality holds for $q \geq 2$.

- $||A||_q \leq ||\,[\ ||A_{ij}||_q\ ]\,||_q$ holds for *integer* q and any partitioning.
- Generalisation to higher numbers of blocks does not hold for non-integer q.
- Counterexample: 4×4 partitioning, blocks A_{ij} are scalars,

$$A = \begin{pmatrix} 3 & 0 & -2 & -2 \\ 0 & 3 & 2 & -1 \\ -2 & 2 & 4 & 0 \\ -2 & -1 & 0 & 3 \end{pmatrix} \geq 0, \qquad \begin{cases} ||\,A\,||_{1.5} & = \ 9.49929 \\ ||\,[\ |A_{ij}|\ ]\,||_{1.5} & = \ 9.63184 \end{cases}$$

Strong Sharpness

- In the setting of NCI's, it makes sense to ask for the *sharpest possible* bounds.
- Strongly sharp NCI's can be saturated for any allowed choice of the constituent quantities of the bound.
- Both the Pinching and H&J inequality are strongly sharp: for any possible choice of $a_i \geq 0$, there is an A such that $||A_{ii}||_q = a_i$ and equality holds in the NCI.
- King's bound is also strongly sharp. Take positive, scalar blocks.
- Bhatia and Kittaneh's bounds are *not* strongly sharp.
- My goal in life: find some more strongly sharp NCI's

2. Another NCI for PSD matrices

A Companion for Chris' Inequality

- Recall, for $A \geq 0$, and $1 \leq q \leq 2$,

$$\left\| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right\|_q \geq \left\| \begin{pmatrix} ||A_{11}||_q & ||A_{12}||_q \\ ||A_{21}||_q & ||A_{22}||_q \end{pmatrix} \right\|_q ,$$

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- This invites the question: What about an *upper bound* for $1 \leq q \leq 2$, and a lower bound for $q \geq 2$?

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- This invites the question: What about an *upper bound* for $1 \leq q \leq 2$, and a lower bound for $q \geq 2$?

- The answer is:

$$\left|\left|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right|\right|_q \leq \left(||A_{11}||_q^q + ||A_{22}||_q^q + (2^q - 2)||A_{12}||_q^q\right)^{1/q} ,$$

  and the reversed inequality for $q \geq 2$.

That strange factor...

- Factor $(2^q - 2)$ looks rather weird, but...
- For $q = 1$ and $q = 2$ we know

$$\left|\left|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right|\right|_1 = ||A_{11}||_1 + ||A_{22}||_1$$

$$\left|\left|\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right|\right|_2^2 = ||A_{11}||_2^2 + ||A_{22}||_2^2 + 2||A_{12}||_2^2.$$

- The factor $2^q - 2$ on $||A_{12}||_q^q$ interpolates between 0 and 2.
- One can make even more sense out of it by reformulating the inequality.
- W.l.o.g. one can take $A_{12} = A_{21} \geq 0$ (use polar decomposition).
- Put $A_{11} = B$, $A_{12} = A_{21} = C$, $A_{22} = D$, all PSD.

Reformulation

- Now note the following:

$$\operatorname{Tr}\begin{pmatrix} C & C \\ C & C \end{pmatrix}^q = 2^q \operatorname{Tr} C^q, \quad \operatorname{Tr}\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}^q = 2 \operatorname{Tr} C^q.$$

- The inequality can thus be rewritten as

$$\operatorname{Tr}\begin{pmatrix} B & C \\ C & D \end{pmatrix}^q - \operatorname{Tr}\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}^q \leq \operatorname{Tr}\begin{pmatrix} C & C \\ C & C \end{pmatrix}^q - \operatorname{Tr}\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}^q.$$

- Both sides are non-negative, since $\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$ is a pinching of $\begin{pmatrix} B & C \\ C & D \end{pmatrix}$.
- The left-hand side is the amount of norm decrease caused by this pinching.
- The inequality says that, when fixing C and constraining B and D to keep A positive, this norm decrease is maximal when $B = D = C$.

Strong Sharpness

- For scalar blocks, my bound is *not* strongly sharp, because Chris's bound is an equality in that case.
- My bound is strongly sharp when going to blocks of size at least 2×2, provided $||C||_q \leq ||B||_q, ||D||_q$.
- By a theorem of Horn and Mathias, positivity of A implies $||C||_q^2 \leq ||B||_q ||D||_q$.
- For $||B||_q \leq ||C||_q \leq (||B||_q ||D||_q)^{1/2}$ one can find a better bound than mine (and I have a conjecture!).

Strong Sharpness

- Witnesses of strong sharpness:

$$B = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \quad D = \begin{pmatrix} \delta & 0 \\ 0 & \gamma \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix},$$

  for β, γ and δ non-negative numbers.

- Then $||B||_q^q = \beta^q + \gamma^q$, $||D||_q^q = \delta^q + \gamma^q$, $||C||_q = \gamma$ and

$$\left|\left| \begin{pmatrix} B & C \\ C^* & D \end{pmatrix} \right|\right|_q^q = \beta^q + \delta^q + 2^q \gamma^q$$
$$= ||B||_q^q + ||D||_q^q + (2^q - 2)||C||_q^q.$$

Proof of the inequality

- General Idea: fix C and show that the maximum of

$$\mathrm{Tr}\begin{pmatrix} B & C \\ C & D \end{pmatrix}^q - \mathrm{Tr}\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}^q$$

  over all B, D such that $A \geq 0$ is obtained in $B = D = C$.

- Step 1: Fix D and maximise over B. Allowed B obey $B \geq B_0 := CD^{-1}C$.
- Put $B = B_0 + t\Delta$, with $\Delta \geq 0$, and define

$$f(t) := \mathrm{Tr}\begin{pmatrix} B_0 + t\Delta & C \\ C & D \end{pmatrix}^q - \mathrm{Tr}\begin{pmatrix} B_0 + t\Delta & 0 \\ 0 & D \end{pmatrix}^q .$$

- We find that $f'(t) \leq 0$ so that $f(t)$ is maximal in 0.
- Therefore, maximum obtained for $B = CD^{-1}C$.

Proof of the inequality

- Step 2: Maximise

$$f(D) := \mathrm{Tr}\begin{pmatrix} CD^{-1}C & C \\ C & D \end{pmatrix}^q - \mathrm{Tr}\begin{pmatrix} CD^{-1}C & 0 \\ 0 & D \end{pmatrix}^q$$

  over all $D \geq 0$.

- Easy part: $D = C$ is a stationary point of $f(D)$.
- Hard part:
  - $D = C$ is the *only* stationary point;
  - $f(C) \geq f(D)$ for D on the *boundary* of the set $D \geq 0$.

Proof of Dual Case

- All the above was for $1 \leq q \leq 2$.
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- Well, that's what I thought...
- Spot the mistake in the following argument!

Proof of Dual Case

- Define p such that $1/p + 1/q = 1$; if $q \geq 2$ then $1 \leq p \leq 2$.
- The Schatten p-norm is the dual norm of the Schatten q-norm: For $A \geq 0$,

$$||A||_q = \max_X \{\mathrm{Tr}[AX] : X \geq 0, ||X||_p \leq 1\}.$$

- Hölder's inequality: for $A, B \geq 0$,

$$\mathrm{Tr}[AB] \leq ||A||_q ||B||_p.$$

  with equality if $B = A^{p-1}$.
- Duality proof exploits the following facts:
  - For every $A \geq 0$ there is an optimal $B \geq 0$ with $||B||_q = 1$, such that $||A||_p = \mathrm{Tr}[AB]$.
  - For all other such B, one has $||A||_p \geq \mathrm{Tr}[AB]$.

Proof of Dual Case

- Now consider the expression, with $p \geq 2$,

$$||B \oplus D \oplus (2^p - 2)^{1/p} C||_p.$$

- Take $P, Q, R \geq 0$ such that $P \oplus R \oplus (2^q - 2)^{1/q} Q$ is optimal for this norm:

$$\begin{aligned} & ||B \oplus D \oplus (2^p - 2)^{1/p} C||_p \; ||P \oplus R \oplus (2^q - 2)^{1/q} Q||_q \\ = \; & \mathrm{Tr}\left[(B \oplus D \oplus (2^p - 2)^{1/p} C)\,(P \oplus R \oplus (2^q - 2)^{1/q} Q)\right] \\ = \; & \mathrm{Tr}[BP + DR + (2^p - 2)^{1/p} (2^q - 2)^{1/q} CQ] \\ \leq \; & \mathrm{Tr}[BP + DR + 2CQ], \end{aligned}$$

- Here I have used that for all q, $(2^p - 2)^{1/p} (2^q - 2)^{1/q} \leq 2$.

Proof of Dual Case

- Using my bound for $1 \leq q \leq 2$:

$$\left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q \leq \left|\left| P \oplus R \oplus (2^q - 2)^{1/q} Q \right|\right|_q .$$

and Hölder's inequality, gives

$$\begin{aligned} ||B \oplus D \oplus (2^p - 2)^{1/p} C||_p \left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q &\leq \mathrm{Tr}[BP + DR + 2CQ] \\ &= \mathrm{Tr}\left[\begin{pmatrix} B & C \\ C & D \end{pmatrix} \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right] \\ &\leq \left|\left| \begin{pmatrix} B & C \\ C & D \end{pmatrix} \right|\right|_p \left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q \end{aligned}$$

- Dividing out $\left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q$ we get the desired bound in p.

The mistake in the "Proof"

- I took $P, Q, R \geq 0$ such that

$$\begin{aligned} &||B \oplus D \oplus (2^p - 2)^{1/p} C||_p \; ||P \oplus R \oplus (2^q - 2)^{1/q} Q||_q \\ &= \operatorname{Tr}\left[\left(B \oplus D \oplus (2^p - 2)^{1/p} C\right) \left(P \oplus R \oplus (2^q - 2)^{1/q} Q\right) \right]. \end{aligned}$$

- I applied my bound on $\left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q$.
- To do that I need $\begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \geq 0$, which is *not* guaranteed by the assumptions!

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\begin{aligned}
&||B \oplus D \oplus (2^p - 2)^{1/p} C||_p \; ||P \oplus R \oplus (2^q - 2)^{1/q} Q||_q \\
&= \operatorname{Tr}\left[(B \oplus D \oplus (2^p - 2)^{1/p} C) \, (P \oplus R \oplus (2^q - 2)^{1/q} Q) \right].
\end{aligned}
$$

- I applied my bound on $\left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q$.
- To do that I need $\begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \geq 0$, which is *not* guaranteed by the assumptions!
- A duality proof is like *black magic*. If used without care, it kills you!
- To fix the above proof, we need a rather strange, but beautiful inequality.

3. Conjectures, and other pain-killers

A Conjecture, for redemption

- Let $\begin{pmatrix} B & C \\ C & D \end{pmatrix}$ now be a general *Hermitian* matrix.
- The Positive Part of a Hermitian matrix is what one gets by setting negative eigenvalues to 0.
- Let $\begin{pmatrix} P & Q \\ Q & R \end{pmatrix} = \begin{pmatrix} B & C \\ C & D \end{pmatrix}_+$
- The duality proof can be fixed if, for $1 \le q \le 2$,

$$||P \oplus R \oplus (2^q - 2)^{1/q} Q||_q \le ||B \oplus D \oplus \frac{2}{(2^p - 2)^{1/p}} C||_q$$

- I only need the somewhat weaker inequality

$$\left|\left| \begin{pmatrix} P & Q \\ Q & R \end{pmatrix} \right|\right|_q \le ||B \oplus D \oplus \frac{2}{(2^p - 2)^{1/p}} C||_q.$$

A Conjecture, for strength

- Let $b = ||B||_q$, $c = ||C||_q$ and $d = ||D||_q$.
- Recall that my bound was only strongly sharp for the case $c \leq b, d$.
- In the other cases, e.g. $b \leq c \leq (bd)^{1/2}$, I conjecture the bound

$$\left|\left|\begin{pmatrix} B & C \\ C & D \end{pmatrix}\right|\right|_q^q \leq d^q - (c^2/b)^q + (b + c^2/b)^q.$$

- My conjectures are getting stranger by the minute, but...

A Conjecture, for strength

- This is equivalent to

$$\left\|\begin{pmatrix} B & C \\ C & D \end{pmatrix}\right\|_q^q - \left\|\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}\right\|_q^q \le \left\|\begin{pmatrix} x & c \\ c & y \end{pmatrix}\right\|_q^q - \left\|\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right\|_q^q,$$

with:

$$\begin{aligned} x = y = c, &\quad \text{if} \quad c \le b, d \\ x = b, y = c^2/b, &\quad \text{if} \quad b \le c \\ x = c^2/d, y = d, &\quad \text{if} \quad d \le c. \end{aligned}$$

- And I have cases of equality, so the bound will be strongly sharp (if true).

And now for something completely different

- Why do I care?
- Norm compression inequalities feature in proofs of the multiplicativity property of the $1 \to q$ norm of certain classes of completely positive maps.
- A linear map Φ is completely positive (CP) if and only if Φ preserves positive semidefiniteness of $N \times N$ PSD block matrices when it acts on them blockwise, i.e. as $\Phi : [A_{ij}] \mapsto [\Phi(A_{ij})]$, and this for whatever value of N.
- Letting Φ be a completely positive map, its $1 \to q$ norm is defined as

$$||\Phi||_{1 \to q} = \max_{||X||_1 = 1} ||\Phi(X)||_q .$$

Multiplicativity of p-to-q norms

- Multiplicativity of this norm w.r.t. the tensor product is the statement that, for two CP maps Φ_1 and Φ_2:

$$||\Phi_1 \otimes \Phi_2||_{1\to q} = ||\Phi_1||_{1\to q} \, ||\Phi_2||_{1\to q}.$$

- Shown for various classes of CP maps within various ranges of q.
- Unfortunately, there exist counterexamples when $q > 4.79$.
- It might still be true for any tensor product of CP maps for values of q close to 1. Then one could prove additivity of an entropic counterpart of multiplicativity, and with it a host of other additivity results concerning CP maps.
- That, in turn, would solve a number of long-standing open problems in quantum information theory.

A Conjecture, for qubit CP maps

- For CP maps Φ_1 and Φ_2, when Φ_1 acts on 2×2 matrices, multiplicativity would follow from the NCI-like inequality:
- Let A be a PSD matrix with trace 1, block partitioned as $A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}$, with $\beta = ||B||_q$ and $\delta = ||D||_q$. Then

$$|| \, (\Phi_1 \otimes \mathbf{I})(A) \, ||_q \leq \max_\theta \left|\left| \, \Phi_1(\begin{pmatrix} \beta & \exp(i\theta)\sqrt{\beta\delta} \\ \exp(-i\theta)\sqrt{\beta\delta} & \delta \end{pmatrix}) \right|\right|_q .$$

- I have a proof when $\Phi_1(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \geq 0$. Optimal θ is then 0.
- Interesting Corollary of the Conjecture: Hanner's inequality for matrices.

Hanner's Inequality

- For general matrices A and B, and $1 \leq p \leq 2$, Hanner's inequality reads

$$||A+B||_p^p + ||A-B||_p^p \geq (||A||_p + ||B||_p)^p + |\,||A||_p - ||B||_p|^p,$$

while for $2 \leq p$, the inequality is reversed.

- Proven in the following instances (Ball, Carlen and Lieb):

  1. For all $1 \leq p \leq \infty$ when $A+B$ and $A-B$ are positive semidefinite.
  2. For all $1 \leq p \leq 4/3$, $p=2$, and $4 \leq p \leq \infty$ when A and B are general matrices.

- I have been able to show that the full statement would follow from the conjecture on the previous page.
- It would also follow as a special case of my Big Fat Conjecture...

Final Conjecture, the biggest of them all!

- Let T be a general matrix partitioned in $2 \times N$ blocks:

$$T = \begin{pmatrix} A_1 & A_2 & \cdots & A_N \\ B_1 & B_2 & \cdots & B_N \end{pmatrix}.$$

Then I conjecture the following NCI, for $p \geq 2$:

$$||T||_p \leq \left|\left| \begin{pmatrix} ||A_1||_p & ||A_2||_p & \cdots & ||A_N||_p \\ ||B_1||_p & ||B_2||_p & \cdots & ||B_N||_p \end{pmatrix} \right|\right|_p$$

while for $1 \leq p \leq 2$ the ordering of the inequality is reversed.

- Hanner's inequality: $N = 2$, $B_2 = A_1$, $B_1 = A_2$.
- Chris' NCI: $N = 2$, $T \geq 0$.

Final Conjecture, the biggest of them all!

- I have proofs for special cases:
- All B_k zero
- All A_k and B_k are row vectors
- All A_k and B_k are column vectors
- $A_k = \alpha_k X$, for some scalars α_k and some matrix X, and similarly, $B_k = \beta_k Y$.
- $p \geq 4$; follows easily from Chris' NCI.
- Maybe my own NCI has a bearing on the $2 \leq p \leq 4$ case?