PROFINITE GROUPS AND KAC–MOODY THEORY: SOME OPEN PROBLEMS

PIERRE-EMMANUEL CAPRACE*

1. Introduction

Kac–Moody theory provides examples of discrete and complete totally disconnected locally compact groups which are non-linear. Their properties suggest to pursue their study by comparing (and contrasting) them with the arithmetic groups of characteristic p and their completions. This should also shed some light on the latter.

For simplicity, we shall here focus on Kac–Moody groups of simply laced type. The extensions of the problems exposed below to other types of Kac–Moody groups is straightforward.

The data indexing the construction of (split simply connected) Kac–Moody groups of simply laced type consist of pairs $(D, \mathbb{F}_q)$, where $D = (V, E)$ is a finite simple graph without loops with vertex set $V = \{1, \dots, n\}$, and $\mathbb{F}_q$ is the finite field of order q. Two finitely presented groups, denoted respectively by $\mathcal{G}_D(\mathbb{F}_q)$ and $\mathcal{U}_D(\mathbb{F}_q)$, are associated with $(D, \mathbb{F}_q)$ in the following way.

Construction of $\mathcal{G}_D(\mathbb{F}_q)$. For all $v \in V$ and $e \in E$, let $X_v \cong \mathrm{SL}_2(\mathbb{F}_q)$ and $X_e \cong \mathrm{SL}_3(\mathbb{F}_q)$. The group $\mathcal{G}_D(\mathbb{F}_q)$ is defined by the following presentation.

- Generators: $\bigcup_{v\in V} X_v \cup \bigcup_{e\in E} X_e$.
- Relations:
  - Those of X_v and X_e for all $v \in V$ and $e \in E$.
  - For all $v, v' \in V$ with $v \neq v'$, the commuting relation $[X_v, X_{v'}] = 1$.
  - For each edge $\{v, v'\} \in E$ with $v < v'$, the relation which identifies X_v (resp. $X_{v'}$) with the top-left (resp. bottom-right) block-diagonal subgroup of $X_{\{v,v'\}}$.

The group $\mathcal{G}_D(\mathbb{F}_q)$ is a **split simply connected Kac–Moody group** of type D over $\mathbb{F}_q$ (see [2]). The group functor $\mathcal{G}_D$ on the category of fields is called a **Tits functor**. The transpose-inverse involution of each vertex and edge group extends naturally to an involution of $\mathcal{G}_D(\mathbb{F}_q)$ called the **Cartan–Chevalley involution** and denoted by σ.

Construction of $\mathcal{U}_D(\mathbb{F}_q)$. For all $v \in V$ and $e \in E$, let U_v be a copy of the additive group of $\mathbb{F}_q$ and U_e be a copy of the 3-dimensional Heisenberg group over $\mathbb{F}_q$. The group $\mathcal{U}_D(\mathbb{F}_q)$ is defined by the following presentation.

- Generators: $\bigcup_{v\in V} U_v \cup \bigcup_{e\in E} U_e$.
- Relations:
  - Those of U_v and U_e for all $v \in V$ and $e \in E$.
  - For all $v, v' \in V$ with $v \neq v'$, the commuting relation $[U_v, U_{v'}] = 1$.
  - For each edge $\{v, v'\} \in E$ with $v < v'$, the relation which identifies U_v (resp. $U_{v'}$) with the top-left (resp. bottom-right) one-dimensional subgroup of $U_{\{v,v'\}}$.

Date: December 17, 2008.

Key words and phrases. Kac-Moody group, profinite group.

*F.N.R.S. Research Associate.

The group $\mathcal{U}_D(\mathbb{F}_q)$ is called a **unipotent Kac–Moody group** of type D over $\mathbb{F}_q$. There are two morphisms of functors $u_+ : \mathcal{U}_D \to \mathcal{G}_D$ and $u_- : \mathcal{U}_D \to \mathcal{G}_D$ which are 'swapped' by post-composing with the Cartan–Chevalley involution. It follows from [3] that these morphisms are injective over $\mathbb{F}_q$ provided $q \geq 16$ and D has no triangle.

Completions. Set $\Gamma = \mathcal{G}_D(\mathbb{F}_q)$, $U_+ = u_+(\mathcal{U}_D(\mathbb{F}_q))$ and $U_- = u_-(\mathcal{U}_D(\mathbb{F}_q))$. One verifies that Γ commensurates both U_+ and U_-; furthermore U_+ and U_- are both residually-p, where $p = \mathrm{char}(\mathbb{F}_q)$.

Consider the group topology on Γ defined by declaring that the Γ-conjugacy class of U_+ is a sub-base of identity neighbourhoods. Completing Γ with respect to this topology yields a totally disconnected locally compact $\mathbf{G}^+_D(\mathbb{F}_q)$. A similar group $\mathbf{G}^-_D(\mathbb{F}_q)$ is obtained using the conjugacy class of U_-. The Chevalley involution induces a continuous isomorphism of $\mathbf{G}^+_D(\mathbb{F}_q)$ to $\mathbf{G}^-_D(\mathbb{F}_q)$; the group $\mathbf{G}_D(\mathbb{F}_q) := \mathbf{G}^+_D(\mathbb{F}_q) \cong \mathbf{G}^-_D(\mathbb{F}_q)$ is called the **complete (simply connected) Kac–Moody group** of type D over $\mathbb{F}_q$. The closure of U_+ in $\mathbf{G}^+_D(\mathbb{F}_q)$ is denoted $\mathbf{U}^+_D(\mathbb{F}_q)$ and a group $\mathbf{U}^-_D(\mathbb{F}_q)$ is defined similarly. The group $\mathbf{U}_D(\mathbb{F}_q) := \mathbf{U}^+_D(\mathbb{F}_q) \cong \mathbf{U}^-_D(\mathbb{F}_q)$ is called the **pro-p (simply connected) Kac–Moody group** of type D over $\mathbb{F}_q$; it is indeed a pro-p group.

2. Problems

In this section, it is assumed that the diagram D is connected and not of spherical or affine type; in other words D is *any* connected graph *except* a classical or extended Dynkin diagram.

Problem 1. Compute the pro-p completion of U_+. Does it coincide with $\mathbf{U}_D(\mathbb{F}_q)$?

A more elementary question related to the latter is the following.

Problem 2. Let $G = G_1 \times G_2$ be a profinite group containing a dense copy of $\mathcal{U}_D(\mathbb{F}_q)$. Is is true that G_1 or G_2 is finite?

Problem 3. (Y. Barnea, M. Ershov and Th. Wiegel [1]) Compute the abstract commensurator of $\mathbf{U}_D(\mathbb{F}_q)$. Does it coincide with $\mathrm{Aut}(\mathbf{G}_D(\mathbb{F}_q))$?

Problem 4. Compute $\mathrm{Aut}(\mathbf{G}_D(\mathbb{F}_q))$. Does it contain $\mathbf{G}_D(\mathbb{F}_q)$ as a finite index subgroup?

Any automorphism of $\mathbf{G}_D(\mathbb{F}_q)$ is continuous and the group $\mathrm{Aut}(\mathbf{G}_D(\mathbb{F}_q))$ embeds in the full automorphism group of the building associated with $\mathbf{G}_D(\mathbb{F}_q)$. It is known that $\mathrm{Out}(\mathbf{G}_D(\mathbb{F}_q))$ is compact; it contains the finite automorphism group of the diagram D. In the affine case, the group $\mathrm{Out}(\mathbf{G}_D(\mathbb{F}_q))$ coincides (up to finite index) with the (infinite!) Galois group of the field of formal power series over $\mathbb{F}_q$.

Problem 5. Classify the just-infinite quotients of $\mathbf{U}_D(\mathbb{F}_q)$.

Every parabolic subgroup of $\mathbf{G}_D(\mathbb{F}_q)$ is an open subgroup admitting a Levi decomposition with compact unipotent radical. Parabolic subgroups containing $\mathbf{U}_D(\mathbb{F}_q)$ therefore yield quotients of $\mathbf{U}_D(\mathbb{F}_q)$; in the case of parabolic subgroups of affine type, these quotients are just-infinite. I expect $\mathbf{U}_D(\mathbb{F}_q)$ to have few infinite quotients.

Problem 6. What is the minimal q such that $\mathcal{U}_D(\mathbb{F}_q)$ (resp. $\mathcal{G}_D(\mathbb{F}_q)$, $\mathbf{G}_D(\mathbb{F}_q)$) has Kazhdan's property (T)?

It is shown in [4] that a sufficient lower bound on q for property (T) for any of the above groups is $\frac{1764^d}{25}$, where d is the dimension of the building associated with $\mathcal{G}_D(\mathbb{F}_q)$ (d is bounded above by the size of a maximal classical Dynkin subdiagram of D).

References

[1] Yiftach Barnea, Mikhail Ershov, and Thomas Weigel. Abstract commensurators of profinite groups. Preprint, 2008.

[2] Pierre-Emmanuel Caprace. On 2-spherical Kac-Moody groups and their central extensions. *Forum Math.*, 19(5):763–781, 2007.

[3] Alice Devillers and Bernhard Mühlherr. On the simple connectedness of certain subsets of buildings. *Forum Math.*, 19(6):955–970, 2007.

[4] Jan Dymara and Tadeusz Januszkiewicz. Cohomology of buildings and their automorphism groups. *Invent. Math.*, 150(3):579–627, 2002.

UCL – Math, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
E-mail address: pe.caprace@uclouvain.be