Towards a classification of p-groups of maximal class using periodicity

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Heiko Dietrich

Bettina Eick

Technische Universität Braunschweig

Institut Computational Mathematics

Pockelsstr. 14

38106 Braunschweig

Germany

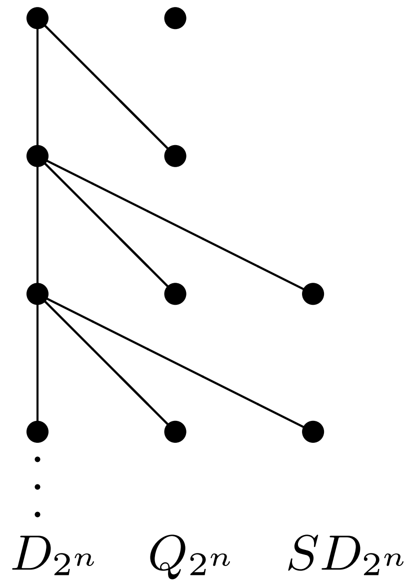
Introduction

Question: Can the p-groups of maximal class be classified using periodicities of coclass trees?

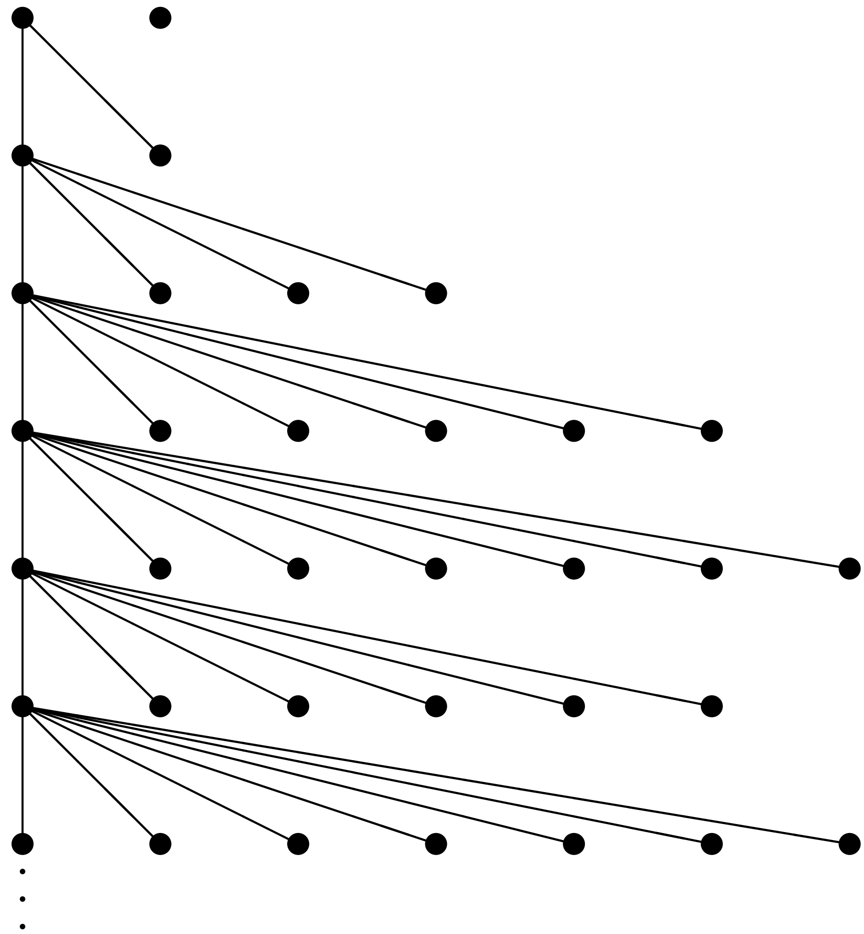
Definition: Let $\mathcal{G} = \mathcal{G}(p)$ be the graph corresponding to p-groups of maximal class.

Recall: $\mathcal{G}$ consists of one isolated group C_{p^2} and one coclass tree $\mathcal{T} = \mathcal{T}(p)$.

The graphs $\mathcal{G}(2)$ and $\mathcal{G}(3)$



$\boldsymbol{\mathcal{G}(2)}$



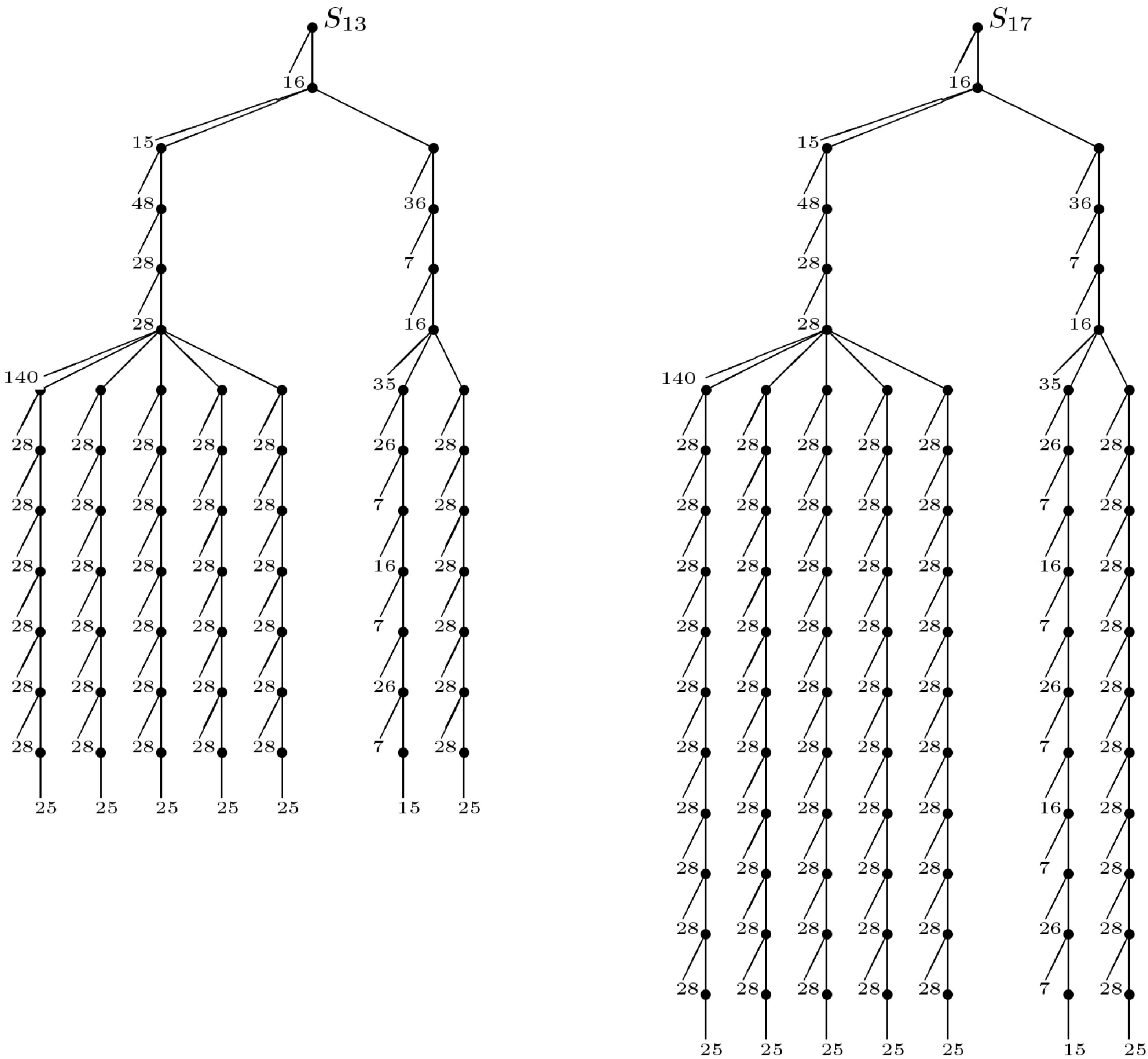
$\boldsymbol{\mathcal{G}(3)}$

– Branches are isomorphic with periodicity $p - 1$.

– Groups are classifiable (Blackburn, 1958).

– Covered by Du Sautoy and Eick & Leedham-Green.

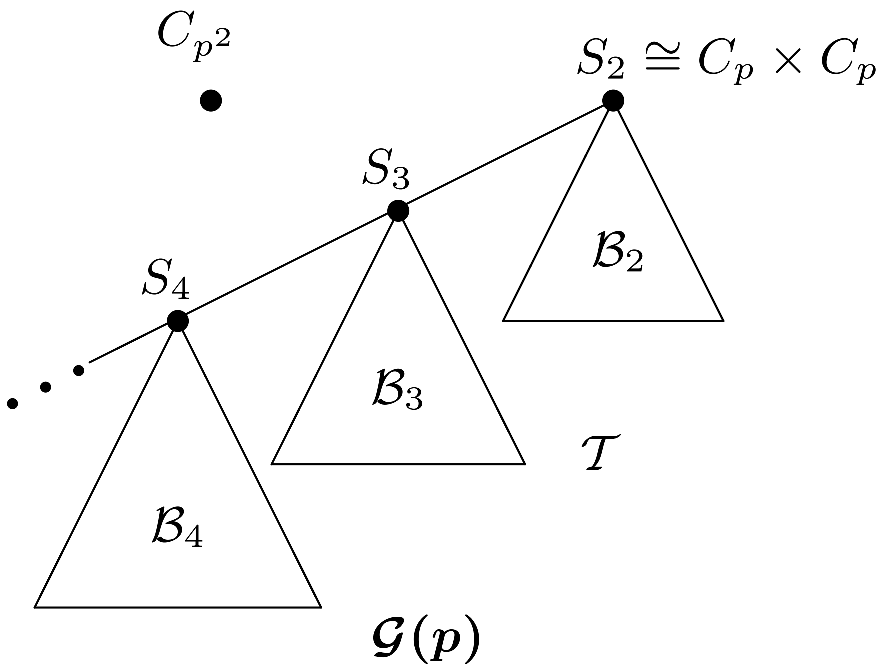
The graph $\mathcal{G}(5)$



– Depth of branches is not bounded.

– Groups are conjectured to be classifiable (Leedham-Green & McKay, Newman,...).

The graph $\mathcal{G}(p)$ for $p \geq 7$

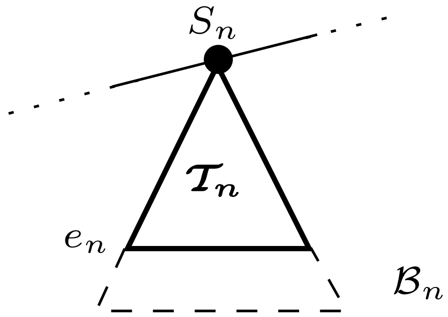


S space group of maximal class, $S_n = S/\gamma_n(S)$.

– Width of branches is not bounded.

– Classification completely open (?).

The body of a branch



Definition:

- For $G \in \mathcal{B}_n$ let $\mathcal{D}_k(G)$ be the subtree of $\mathcal{B}_n$ induced by the descendants of G in $\mathcal{B}_n$ of distance $\leq k$.
- The *body* $\mathcal{T}_n$ of $\mathcal{B}_n$ is $\mathcal{T}_n = \mathcal{D}_{e_n}(S_n)$ with

$$e_n = \left\{ \begin{array}{ll} \max\{0, n-2p+8\} & (p \geq 7), \\ \max\{0, n-4\} & (p = 5) \end{array} \right\} \approx n.$$

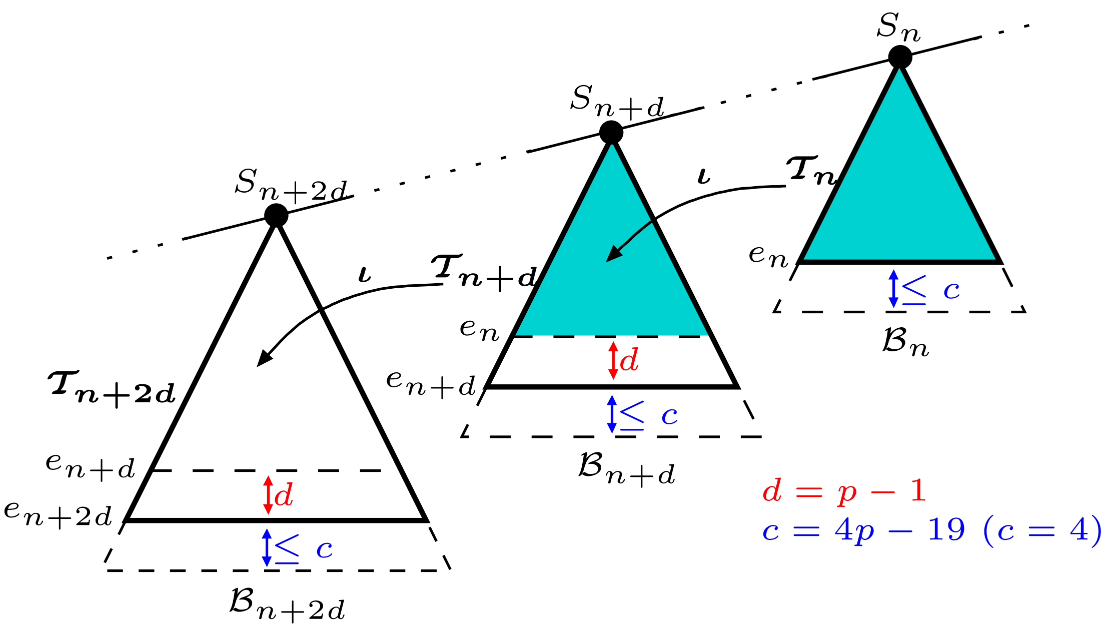
Restriction: Consider only the bodies $\mathcal{T}_n$, $n \geq p+1$.

Theorem: $\mathrm{d}(\mathcal{T}_n) = e_n$ and
$\mathrm{d}(\mathcal{B}_n) - \mathrm{d}(\mathcal{T}_n)$ is bounded by a constant.

Results I

Theorem: Let $n \geq p+1$ and $d = p-1$.
There is an embedding $\iota = \iota_n : \mathcal{T}_n \to \mathcal{T}_{n+d}$ such that

$$\iota(\mathcal{T}_n) = \mathcal{D}_{e_n}(S_{n+d}).$$



Remark: Improvement of results of Du Sautoy and Eick & Leedham-Green for $\mathcal{G}(p)$; close to best possible.

Theorem: If $G \in \mathcal{T}_n$, then the groups in

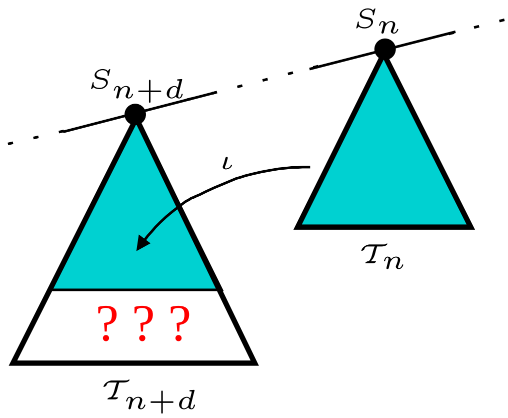
$$\{G, \iota(G), \iota^2(G), \ldots\}$$

can be described by a single parameterized presentation.

Periodic parents

Let $n \geq p + 1$ and $d = p - 1$.

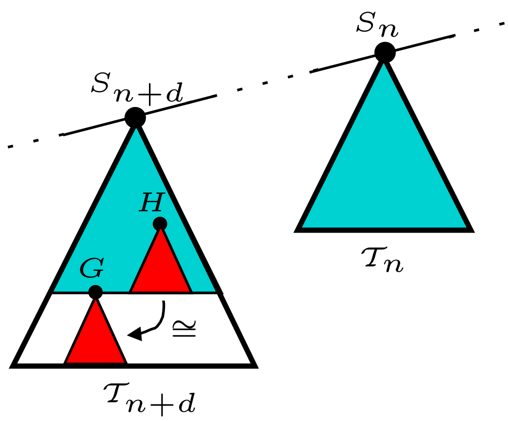
Problem: Describe growth of the bodies:



Conjecture: Every capable group $G \in \mathcal{T}_{n+d}$ of depth e_n has a *periodic parent* H of depth $e_n - d$ with

$$\mathcal{D}_d(G) \cong \mathcal{D}_d(H).$$

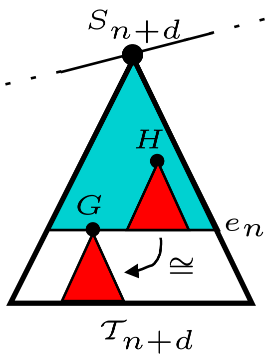
$\mathcal{D}_d(G)$: subtree induced by descendants of G of distance $\leq d$.

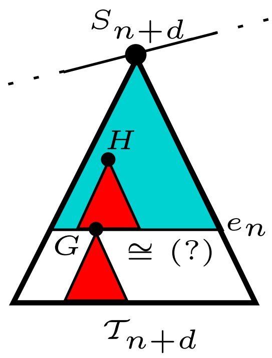


Results II

Let $n \geq p + 1$ and $d = p - 1$.

Problem: How to choose periodic parents?





First idea: Take the d-step parent.

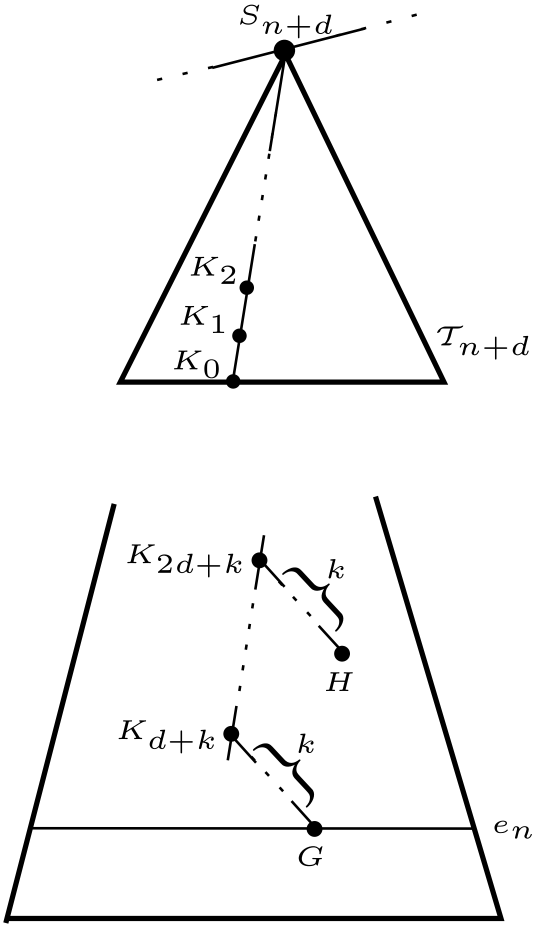
Theorem: For $p \equiv 5 \bmod 6$ and big enough n:
If $\mathrm{Aut}(H)$ is a p-group, then H is a periodic parent of G.

But: The d-step parent is *not* always the periodic parent!

Results II

Let $n \geq p+1$ and $d = p+1$, and

- $p \equiv 5 \bmod 6$,
- K_0 a leaf of $\mathcal{T}_{n+d}$ with ancestors $K_1, K_2, \ldots$ such that
- p'-part of $|\mathrm{Aut}(K_j)|$ is the same for $0 \leq j \leq 3d$.



(G capable)

Theorem: For big enough n and $k \leq d$:
The group H is a periodic parent of G.