Applications of the classification of 2-groups by coclass

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Introduction

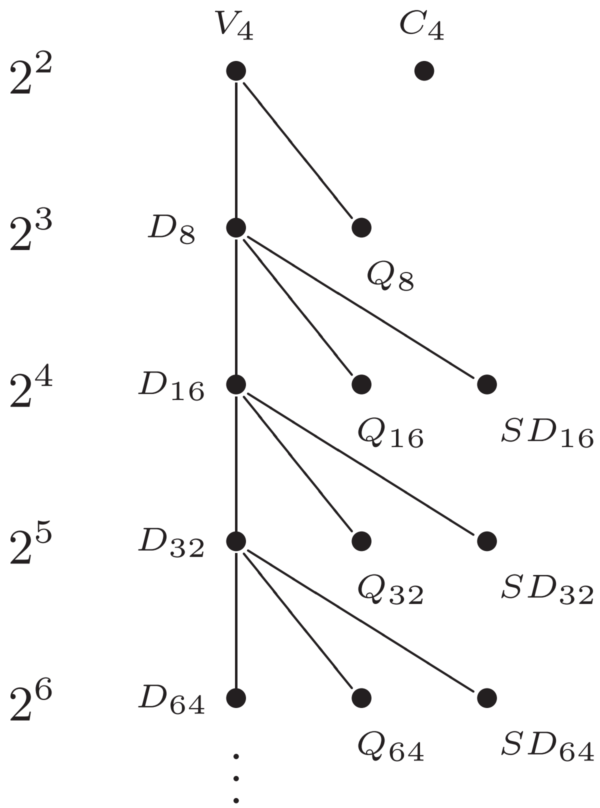
Classifying 2-groups by coclass [du Sautoy, Eick & Leedham-Green]:
Can sort 2-groups of fixed coclass into periodicity classes (up to finitely many exceptions).

Theorem: $\mathcal{T} \subset \mathcal{G}(2,r)$ maximal coclass tree of dim.d.
$\Rightarrow$ exists $f \in \mathbb{N}$ such that
$\forall i \geq f$ exists isomorphism $\pi : \mathcal{B}_{i+d} \to \mathcal{B}_i$.

$\mathcal{T} \subseteq \mathcal{G}(2,r)$ maximal coclass tree. $G \in \mathcal{B}_i$ and $i \geq f$
$\Rightarrow \mathcal{F}_G = (G, \pi(G), \pi^2(G), \ldots)$ is **coclass family**.

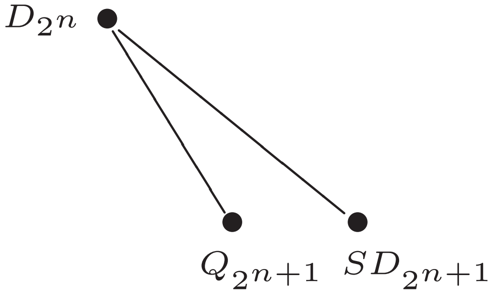
$\mathcal{F}_G$ can be described by a single parametrised presentation.

Example $\mathcal{G}(2,1)$



Example $\mathcal{G}(2,1)$

Periodic part:

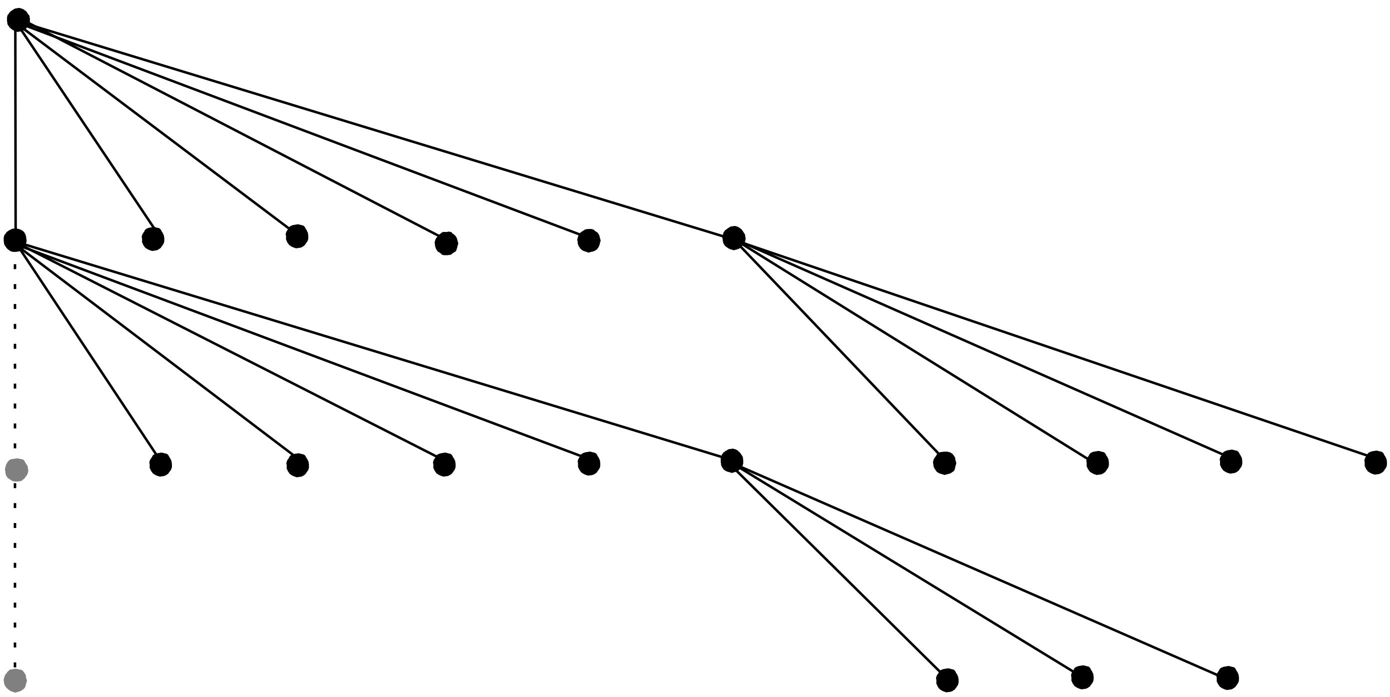


$$D_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1,\ y^2 = 1,\ x^y = x^{-1} \rangle \quad (n \geq 3),$$
$$SD_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1,\ y^2 = 1,\ x^y = x^{2^{n-2}-1} \rangle \quad (n \geq 4),$$
$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1,\ y^2 = x^{2^{n-2}},\ x^y = x^{-1} \rangle \quad (n \geq 4).$$

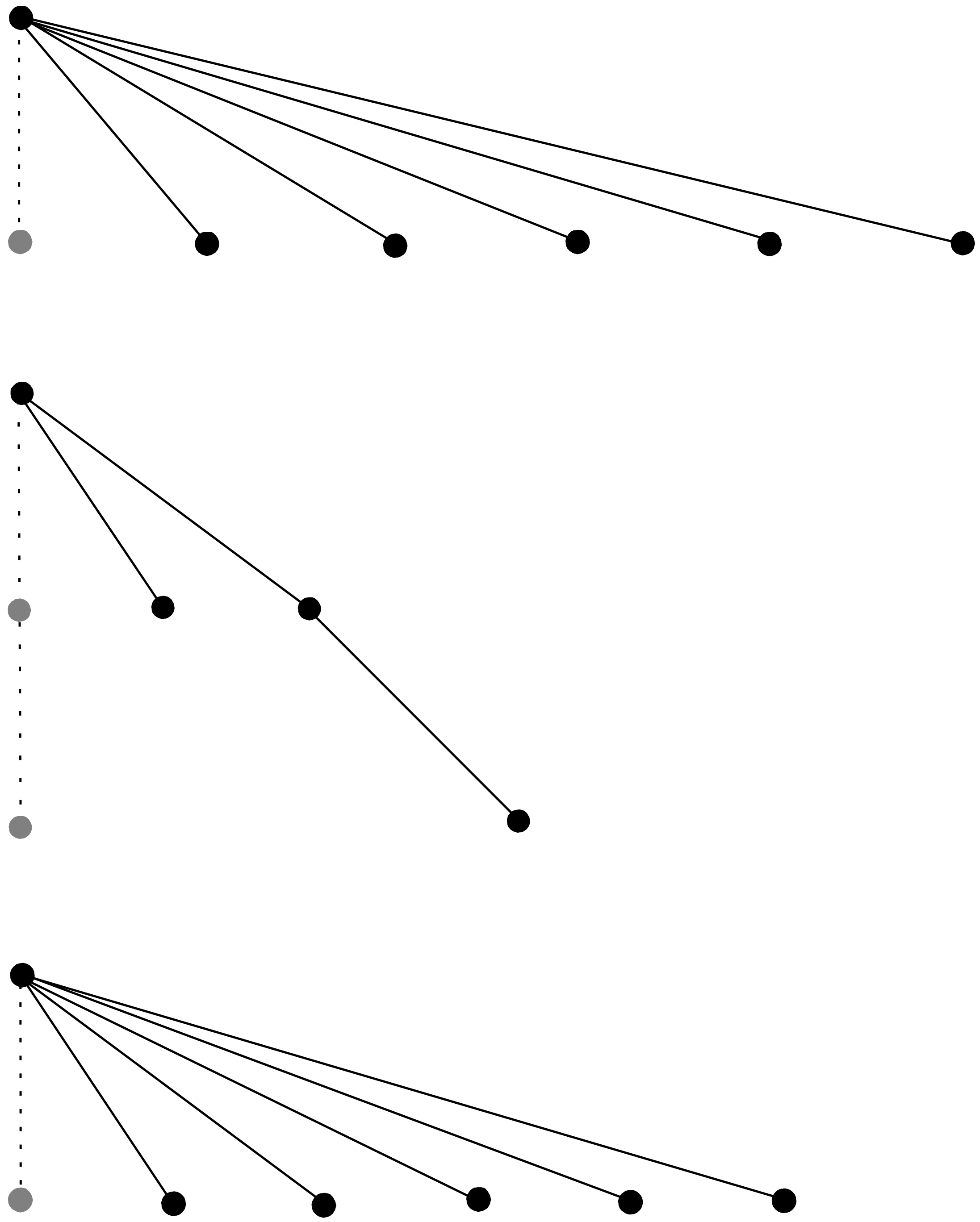
Example $\mathcal{G}(2,2)$

- 5 coclass trees;
- 51 coclass families;
- 27 sporadic groups.

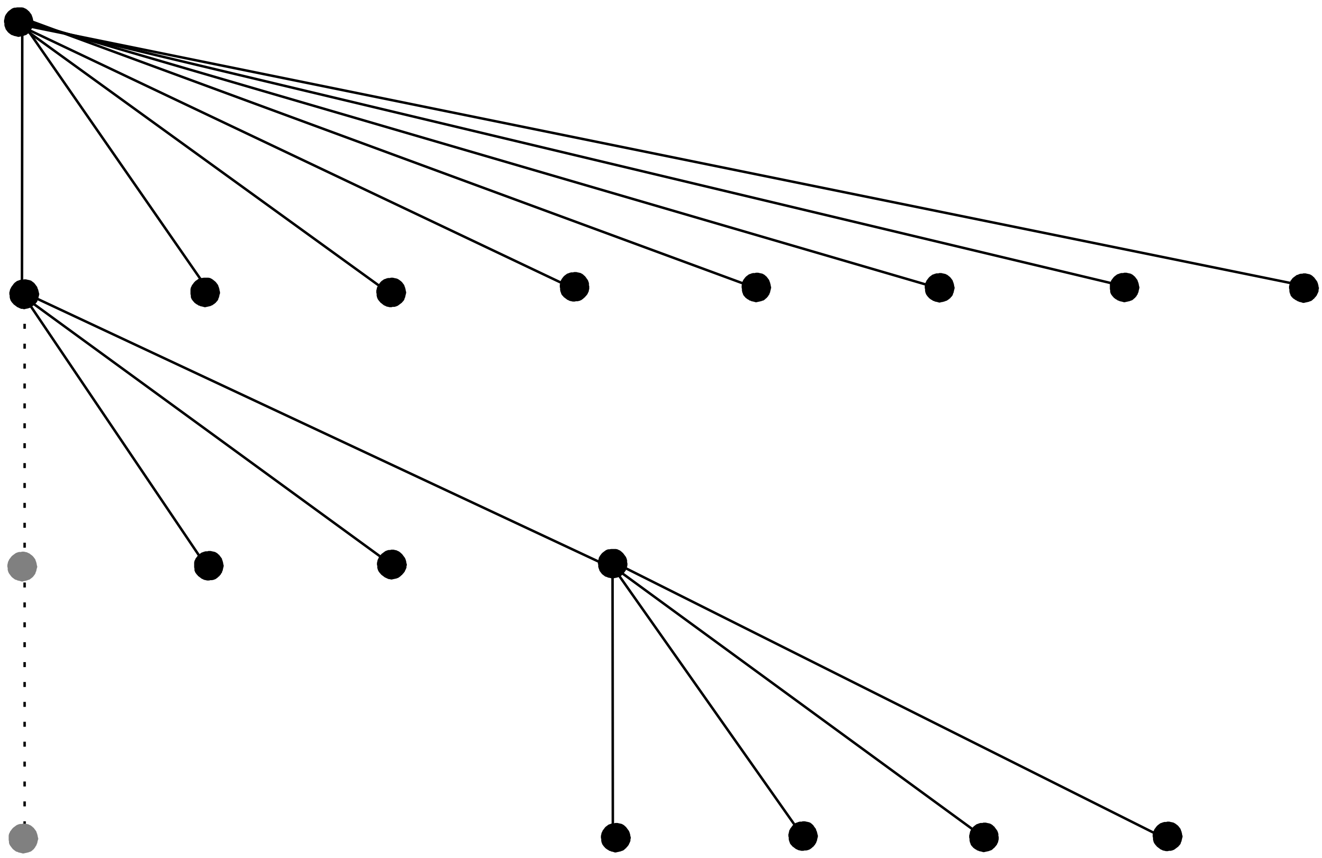
(Investigated by James and O'Brien & Newman)



Example $\mathcal{G}(2,2)$



Example $\mathcal{G}(2,2)$



Properties of 2-groups of fixed coclass

What about the properties of 2-groups of fixed coclass?

Conjecture: Let $\mathcal{F}_G$ be a periodicity class of 2-groups. We can describe certain properties of groups in $\mathcal{F}_G$ by a parametrised presentation.

We want to investigate

- cohomology (rings),
- characters.

Carlson's result: Coclass and cohomology

Theorem: [Carlson]

- $\boxed{p = 2}$.
- k field with $\mathrm{char}(k) = p$.
- $r \in \mathbb{N}$.

$\Rightarrow$ exist only finitely many isomorphism types of cohomology rings $H^*(G, k)$ (G is p-group of coclass r).

Conjecture: The same holds for p an odd prime.

Conjecture cohomology

Conjecture: Let k be a field with $\mathrm{char}(k) = 2$ and $\mathcal{F}_G$ a periodicity class of 2-groups.

- $H^*(G, k) \cong H^*(H, k)$ for all $H \in \mathcal{F}_G$.
- There is a 1-parameter presentation for $H^n(H, \mathbb{Z})$ for $H \in \mathcal{F}_G$.

Example $\mathcal{G}(2,1)$ and cohomology

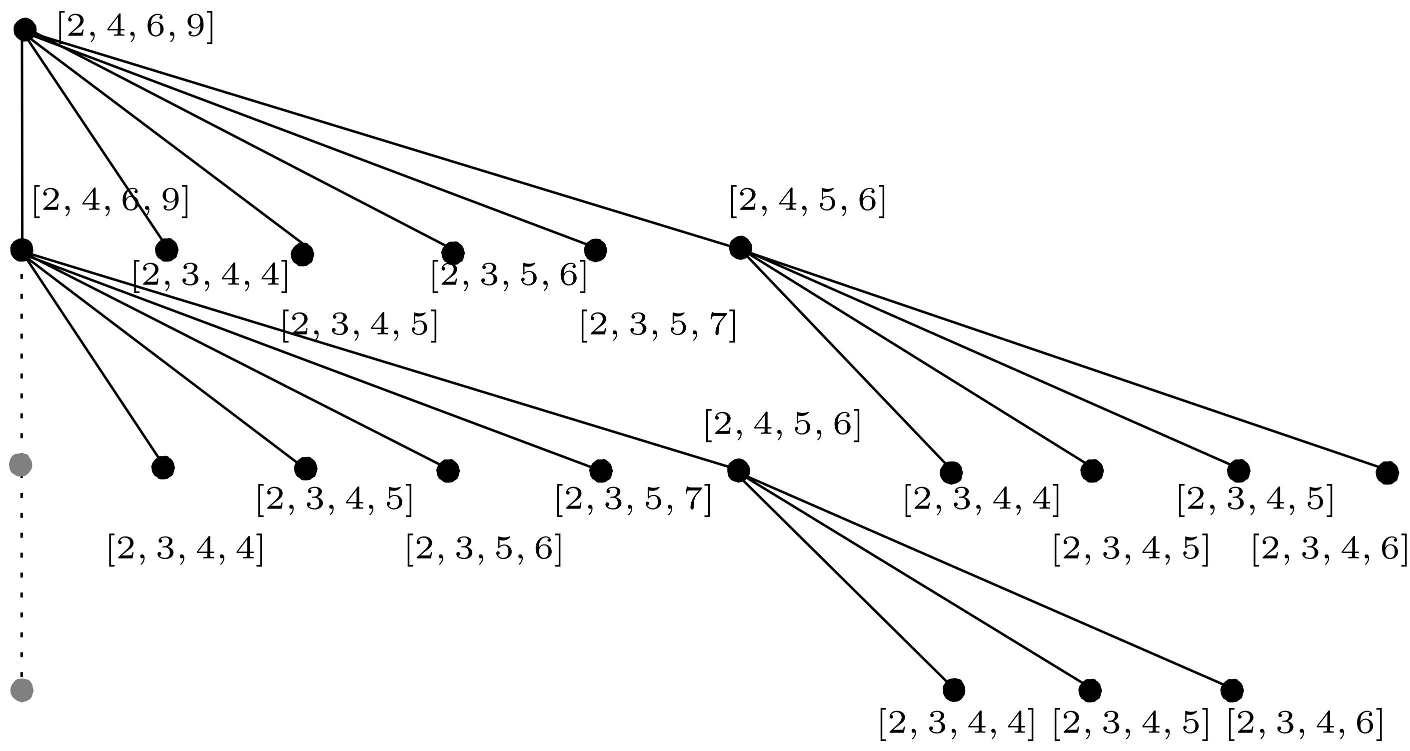
Theorem: If $i, j \geq 4$, then

- $H^*(D_{2^i}, \mathbb{F}_2) \cong H^*(D_{2^j}, \mathbb{F}_2)$;
- $H^*(Q_{2^i}, \mathbb{F}_2) \cong H^*(Q_{2^j}, \mathbb{F}_2)$;
- $H^*(SD_{2^i}, \mathbb{F}_2) \cong H^*(SD_{2^j}, \mathbb{F}_2)$.

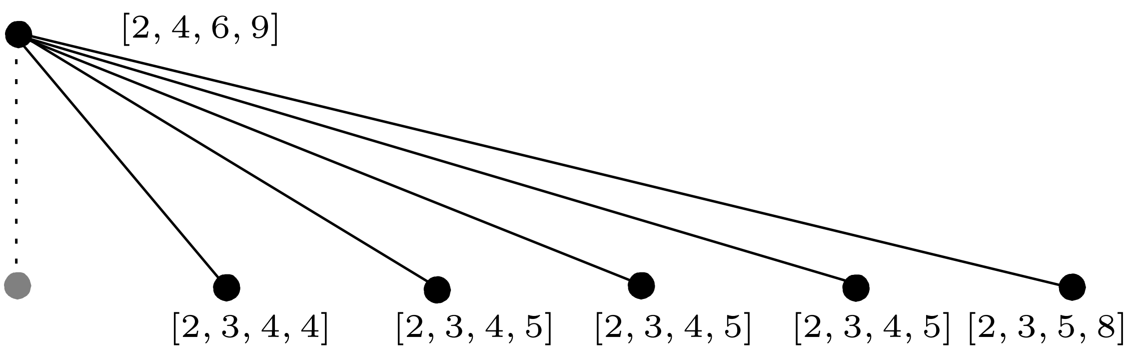
Furthermore,

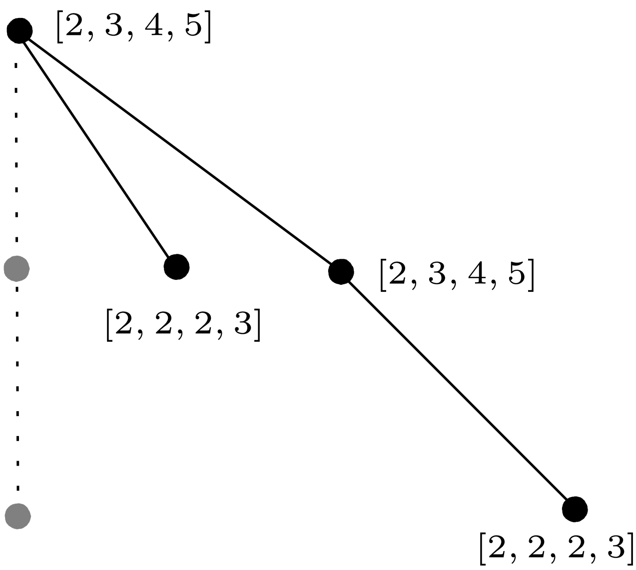
- $H^*(D_{2^i}, \mathbb{Z})$ has a presentation depending only on i;
- $H^*(Q_{2^i}, \mathbb{Z})$ has a presentation depending only on i;
- $H^*(SD_{2^i}, \mathbb{Z})$ has a presentation depending only on i.

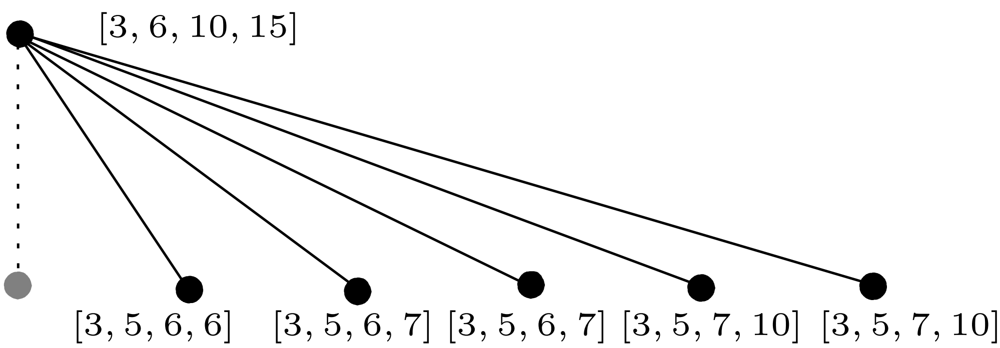
$\mathcal{G}(2,2)$ and cohomology (conjectured)



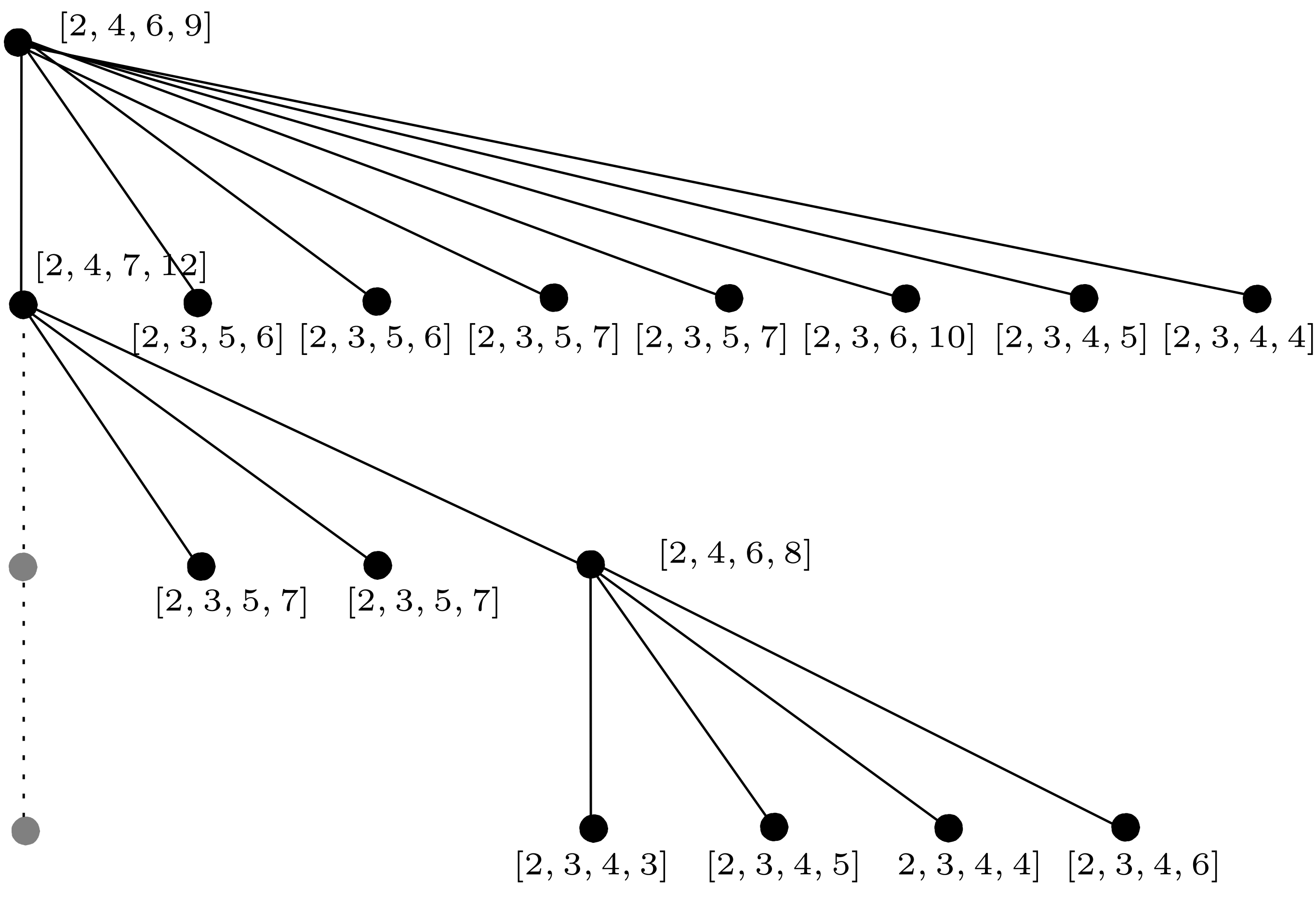
$\mathcal{G}(2,2)$ and cohomology (conjectured)







$\mathcal{G}(2,2)$ and cohomology (conjectured)



Characters and coclass

Known: There exist generic formulas for the character tables of dihedral D_{2^n}, quaternion Q_{2^n} and semi-dihedral groups SD_{2^n}.

Conjecture: There exist 1-parameter presentations for the characters of groups in the same periodicity class.

Conjugacy classes

- G be a pro-2-group, T its translation subgroup;
- maximal G-invariant series $T > 2T > 2^2T > \ldots$;
- $x \in G$: $\quad C = C_G(x)$, $C_i = C_G(xp^iT)$.

1-1 correspondence:
conjugacy classes $\leftrightarrow$ C_i-orbits on $2^iT/2^{i+1}T$.

Conjecture:
Exists $l \in \mathbb{N}$ (l independent of x) such that for $i \geq l$:
C_i-orbits on $2^iT/2^{i+1}T \leftrightarrow C_{i+1}$-orbits on $2^{i+1}T/2^{i+2}T$.