On the classification of p-groups by coclass: Results and open problems

Cambridge, July 2008

Bettina Eick
Braunschweig

Introduction

Leedham-Green and Newman (1980):
Proposed to classify p-groups by coclass

Definition: If G has order p^n and class c, then its coclass is

$$cc(G) = n - c.$$

Problem: Given p and r, produce a classification of the p-groups of coclass r (up to isomorphism).

Introduction

Question 1:
Is this possible?

Question 2:
If yes, then would such a classification be useful?

That is:

- We want to solve the problem.
- We want interesting applications.

The coclass graph $\mathcal{G}(p, r)$

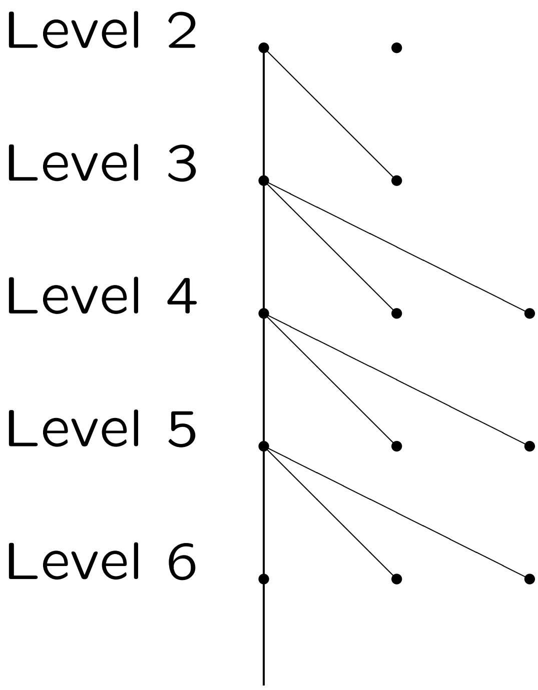
Visualise the p-groups of coclass r in $\mathcal{G}(p, r)$:

- vertices at level l
  $\leftrightarrow$ groups of coclass r and order p^l
- G and H are adjoined by an edge
  $\leftrightarrow$ exists $N \trianglelefteq H$ with $|N| = p$ and $H/N \cong G$

Then:

- $\mathcal{G}(p, r)$ is an infinite graph
  (there are infinitely many p-groups of coclass r for each prime p and each $r \in \mathbb{N}$)
- $\mathcal{G}(p, r)$ is a collection of trees
  (N is the smallest non-trivial subgroup of the lower central series of H and thus is unique)

Example: $\mathcal{G}(2,1)$



Pro-p-groups of coclass r

Notation:

- Let $G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots$ denote the lower central series of a group G.
- Let $G_i = G/\gamma_i(G)$ corresponding quotient.
- The coclass of an infinite pro-p-group G is defined by $cc(G) = \lim_{i\to\infty} cc(G_i)$.

Infinite paths in $\mathcal{G}(p,r)$:

- The inverse limit of the groups on an infinite path is an infinite pro-p-group of coclass r.
- An infinite pro-p-group G of coclass r defines the infinite path $G_i, G_{i+1}, \ldots$.

$\Rightarrow$ The infinite paths in $\mathcal{G}(p,r)$ correspond to the infinite pro-p-groups of coclass r.

The coclass-conjectures

Leedham-Green and Newman (1980):

Conjecture A: There is an $f = f(p, r)$ such that every p-group of coclass r has a normal subgroup of class 2 with index at most p^f.

Conjecture B: There is a $g = g(p, r)$ such that every p-group of coclass r has soluble length at most g.

Conjecture C: Every pro-p-group of finite coclass is soluble.

Conjecture D: There are only finitely many pro-p-groups of coclass r.

Conjecture E: There are only finitely many soluble pro-p-groups of coclass r.

The coclass-conjectures

- These conjectures have lead to a major research project
- They are all proved now; many people have been involved
- The strongest Conjecture A was proved by Leedham-Green (1994) and Shalev (1994) (independently)
- The book by Leedham-Green and McKay (2002) contains detailed information and references

Results on the structure of $\mathcal{G}(p,r)$

Let G be an infinite pro-p-group of coclass r. Choose i minimal with:

- G_i has coclass r (and thus is in $\mathcal{G}(p,r)$).
- The tree of descendants of G_i in $\mathcal{G}(p,r)$ contains only one maximal infinite path.

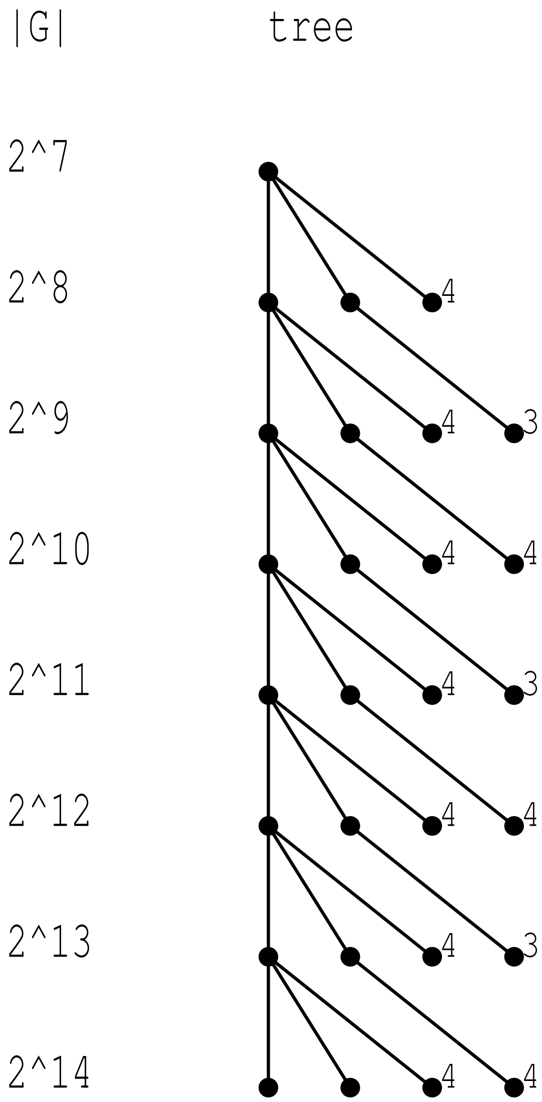
The subtree $\mathcal{T}(G)$ of $\mathcal{G}(p,r)$ of all descendants of G_i is the *coclass tree* defined by G

By the coclass-conjectures $\mathcal{G}(p,r)$ consists off:

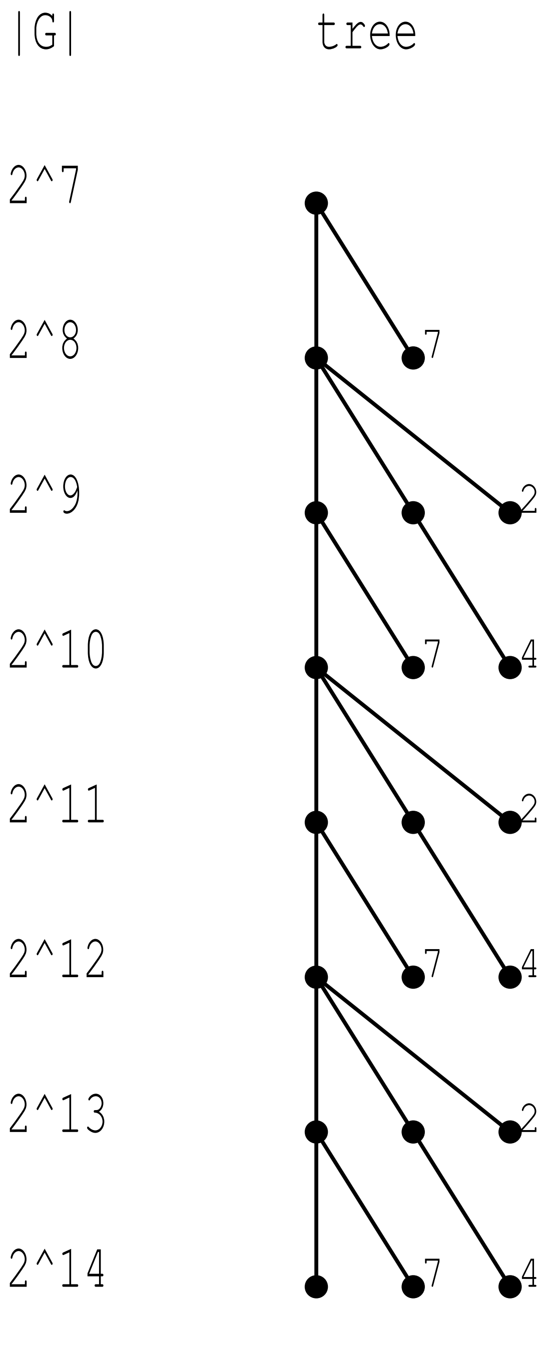
- finitely many coclass trees, and
- finitely many 'sporadic' groups outside.

Problem: Given an infinite pro-p-group G, is it possibly to classify the groups in $\mathcal{T}(G)$?

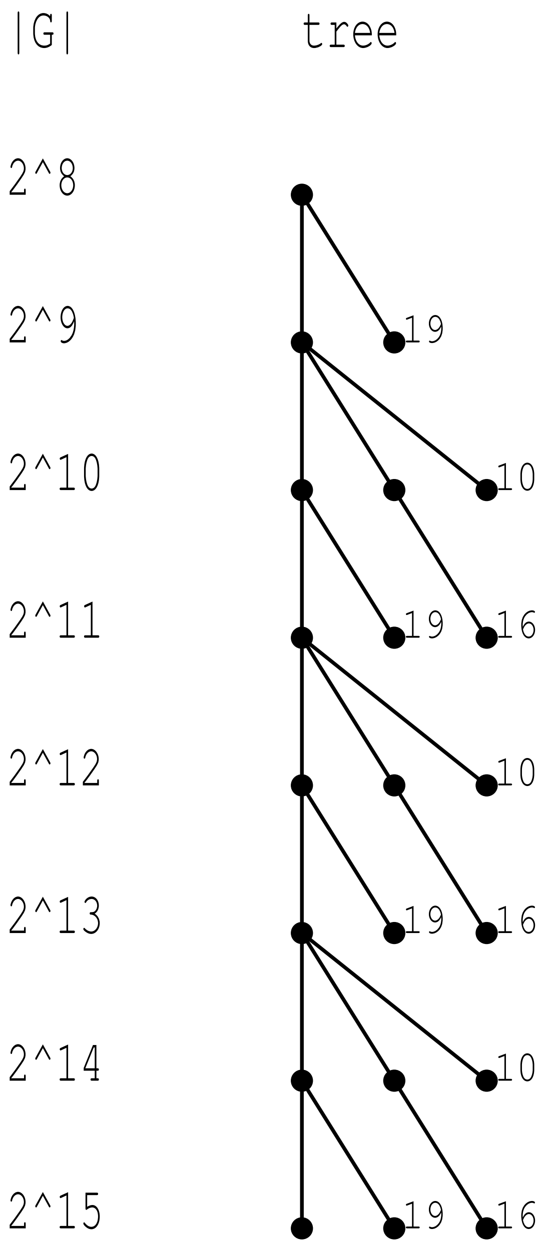
An example computed with GAP
(Prime 2, coclass 2)



An example computed with GAP (Prime 2, coclass 2)



An example computed with GAP (Prime 2, coclass 3)



Notation for coclass trees

Let G be an infinite pro-p-group of coclass r.

- *Main line:*
  consists of the groups $G_i, G_{i+1}, \ldots$ on the (unique) infinite path of $\mathcal{T}(G)$.
- *Branch $\mathcal{B}_j(G)$:*
  this subtree of $\mathcal{T}(G)$ consists of the descendants of G_j which are not descendants of G_{j+1}.
- *Shaved branch $\mathcal{B}_j^k(G)$:*
  consists of all groups of distance at most k from G_j in $\mathcal{B}_j(G)$.

Periodicity of Type I

Let G be an infinite pro-p-group of coclass r and dimension d. (Arbitrary prime p and $r \in \mathbb{N}$)

Theorem: For every $k \in \mathbb{N}$ there exists $f(k) \in \mathbb{N}$ so that for every $j \geq f(k)$

$$\rho_j : \mathcal{B}_j^k(G) \to \mathcal{B}_{j-d}^k(G)$$

is a graph isomorphism.

- First conjectured by Newman and O'Brien for $p = 2$ (1999).
- First proved by Du Sautoy using Zeta-functions (2001)
- New proof by Eick and Leedham-Green using an explicit group-theoretic construction (2005)

Note: there exists k with $\mathcal{B}_j^k(G) = \mathcal{B}_j(G)$ for all j if and only if $p = 2$ or $(p, r) = (3, 1)$.

The proof by Eick and Leedham-Green

We write all groups in $\mathcal{B}_j^k(G)$ as extensions:

- Let $R := G_l$ for suitable, large l.
- Let $T := \gamma_l(G)$ and $T_n = \gamma_{l+n}(G)$.
- Implies that $T \cong \mathbb{Z}_p^d$.

Theorem: For large enough j every group H in $\mathcal{B}_j^k(G)$ satisfies:

- $H_l \cong R$.
- $\gamma_l(H) \cong T/T_n$ as R-module for suitable n.

$\Rightarrow$ H is an extension of T/T_n by R.

The proof by Eick and Leedham-Green

Use cohomology theory:

Theorem: For large enough n:

$$H^2(R, T/T_n) \cong H^2(R, T) \oplus H^3(R, T_n).$$

Corollary.

$$\begin{array}{rcl} H^2(R, T/T_n) & \cong & H^2(R, T) \oplus H^3(R, T_n) \\ & & \quad\downarrow id \qquad\quad \downarrow \mu \\ H^2(R, T/T_{n+d}) & \cong & H^2(R, T) \oplus H^3(R, T_{n+d}) \end{array}$$

where $\mu : H^3(R, T_n) \to H^3(R, T_{n+d})$ is induced by multiplication $T_n \to T_{n+d} : s \mapsto ps$.

$\Rightarrow$ The isomorphism $id \oplus \mu$ induces the graph isomorphism $\mathcal{B}_j^k(G) \to B_{j+d}^k(G)$.

The proof by Eick and Leedham-Green

- Yields explicit bounds for $f(k)$.
- Exhibits structure of the groups in $\mathcal{B}_j^k(G)$.
- Splits the groups in $\mathcal{T}(G)$ in finitely many *periodicity classes* (sequences $(H_0, H_1, \ldots)$, where H_{i+1} is the image of H_i under $id \oplus \mu$).

Further Result: The groups in a periodicity class can be described by a single parametrised presentation.

Summary

Result: The 2-groups of coclass r (and the 3-groups of coclass 1) can be classified up to isomorphism.

Question 1: Is this classification useful?

Question 2: What can we do for $p > 2$?

Applications I

Let $(H_0, H_1, \ldots)$ be a periodicity class of 2-groups.

Theorem: There exists an $l > 0$ with

$$|Aut(H_{i+1})| = 2^{d+l}|Aut(H_i)|.$$

⇒ The automorphism groups also satisfy a periodic pattern

Corollary: There are only finitely many 2-groups of coclass r with $|H| \nmid |Aut(H)|$, since

$$\frac{|Aut(H_{i+1})|}{|H_{i+1}|} = \frac{2^{l+d}|Aut(H_i)|}{2^d|H_i|} = 2^l \frac{|Aut(H_i)|}{|H_i|}$$

Applications II

Let $(H_0, H_1, \ldots)$ be a periodicity class of 2-groups.

Conjecture: The abelian invariants of the Schur multiplicators $M(H_i)$ have the form:

$$(a_1 + ib_1, \ldots, a_l + ib_l).$$

Theorem:

- If $M(G)$ is non-trivial, then there are at most finitely many p-groups in $\mathcal{T}(G)$ with trivial Schur multiplicator.
- If $p > 2$, then $M(G)$ is non-trivial.

Further Talks

- See Dörte's talk for further applications of the classification for $p = 2$.
- See Heiko's talk for a stronger version of the classification for $r = 1$.
- See Tobias' talk for experiments with a similar classification by width, rank and obliquity rather than coclass.

The case $p > 2$

We have computer experiments for:

- $\mathcal{G}(5, 1)$
  (Newman, and others)
- $\mathcal{G}(3, 2)$
  (Eick, Leedham-Green, Newman, O'Brien)
- $\mathcal{G}(7, 1)$ and $\mathcal{G}(11, 1)$
  (Dietrich)

These suggest conjectures...

The case $p > 2$

Conjecture. For every prime p and every $r \in \mathbb{N}$:

- The p-groups of coclass r can be classified by finitely many infinite coclass families and finitely many 'sporadic' groups.
- The groups in a coclass family share the same structure and can be described by a single parametrised presentation.

Conjecture. Many structural invariants of the groups in a coclass family can be described in a uniform way. In particular:

- their Schur multiplicators,
- their cohomology rings over a ring R,
- their automorphism groups, and
- their conjugacy classes and character tables.

Periodicity of Type II

- Let G be an infinite pro-p-group of coclass r and dimension d.
- Let $U_n = (\mathcal{B}_n, \mathcal{B}_{n+d}, \ldots)$ be a sequence of branches of unbounded depth in $\mathcal{T}(G)$.

Conjecture. There exist $k, l \in \mathbb{N}$ with $l \geq f(k)$ such that, for every $m \in \mathbb{N}$ with $j = n + md \geq l$, there exists a map $\pi_j(k)$ from the set of groups of depth k in $\mathcal{B}_j$ into the set of groups of depth $k - d$ in $\mathcal{B}_{j-d}$ with:

- ρ and π are compatible: for every H of depth k in $\mathcal{B}_j$ we obtain that

$$\pi_j(H) = \rho_{j-d}^{-1}(\pi_{j-d}(\rho_j(H))).$$

- the descendant trees of H and $\pi_j(H)$ are isomorphic for every H of depth k in $\mathcal{B}_j$.

The case $p > 2$

- Conjecture is due to Eick, Leedham-Green, Newman, O'Brien
- Experiments suggest that it holds in $\mathcal{G}(5, 1)$ and $\mathcal{G}(3, 2)$
- Proving it in general proves difficult, since π_j is difficult to describe
- Dietrich is making good progress on a proof for $p \equiv 5 \bmod 6$ and $r = 1$.