Narrow Lie algebras

Sandro Mattarei

Cambridge, 11 July 2008

Summary

A (Graded) Lie algebras of maximal class

– in characteristic zero

– in prime characteristic

B Some tools: loop algebras; certain simple modular Lie algebras.

C Thin Lie algebras (width 2 and obliquity 0)

– generalities

– second diamond in degree $2q-1$

– second diamond in degree q

A Lie algebras of maximal class (Lamc)

A residually nilpotent Lie algebra L (over a field F) is *of maximal class* if
$\dim L/L^2 = 2$, and $\dim L^i/L^{i+1} \le 1$ for $i \ge 2$.

M. Vergne (1966-68) showed that, already over $\mathbb{C}$, there are "too many" of those.

Assume then that L is $\mathbb{N}$-graded (gLamc):

$$L = \bigoplus_{i=1}^{\infty} L_i, \qquad [L_i, L_j] \subseteq L_{i+j}.$$

Example (metabelian). $M = \bigoplus_{i \ge 1} M_i$,
with basis $\{x, y_i : i \ge 1\}$, $\deg x = 1$, $\deg y_i = i$,

$$[y_i, x] = y_{i+1}, \qquad [y_i, y_j] = 0.$$

M is generated by M_1, and $\dim M_1 = 2$.
Up to isomorphism, M is the unique infinite-dim gLamc generated by M_1.

Its subalgebra generated by x and y_t is $\cong M$ as a Lie algebra, but as a *graded* Lie algebra it is generated in degrees 1 and t.

Glamc generated in degrees 1 and 2

Example (metabelian, too). $K = \bigoplus_{i \geq 1} K_i$, with basis $\{x, y_i : i \geq 2\}$, $\deg x = 1$, $\deg y_i = i$,

$$[y_i, x] = y_{i+1},$$
$$[y_i, y_2] = y_{i+2} \qquad (i \geq 3),$$
$$[y_i, y_j] = 0 \qquad (i, j \geq 3).$$

Example (insoluble). $W_1^+ = \bigoplus_{i \geq 1} F e_i$,

$$[e_i, e_j] = (j - i) e_{i+j}.$$

W_1^+ is the *positive part* of the first Witt algebra

$$W_1 := \mathrm{Der}(F[x]) = \bigoplus_{i \geq -1} F e_i,$$

where $e_i = x^{i+1} \frac{d}{dx}$.

[Shalev-Zelmanov 1997] If L is an infinite-dim gLamc generated by L_1 and L_2, then L is isomorphic with one of M, K, W_1^+.

A. Shalev and E.I. Zelmanov, *Narrow Lie algebras: a co-class theory and a characterization of the Witt algebra*, J. Algebra **189** (1997), 294–331.

A coclass theory in characteristic zero

From now on consider *only* gLa $L = \bigoplus_{i \geq 1} L_i$ generated by L_1. Note that $L^j = \bigoplus_{i \geq j} L_i$ then.

Such L has *finite coclass* if there is an integer r such that $\dim(L/L^i) \leq i + r$ for all i. The smallest such r is the *coclass* of L.

Assume $\operatorname{char} F = 0$.

Vergne proved that if L is a gLa of coclass one and dimension d, then $L/Z(L) \cong M/M^{d-1}$ (and even $L \cong M/M^d$ when d is odd).

[Shalev-Zelmanov 1997] There is a function f such that, if L is a gLa of coclass r then $L/Z_{f(r)}(L) \cong M/M^d$, for some d.

As a consequence, analogues of the coclass conjectures for pro-p groups hold for gLa, and even in a *stronger* form.
This extends to char $p > 0$ (only) assuming that L is *restricted* (Riley-Semple 1994).

Some insoluble gLamc in prime characteristic

[Shalev 1994] For each prime p there exists countably many insoluble gLamc L over $\mathbb{F}_p$.

Thus, the analogues of the coclass conjectures fail for gLa of characteristic $p > 0$, already in the coclass one case.

A. Shalev, *Simple Lie algebras and Lie algebras of maximal class*, Arch. Math. **63** (1994), 297–301.

Shalev's algebras are built as *loop algebras* of certain simple finite-dimensional Lie algebras, of dim p^n-1 ($n > 1$), with a certain *nonsingular derivation*.

By construction, they have a "periodic" structure. One may still hope that all infinite-dim gLmc arise from a loop algebra construction, and that those given by Shalev are the only insoluble ones. Both assertions are false.

Many gLamc in prime characteristic

[Caranti-M-Newman 1997] Over F of positive char there are $|F|^{\aleph_0}$ isomorphism types of (non-periodic) gLamc.

A. Caranti, S. Mattarei, and M.F. Newman, *Graded Lie algebras of maximal class*, Trans. Amer. Math. Soc. **349** (1997), 4021–4051.

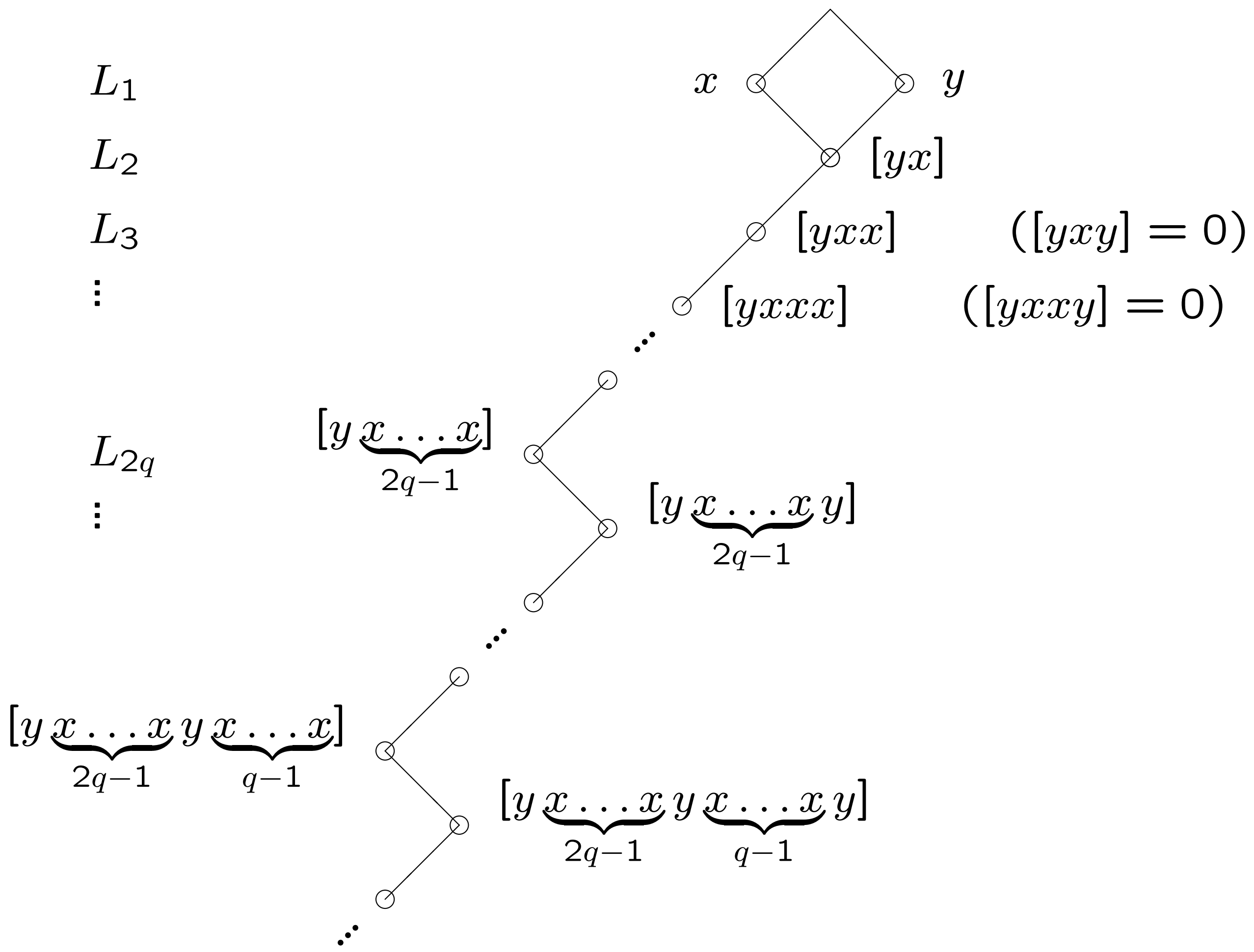
For a gLamc $L = \bigoplus_{i\geq 1} L_i$, the centralizers $\mathbf{C}_{L_1}(L_i)$, for $i \geq 2$, are one-dimensional subspaces of L_1. Their sequence

$$\left(\mathbf{C}_{L_1}(L_i)\right)_{i\geq 2}$$

determines L up to isomorphism.

Say $\mathbf{C}_{L_1}(L_2) = Fy$ is the first centralizer in order of occurrence, then the earliest occurrence of a second centralizer Fx (if $L \not\cong M$) is in degree $2q$ with q a power of p: $\mathbf{C}_{L_1}(L_{2q}) = Fx$.

Centralizers other than the first occur at isolated places in the sequence, separated by several occurrences of the first centralizer.



The distances between centralizers other than the first have the form $2q$ or $2q - p^t$ $(t \geq 0)$. In Shalev's loop algebras the distances are as follows, for some r, s powers of p:

$$2q, (q^{r-2}, 2q - 1, (q^{r-2}, 2q)^{s-1})^{\infty}.$$

Shalev's algebras are *periodic,* of period $qrs-1$.

A classification of gLamc

Inflation: a way of constructing a new gLamc from a given one by inserting a new (or old) centralizer $p-1$ times between each two consecutive centralizers in the original sequence.

This can be repeated finitely or infinitely many times. In the latter case one gets uncountably many algebras.

[Caranti-Newman 2000] All infinite-dimensional gLamc can be obtained via finitely or infinitely many inflation steps from Shalev's algebras or a (limit) variation of them.

A. Caranti and M. F. Newman, *Graded Lie algebras of maximal class. II*, J. Algebra **229** (2000), 750–784.

[Jurman 2005] A similar result in char two, with an extra type besides Shalev's algebras.

G. Jurman, *Graded Lie algebras of maximal class. III*, J. Algebra **284** (2005), 435–461.

The relevance of finite presentability

A crucial ingredient in the classification is the fact [Carrara 2001] that Shalev's loop algebras are "almost" finitely presented, that is, an extension of them by an infinite-dim centre is finitely presented (with homogeneous relations).

This implies that, *as gLamc,* they are determined by a suitable finite-dim quotient.

C. Carrara, *(Finite) presentations of the Albert-Frank-Shalev Lie algebras*, Boll. Un. Mat. Ital. (Ser. VIII) **4** (2001), 391–427.

A simpler but analogous result [Caranti 1997] for a loop algebra of Witt's p-dimensional simple Lie algebra will play a role in Ershov's proof that the Nottingham group is finitely presented as pro-p group (for odd p).

A. Caranti, *Presenting the graded Lie algebra associated to the Nottingham group*, J. Algebra **198** (1997), 266–289.

B Loop algebras

Take $S = \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} S_i$ finite-dim La, $\quad U \subseteq S_{[1]}$. The (positive part of the twisted) *loop algebra* of S is the Lie subalgebra of $S \otimes \mathbb{F}[t]$ generated by $U \otimes t$.

A special case is the following:

$$S = \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} S_i, \qquad \dim(S_i) = 1$$
$$D \text{ a derivation}, \quad DS_i = S_{i+1}.$$

The loop algebra of $S \oplus FD$ with respect to $U = S_1 \oplus FD$ is then an infinite-dim gLamc:

$$L = \bigoplus_{i \geq 1} L_i \qquad \subseteq (S \oplus FD) \otimes F[t],$$

where $\qquad L_1 = S_{[1]} \otimes t + FD \otimes t,$

and $\qquad L_i = S_{[i]} \otimes t^i \quad \text{for } i > 1.$

Shalev noted that there exist simple Lie algebras S satisfying these requirements. They have dimension $p^n - 1$, for $n > 1$.

Certain Block algebras

Let $q = p^n > p$, $\mathbb{F}_q \subseteq F$, and let $f : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ a non-degenerate alternating $\mathbb{F}_p$-bilinear form. Then

$$S = \bigoplus_{\alpha \in \mathbb{F}_q^*} F u_\alpha, \qquad [u_\alpha, u_\beta] = f(\alpha, \beta) u_{\alpha+\beta},$$

is a simple Lie algebra over F, of dim $q - 1$, a special case of a *Block algebra*.

The linear map defined by $D u_\alpha = \alpha \cdot u_\alpha$ is a derivation, and is nonsingular, as $D^{q-1} = \mathrm{id}$.

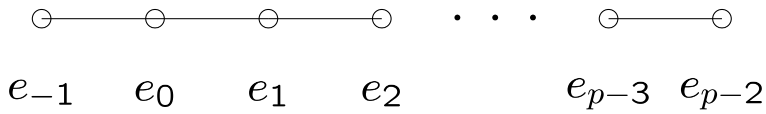
One can produce a grading of S over $\mathbb{Z}/(q-1)\mathbb{Z}$ such that $DS_i = S_{i+1}$. The corresponding loop algebra is the gLamc constructed by Shalev.

In modern notation this S is $H(2; \mathbf{n}; \Phi(\tau))^{(1)}$. Here the pair $\mathbf{n} = (n_1, n_2)$ depends on the alternating form f, with $p^{n_1+n_2} = q$.

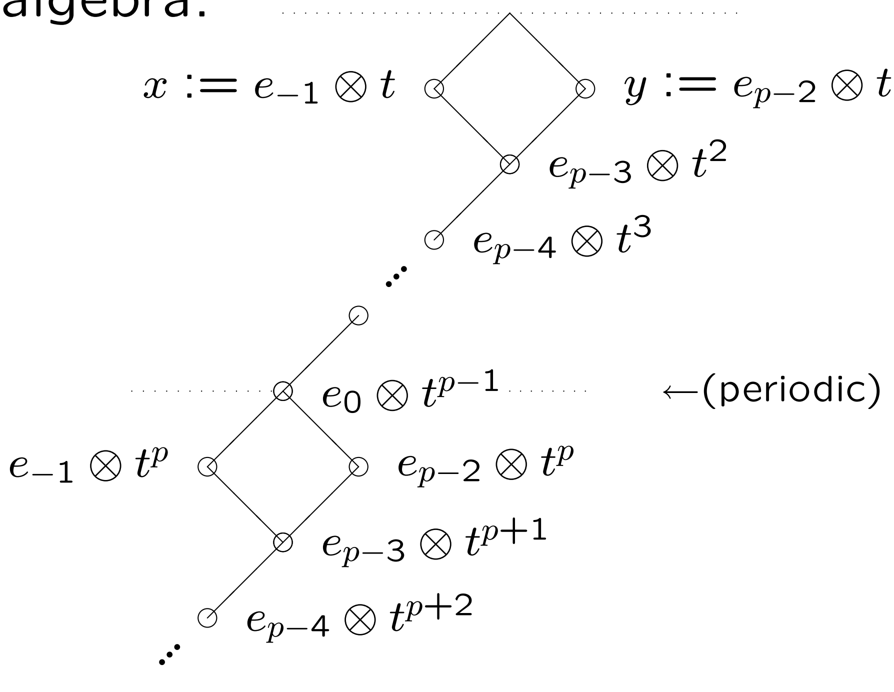
Another example of a loop algebra

Consider the Witt algebra

$$W(1;1) = \bigoplus_{i=-1}^{p-2} Fe_i, \quad [e_i, e_j] = (j-i)e_{i+j}.$$



Now view its $\mathbb{Z}$-grading modulo $p-1$, and build the loop algebra:



This is the graded Lie algebra associated with the lower c. s. of the Nottingham group.

Some modular simple Lie algebras we will need to mention

According to the classification of simple modular Lie algebras (completed in the 1990's), every simple modular Lie algebra over an algebraically closed field of char > 3 is of classical, *Cartan*, or (only in char 5) *Melikian* type.

Out of the four series of Cartan type W, S, H, K we will use only a few algebras, namely,

- the *Zassenhaus algebra* $W(1; n)$, of dim p^n,

and three types of the Hamiltonian series,

- *the graded Hamiltonian algebras* $H(2; \mathbf{n})^{(2)}$, of dim $p^n - 2$,

- the *Block algebras* $H(2; \mathbf{n}; \Phi(\tau))^{(1)}$, of dim $p^n - 1$, already mentioned, and

- the *Albert-Zassenhaus algebras* $H(2; \mathbf{n}; \Phi(1))$, of dim p^n.

C Thin groups and Lie algebras

A *thin* p-group (Brandl-Caranti-Scoppola 1992) is a two-generated p-group G such that every normal subgroup N either contains or is contained in some term $\gamma_i(G)$ of the lower c. s.

A *thin* algebra is a gLa $L = \bigoplus_{i \geq 1} L_i$, with $\dim L_1 = 2$, such that every homogeneous ideal I either contains or is contained in some L^i. Equivalently, a thin algebra is a gLa of *width* two and *obliquity* zero.

A p-group G is thin iff the graded Lie algebra associated to $\{\gamma_i(G)\}$ is thin.

The condition on the homogeneous ideals can be replaced by the *covering property*

$$L_{i+1} = [u, L_1] \text{ for all } 0 \neq u \in L_i, \text{ for all } i \geq 1.$$

A *diamond* is a two-dimensional homogeneous component. L_1 is the first diamond. If there are no other diamonds then L is a gLamc, a case we may as well exclude by definition.

The second diamond in thin groups

[Caranti-M-Newman-Scoppola 1996]

1. Let G be an infinite thin pro-p group, and let L be the gLa associated to $\{\gamma_i(G)\}$. If L_k is the second diamond, then k can only be 3, 5, or p.

2. If $k = 3$ and L_4 is not a diamond, or if $k = 5$, then L belongs to three possible isomorphism types.

A. Caranti, S. Mattarei, M.F Newman and C.M Scoppola, *Thin groups of prime-power order and thin Lie algebras*, Quart. J. Math. Oxford Ser. (2) **47** (1996), 279–296.

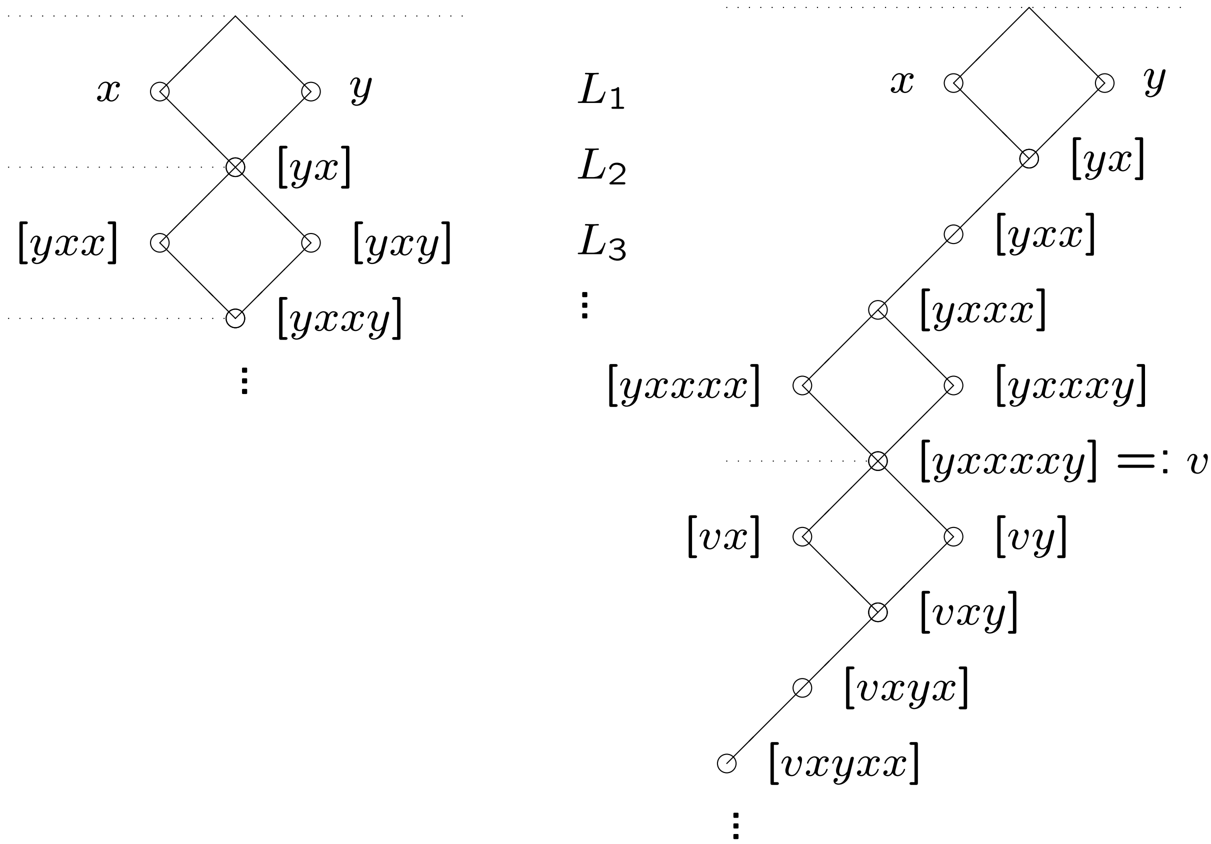
The case $k = p$ occurs for the graded Lie algebra associated with the lower central series of the Nottingham group, for odd p.

Pro-p groups corresponding to the three types with $k = 3$ or 5 have been investigated in

S. Mattarei, *Some thin pro-p-groups*, J. Algebra **220** (1999), 56–72.

“Classical” thin algebras

Two of the three thin algebras mentioned are:



They are loop algebras of classical simple algebras of type A_1 and A_2. For example, the former is a loop algebra of

$$S = \mathfrak{sl}_2 = Ff \oplus Fh \oplus Fe = S_{-1} \oplus S_0 \oplus S_1$$

(with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$) where the degrees are viewed modulo 2.

The second diamond in thin algebras

[Caranti-Jurman 1999; Avitabile-Jurman 2001] Let L be a thin Lie algebra (with $\dim L = \infty$). If L_k is the second diamond, then k can only be 3, 5, q or $2q - 1$, where q is a power of p.

A. Caranti and G. Jurman, *Quotients of maximal class of thin Lie algebras. The odd characteristic case*, Comm. Algebra **27** (1999), 5741–5748.

M. Avitabile and G. Jurman, *Diamonds in thin Lie algebras*, Boll. Unione Mat. Ital. **4** (2001), no. 3, 597–608.

The cases $k = 3$ or 5 have been dealt with in [C-M-N-S 1996], except when both L_3 and L_4 are diamonds. Little is known in that case:

[Gavioli-Monti-Young 2001] The *metabelian* thin algebras L are in bijective correspondence with the quadratic extensions of the ground field F. All L_i except L_2 are diamonds.

N. Gavioli, V. Monti and D.S. Young, *Metabelian thin Lie algebras*, J. Algebra **241** (2001), 102–117.

“Nonclassical” thin algebras

• When $k = q$ we have the *Nottingham* algebras.

Several of them have been obtained as loop algebras of Zassenhaus algebras $W(1;n)$ (of dim p^n), or graded simple Hamiltonian algebras $H(2;\mathbf{n})^{(2)}$ (of dim p^n-2), or Albert-Zassenhaus algebras $H(2;\mathbf{n};\Phi(1))$ (of dim p^n).

• When $k = 2q-1$ we have the (-1)-*algebras*.

Several of them can be realized as loop algebras of Block algebras $H(2;\mathbf{n};\Phi(\tau))^{(1)}$ (of dim $p^n - 1$).

We know that both classes contain also (uncountably many) non-periodic algebras, which can be related to gLamc.

(Almost) finite presentations and explicit constructions

Typical results in this area come in pairs:

• A proof that a certain initial structure determines a unique thin algebra L. In other words, L is uniquely determined (as a thin algebra) by a suitable finite-dim quotient. Usually done by exhibiting a finite presentation for a central extension $\tilde{L}$ of L.

Methods: the Jacobi identity (many times), exploiting the periodicity of the structure.

• An explicit construction of $\tilde{L}$ as a loop algebra of a suitable finite-dimensional Lie algebra S. The centre of $\tilde{L}$ is related to the second cohomology group of the finite-dim algebra S.

Methods: guess the right algebra S (easy) and produce a suitable cyclic grading of it (harder).

Diamond types of (-1)-algebras

Let L be thin with second diamond L_{2q-1}. Choose generators $x, y \in L_1$, with $[L_2, y] = 0$. Then y centralizes all one-dim components not immediately preceding a diamond.

Suppose that L_k is a diamond, that $L_{k-1} = Fv$, that $[vxy] + [vyx] = 0$ and $[vyy] = 0$. Then

$$[vyx] = \lambda[vxx] \quad \text{for some } \lambda \in \mathbb{F} \cup \{\infty\}.$$

We say that L_k is a diamond of type λ.

When $\lambda = 0$ the element $[vy]$ would be central, and so we have a *fake* diamond $L_k = F[vx]$ (which has dim one rather than two).

These structural features of (-1)-algebras were first proved in special cases; now they are proved in full generality in:

S. Mattarei, *Constituents of graded Lie algebras of maximal class and chain lenghts of thin Lie algebras*, in preparation.

It follows that L is uniquely determined by specifying the degrees and types of the diamonds.

All diamonds of infinite type

[Caranti-M 1999] Let L be a (-1)-alg. with all diamonds of type ∞. Then L has a derivation D such that $M = L + FD$ is a graded Lie algebra of maximal class (with respect to a new grading).

A. Caranti and S. Mattarei, *Some thin Lie algebras related to Albert-Frank algebras and algebras of maximal class*, J. Austr. Math. Soc. **67** (1999), 157–184.

Actually, we can go back and obtain a bijective correspondence

(-1)-algebras with all diamonds of type ∞	$\leftrightarrow$	gLamc with only two centralizers

As we know, the class on the right contains $|F|^{\aleph_0}$ isomorphism types. Most of them are non-periodic and, hence, not loop algebras. Nevertheless, they are classified.

All diamonds of finite types

[Caranti-M 1999] There is a unique thin algebra with second diamond L_{2q-1} of finite type. The (possibly fake) diamonds occur in all degrees $\equiv 1 \pmod{q-1}$, and their types form an arithmetic progression.

In this description L_q receives type zero, and can be thought of as a fake diamond.

[Caranti-M 2005] Such L is constructed as a loop algebra of a Block algebra $H(2; \mathbf{n}; \Phi(\tau))^{(1)}$.

A. Caranti and S. Mattarei, *Gradings of non-graded Hamiltonian algebras*, J. Aust. Math. Soc. **79** (2005), 399–440.

The algebra used is the same used in Shalev's construction of the periodic insoluble graded Lie algebras of maximal class, but with respect to a different grading.

Diamonds of both finite and infinite types

[Avitabile-M 2005] There is a thin algebra with (possibly fake) diamonds in all degrees $\equiv 1 \pmod{q-1}$, all of type ∞ except those in degree $\equiv q \pmod{p^s(q-1)}$, whose types run over $0, 1, \ldots, p-1$ cyclically.

M. Avitabile and S. Mattarei, *Thin Lie algebras with diamonds of finite and infinite type*, J. Algebra **293** (2005), 36–64.

The algebra is uniquely determined by a certain finite-dimensional quotient of it (roughly modulo somewhere below the earliest diamond of type 1, to set off the arithmetic progr.).

It is constructed as a loop algebra of a Block algebra $H(2; \mathbf{n}; \Phi(\tau))^{(1)}$.

Instrumental in the construction is an Artin-Hasse exponential of an outer derivation of S, used to switch gradings [M 2005]. This is related to the *toral switching* technique in modular Lie algebras.

S. Mattarei, *Artin-Hasse exponentials of derivations*, J. Algebra **294** (2005), no. 1, 1–18.

Hopes for a classification?

[M] Let L be a (-1)-algebra with diamonds of arbitrary types. Then there is a (-1)-algebra $\bar{L}$ with diamonds in the same degrees as L, but all of type ∞.

S. Mattarei, *Deformations of thin Lie algebras*, in preparation.

Thus, the possible degree patterns in which diamonds occur in a (-1)-algebra are completely determined by the theory of gLamc. How to go back from $\bar{L}$ to L is essentially a deformation problem.

It seems reasonable to expect that as soon as L has at least one diamond of finite type, then it is a loop algebra (one of the types studied). Conditional results on the degree of such diamond are obtained in:

M. Avitabile and S. Mattarei, *Diamonds of finite type in thin Lie algebras*, submitted.

Nottingham algebras

For thin algebras with second diamond L_q one can also attach a type $\in F \cup \{\infty\}$ to each diamond, though in a different way.

Here the type of the second diamond can be set to be -1, and the *fake* diamonds are of two types: 0 and 1.

In several cases periodicity can be proved, and explicit constructions can be given. The diamonds occur at regular intervals (counting the fake ones), and their types follow arithmetic progressions.

According to the number of fake diamonds encountered in a period, the finite-dim algebra S of which L is a loop algebra may have dim p^n or $p^n - 2$.

Some patterns which have been studied are:

Constant sequence $\{-1\}$; $W(1;n)$, of dim p^n.
Arithm. progr. $\subseteq \mathbb{F}_p$; $H(2;\mathbf{n})^{(2)}$, of dim $p^n - 2$.
Arithm. progr. $\not\subseteq \mathbb{F}_p$; $H(2;\mathbf{n};\varPhi(1))$, of dim p^n.

M. Avitabile, *Some loop algebras of Hamiltonian Lie algebras*, Internat. J. Algebra Comput. **12** (2002), no. 4, 535–567.

A. Caranti and S. Mattarei, *Nottingham Lie algebras with diamonds of finite type*, Internat. J. Algebra Comput. **14** (2004), 35–67.

M. Avitabile and S. Mattarei, *Thin loop algebras of Albert-Zassenhaus algebras*, J. Algebra **315** (2007), 824–851.

There are also algebras with finite and infinite types, with the finite ones occurring at regular intervals and following an arithmetic progr.:

M. Avitabile and G. Jurman, *Nottingham Lie algebras with diamonds of finite and infinite type*, in preparation.

Periodicity, and hence uniqueness of these algebras, is proved by giving a presentation for a graded quotient, large enough.

Going a little farther than the third diamond (to set off the arithmetic progr.) is usually enough when its type is finite, but not when that diamond is fake.

The Nottingham algebras where the third "real" diamond occurs "later than it should" have been studied in:

D. S. Young, *Thin Lie algebras with long second chains*, Ph.D. thesis, Canberra, March 2001.

They include a subclass in a bijection with the gLamc, hence of uncountable cardinality.

Much remains to be done here.