Periodicities in trees associated with the classification of p-groups by width, rank and obliquity

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Report on Diploma Thesis

p-Groups: Rank, Width, and Obliquity

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Introduction

Definition. *Let G be a pro-p group.*

- *The* **rank** *of G is*

$$\sup\{\mathrm{d}(U) \mid U \leq_c G\}.$$

- *The* **width** *of G is*

$$\sup_{i\geq 1} \log_p(|\gamma_i(G) : \gamma_{i+1}(G)|).$$

- *The* **obliquity** *of G is*

$$\sup_{i\geq 1} \log_p(|\gamma_i(G) : \mu_i(G)|)$$

  where

$$\mu_i(G) = \bigcap\{N \lhd_c G \mid N \not\leq \gamma_i(G)\}.$$

The graph $\mathcal{G}(p, r, w, o)$

- Vertices correspond to finite p-groups of rank r, width w, and obliquity o (up to isomorphism).
- An edge joins $G/\gamma_c(G)$ to G where c is the class of G.

Project: Analyse $\mathcal{G}(p, r, w, o)$.

(1) Determine the infinite paths.
$\rightsquigarrow$ Classify infinite pro-p groups of rank r, width w, and obliquity o up to isomorphism.

(2) Investigate the branches attached to the infinite paths. Is there a periodicity?

[(3) Consider the remaining "sporadic" groups.]

Infinite pro-p groups

Klaas et al.:

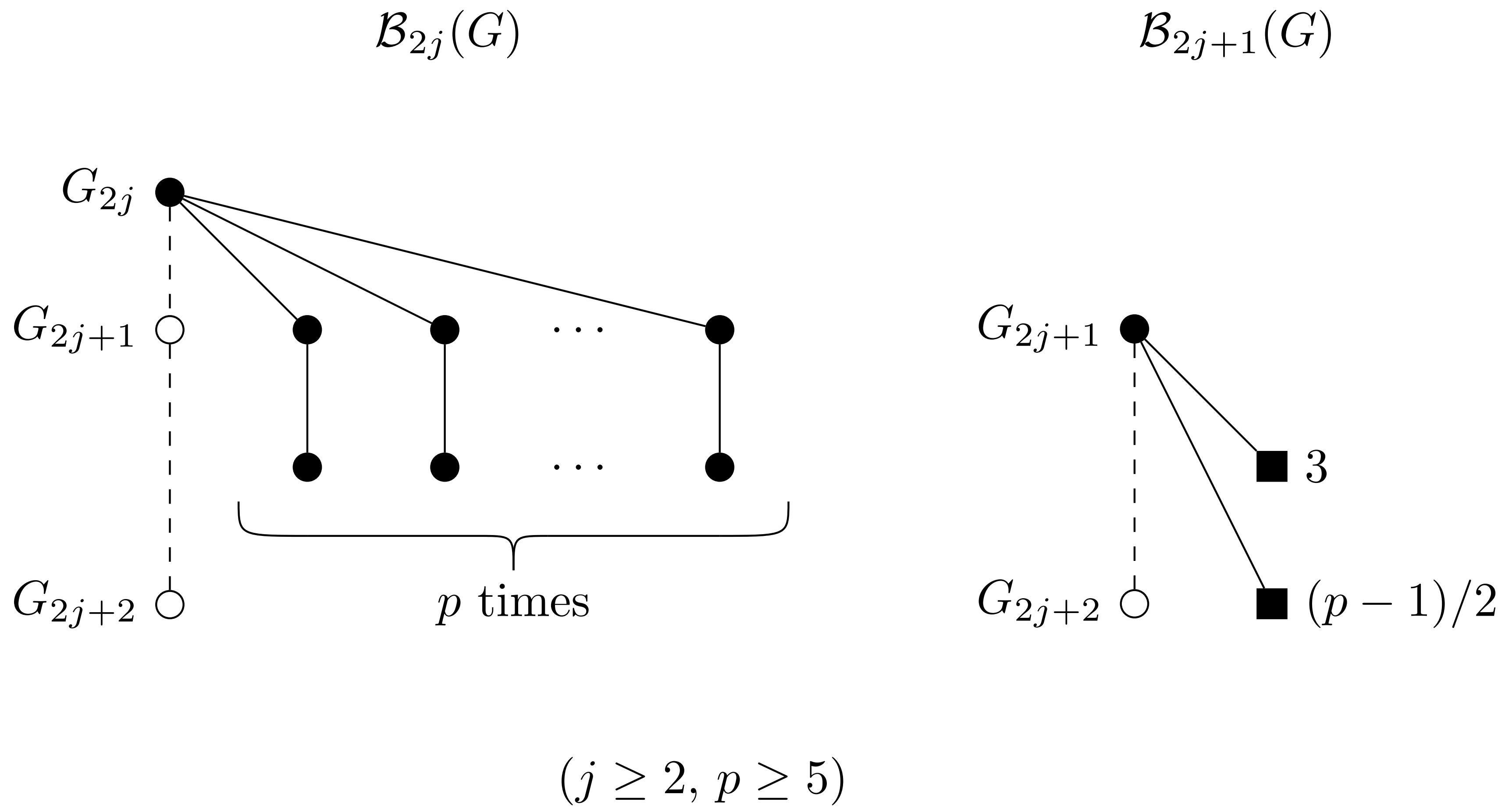
- Classification of infinite pro-p groups of finite rank, width, and obliquity up to isomorphism of their Lie algebras.
- The possible Lie algebras correspond to *maximal* groups.
- Each infinite pro-p group of finite rank, width, and obliquity can be obtained as a subgroup of one of these maximal groups.

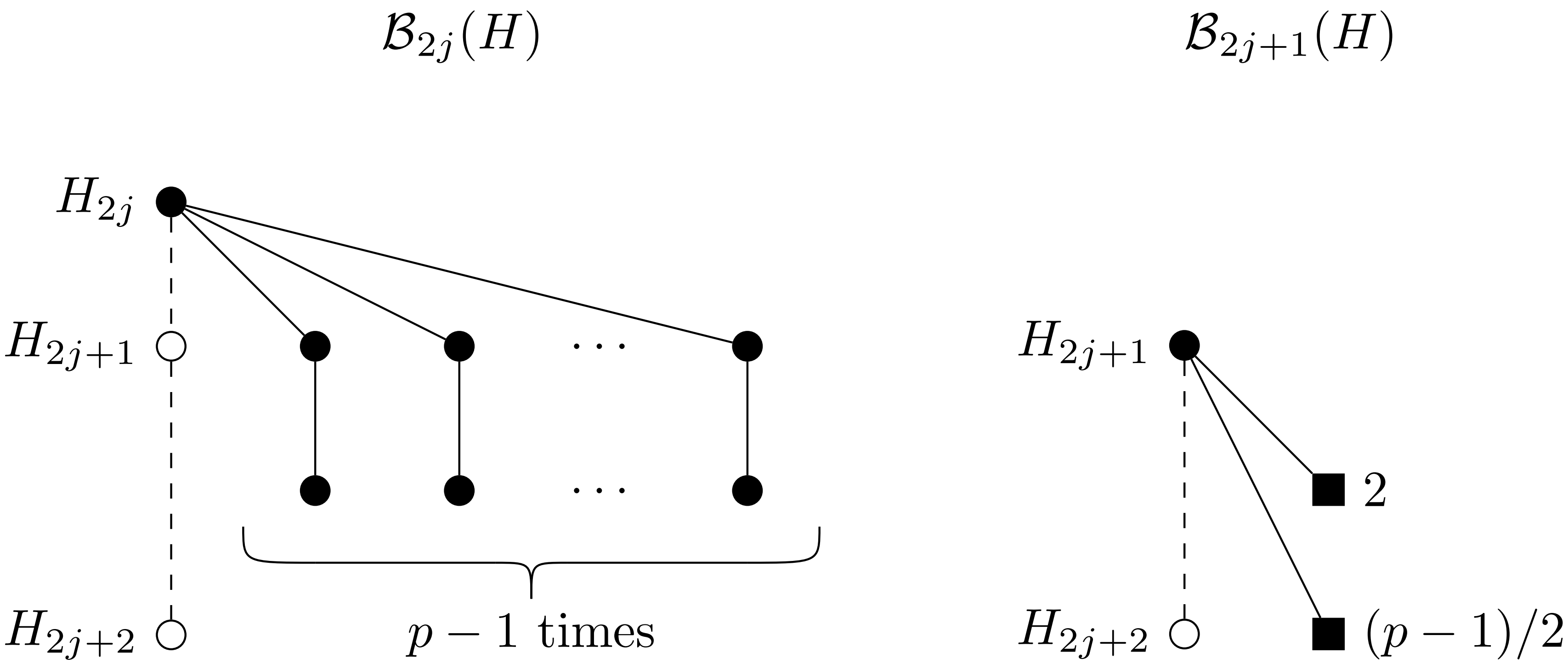
Example:

- There are two simple three-dimensional Lie algebras over $\mathbb{Q}_p$: $\mathfrak{sl}_2(\mathbb{Q}_p)$ and $\mathfrak{sl}_1(D)$.
- Let G and H be the corresponding maximal groups.
- Leedham-Green & McKay: If $p \geq 3$, then G and H represent the two isomorphism classes of infinite pro-p groups of rank 3, width 2, and obliquity 0.

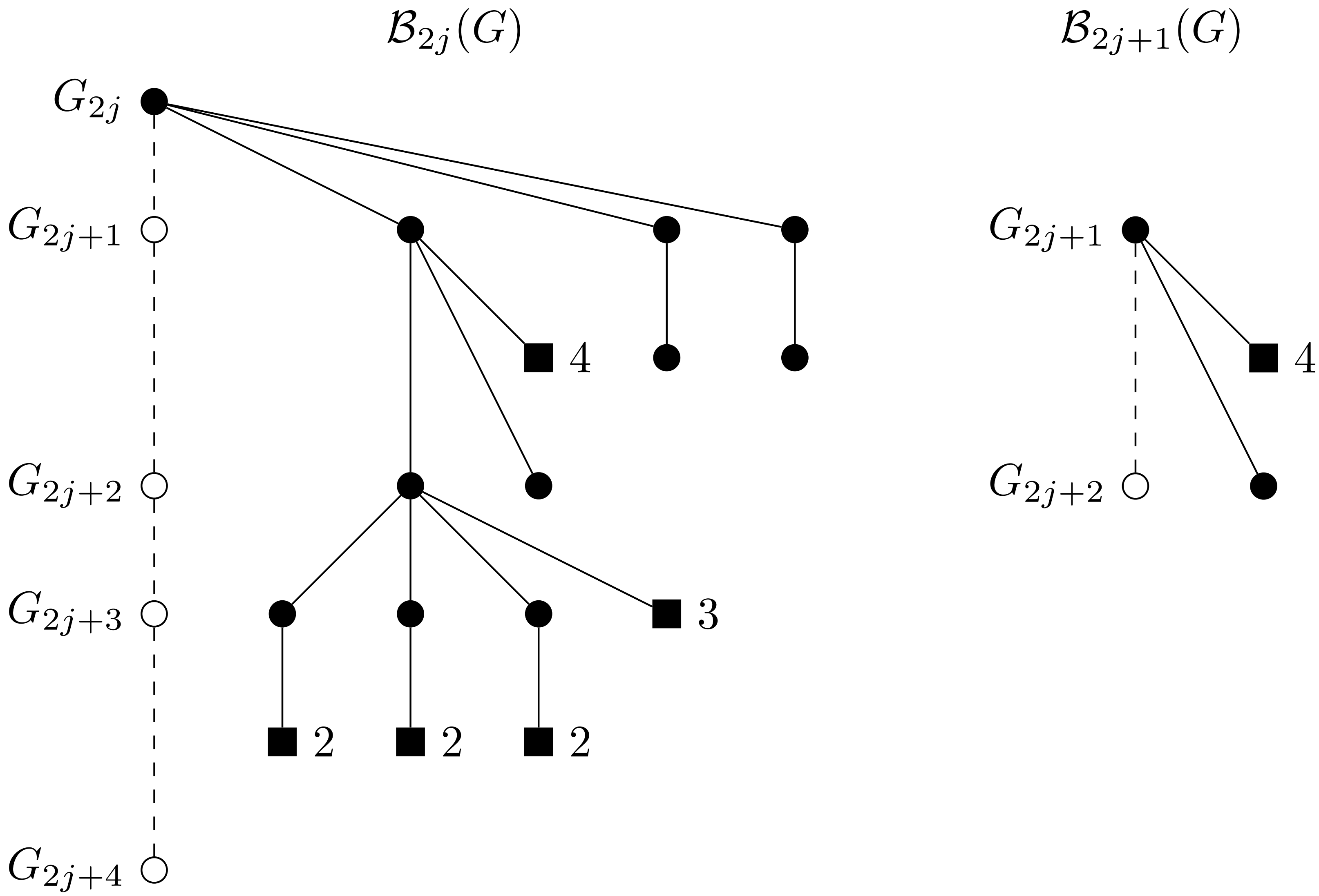
$\mathcal{G}(p,3,2,0)$ for $p \geq 3$

- Sanderson, Leedham-Green & McKay: First investigation of the graph. In particular, they proved that branches have depth at most 2 if $p \geq 5$.
- Detailed analysis using GAP, in particular ANUPQ.
- Result: Branches seem to be periodic with period 2.
- Precise shapes...

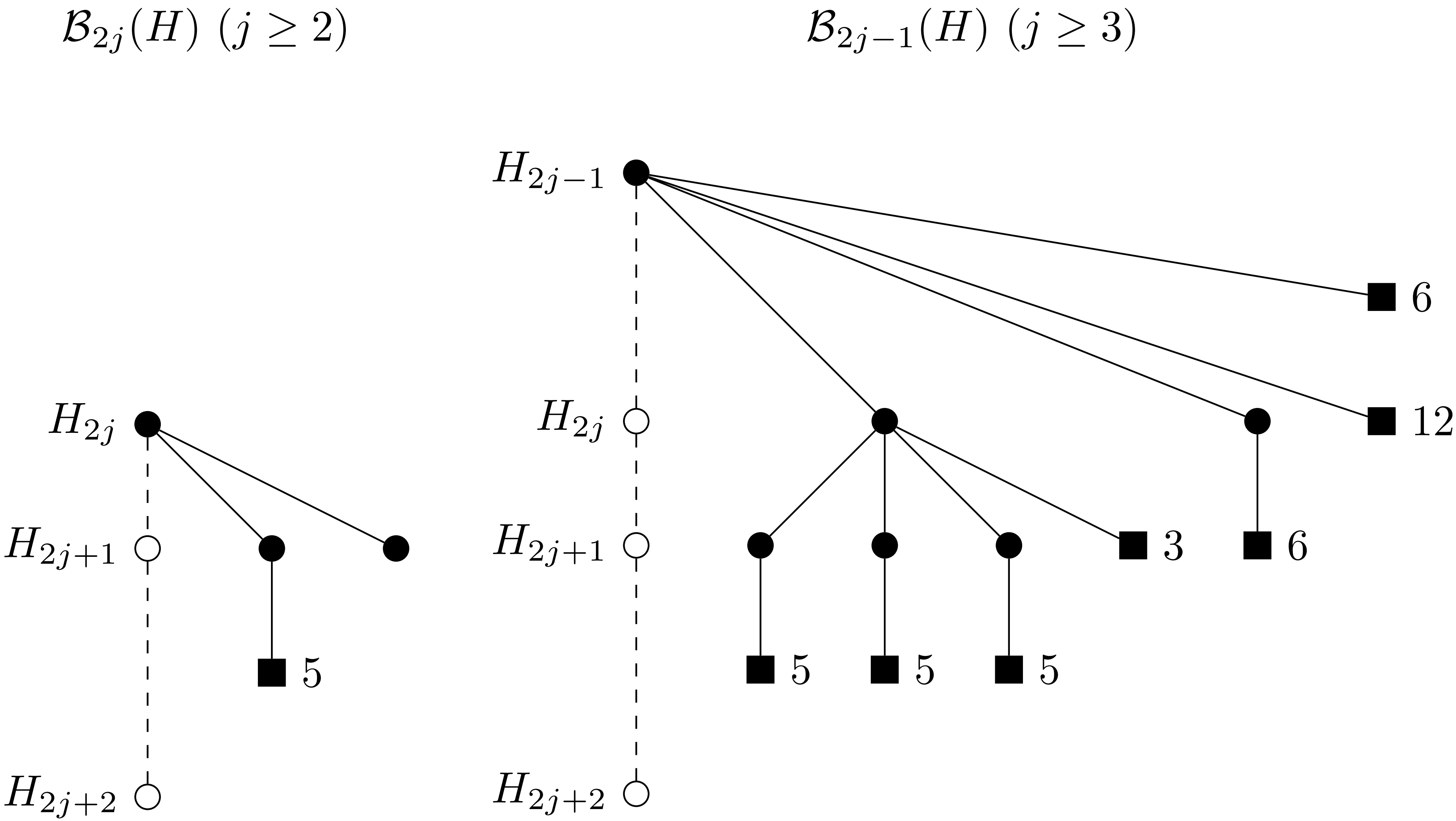

$\mathcal{B}_{2j}(G)$
$\mathcal{B}_{2j+1}(G)$
G_{2j}
G_{2j+1}
G_{2j+2}
$\cdots$
p times
G_{2j+1}
G_{2j+2}
3
$(p-1)/2$
$(j \geq 2,\ p \geq 5)$


$\mathcal{B}_{2j}(H)$
$\mathcal{B}_{2j+1}(H)$
H_{2j}
H_{2j+1}
H_{2j+2}
$p-1$ times
H_{2j+1}
H_{2j+2}
2
$(p-1)/2$

$(j \geq 2,\, p \geq 5)$


$\mathcal{B}_{2j}(G)$
$\mathcal{B}_{2j+1}(G)$
G_{2j}
G_{2j+1}
G_{2j+2}
G_{2j+3}
G_{2j+4}
4
3
2
2
2
G_{2j+1}
G_{2j+2}
4

$(j \geq 2, p = 3)$

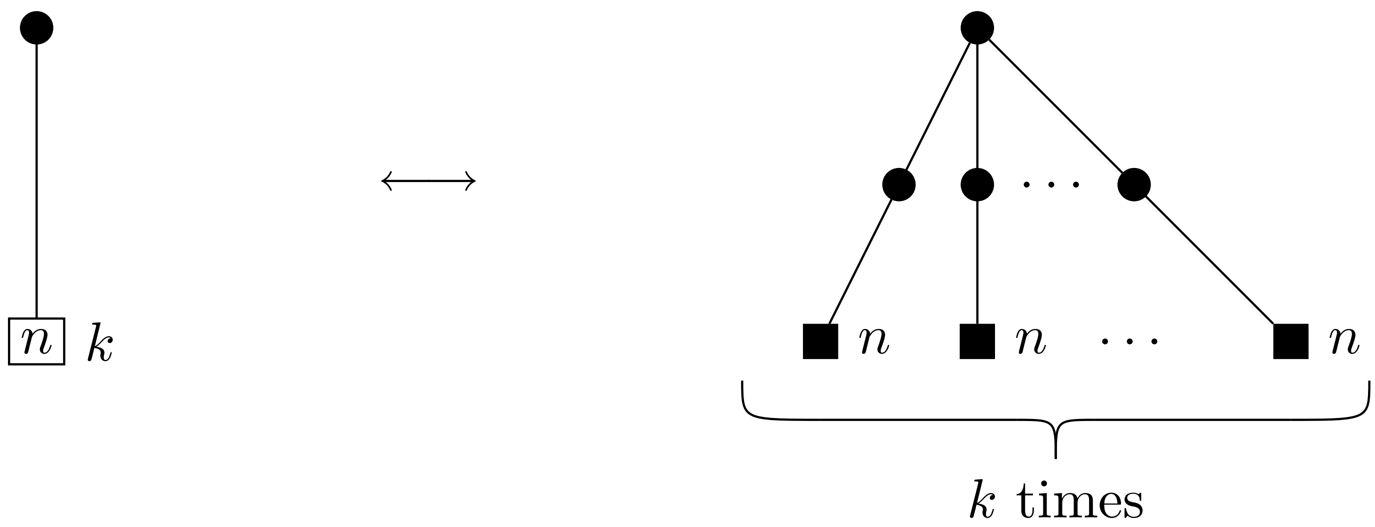

$\mathcal{B}_{2j}(H)$ $(j \geq 2)$
$\mathcal{B}_{2j-1}(H)$ $(j \geq 3)$
H_{2j-1}
H_{2j}
H_{2j+1}
H_{2j+2}
H_{2j}
H_{2j+1}
H_{2j+2}
6
12
3
6
5
5
5
5

$(p = 3)$

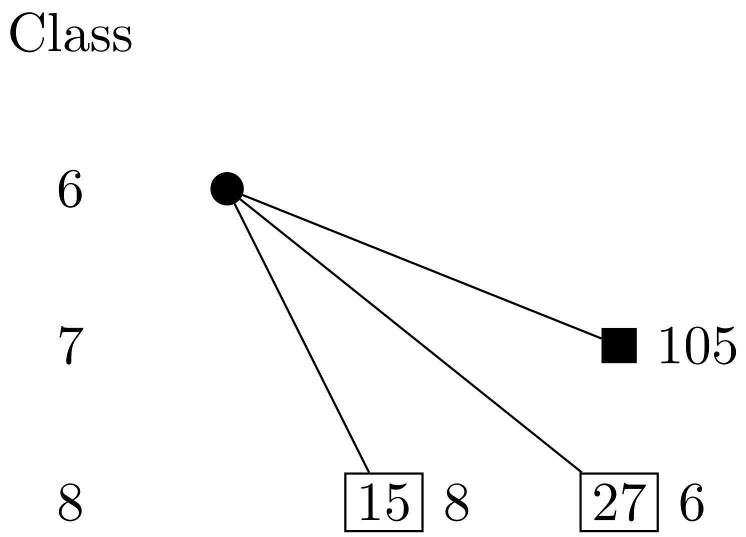
$\mathfrak{sl}_n(K)$

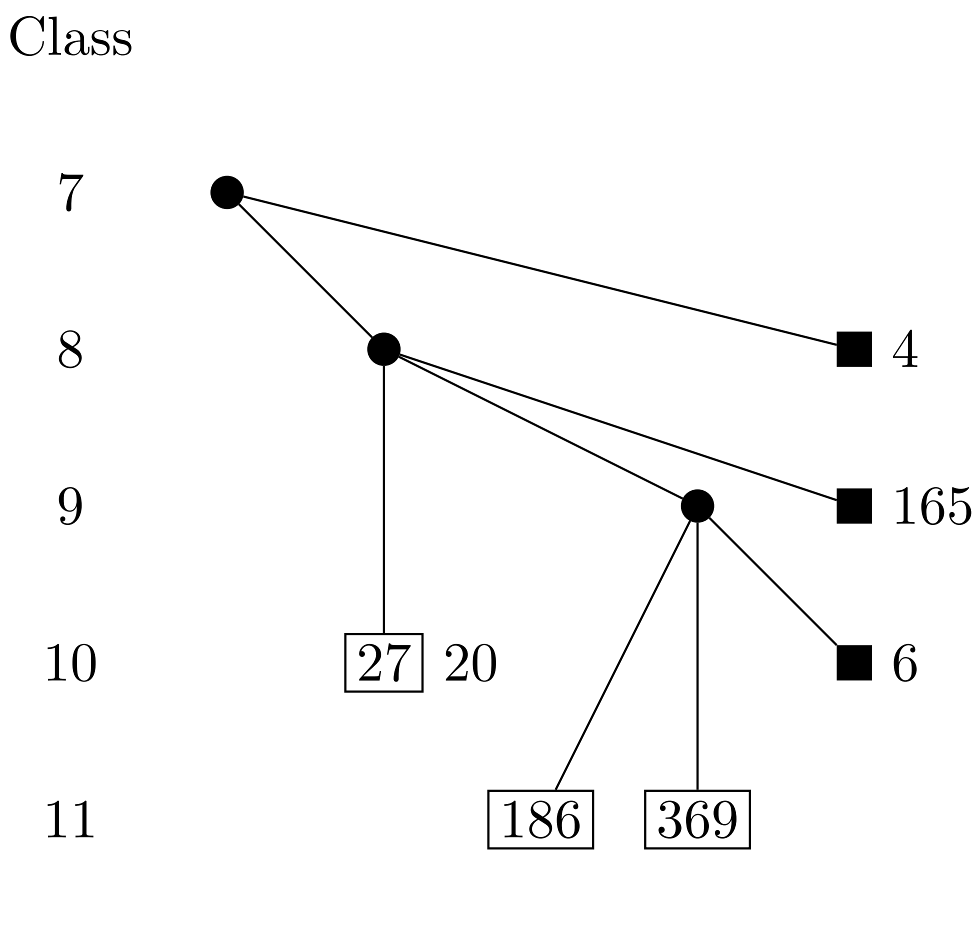
- Another interesting class of pro-p groups: The maximal groups with Lie algebra of the form $\mathfrak{sl}_n(K)$.
- Investigate the associated branches; ignore the rank, width, and obliquity and consider only those groups with the "right" lower central pattern.
- An example: $\mathfrak{sl}_2(K)$ for $p = 3$ and $K = \mathbb{Q}_3(\sqrt{-3})$.
- Result: Graph seems to be periodic with period 4.
- Explicit branches...

Notation

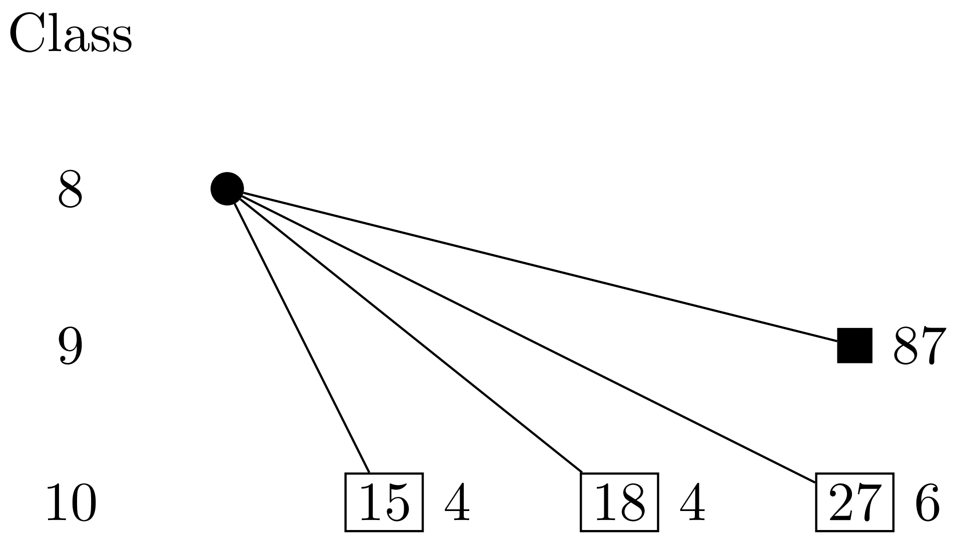


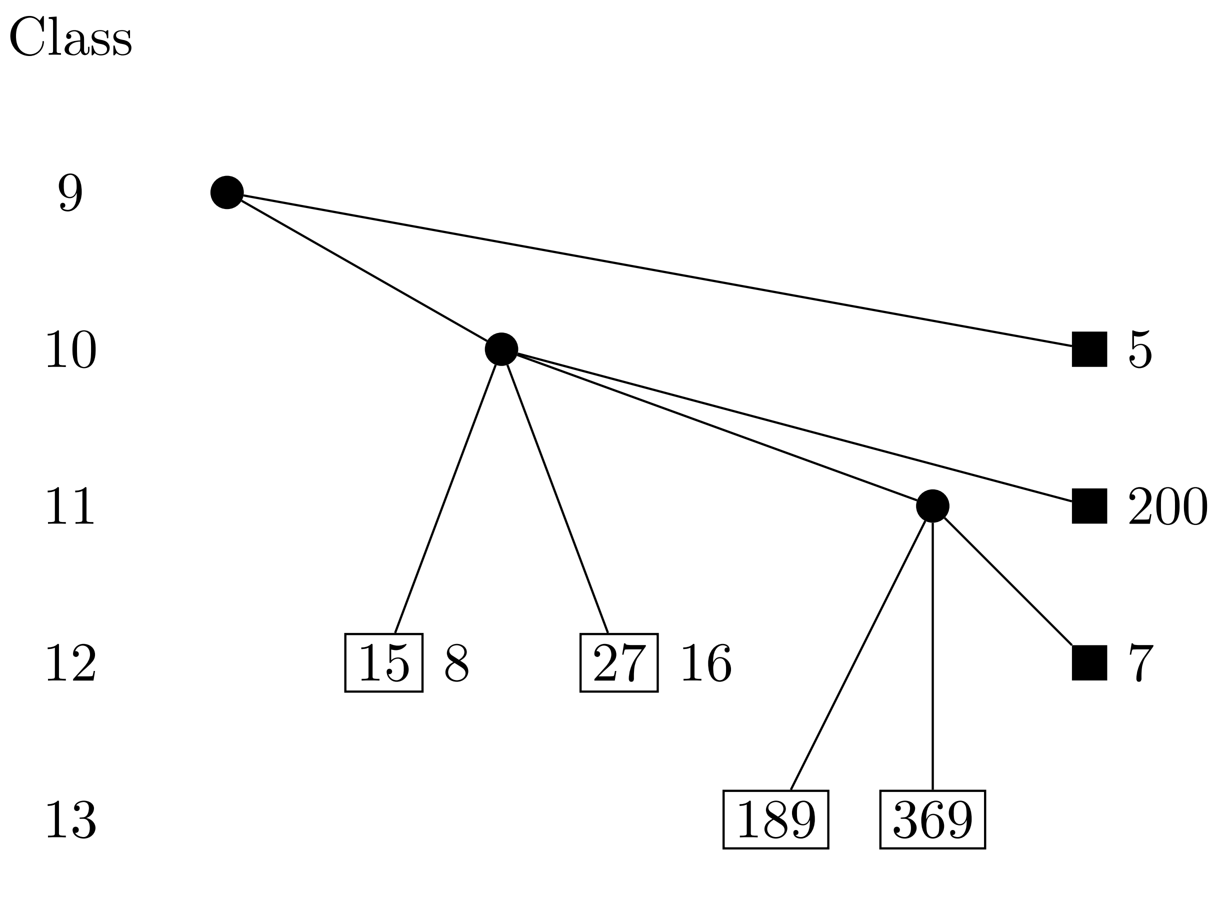
$$\mathfrak{sl}_2(K), \quad p = 3, \quad K = \mathbb{Q}_3(\sqrt{-3})$$





$$\mathfrak{sl}_2(K), \quad p = 3, \quad K = \mathbb{Q}_3(\sqrt{-3})$$





$\mathfrak{sl}_n(K)$

- Klaas et al.: classification of maximal groups of dimension ≤ 14.
- Similar results occur for various choices of p, fields K with $|K : \mathbb{Q}_p| \leq 4$, and $n \leq 3$.
- As these numbers increase, the groups become less accessible.
- (partial) data support the conjecture that these branches *always* repeat periodically

Cohomology and $\mathcal{G}(p, 3, 2, 0)$

- Let $p \geq 5$ and Q be either G or H.
- A group in $\mathcal{B}_i(Q)$ is an extension of a Q_2-module A by Q_i.
- Using the results of Sanderson, A is found to be a Q_2-module quotient of $\gamma_2(Q)/\gamma_4(Q)$ or of $\gamma_3(Q)/\gamma_5(Q)$.

Theorem. *$H^2(Q_i, A) \cong H^2(Q_{i+2}, A)$ for all sufficiently large i.*