Transitive orderings of properties of utility functions

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Abstract

This note considers orderings of properties (or assumptions) on utility functions and specifies domains on which those orderings are transitive or acyclic.

1. Introduction

This note considers the orderings of properties (or assumptions) on utility functions studied in Mandler (2001) and Mandler (2003). We specify domains of properties under which those orderings are transitive – or at least acyclic and hence possessing transitive extensions. The aim is for the set of properties to be rich enough to encompass "compromises" between ordinal and cardinal assumptions. We explain this broader purpose of this agenda in the papers cited.

2. Properties of utility and orderings of properties.

Let X be a nonempty set of consumption bundles and, for any nonempty $A \subset X$, let $\mathbb{R}^A$ denote the set of functions from A to $\mathbb{R}$. An agent is characterized by a nonempty set $\boldsymbol{U} \subset \mathbb{R}^X$, called a *psychology*.

For $\boldsymbol{U} \subset \mathbb{R}^X$, call X the *domain of* $\boldsymbol{U}$ and call a subset A of the domain X *decisive* for $\boldsymbol{U}$ if and only if, for all $u, v \in \boldsymbol{U}$ and $x, y \in A$, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$. Unlike Mandler (2001), we suppose here that psychologies are complete, i.e., that the domain X of $\boldsymbol{U}$ is decisive for $\boldsymbol{U}$.

Definition 2.1 A psychology $\boldsymbol{U}$ is *no stronger than* a psychology $\boldsymbol{V}$ if and only if $\boldsymbol{U} \supset \boldsymbol{V}$. $\boldsymbol{U}$ is *weaker* than $\boldsymbol{V}$ if and only if $\boldsymbol{U}$ is no stronger than $\boldsymbol{V}$ and it is not the case that $\boldsymbol{V}$ is no stronger than $\boldsymbol{U}$.

Since set inclusion is transitive, the "no stronger than" relation on psychologies is as well. When $|X| > 1$, however, the "no strong than" relation is incomplete (simply let $\boldsymbol{U}$ contain only utilities that represent some binary relation $\succeq$ and $\boldsymbol{V}$ contain only utilities that represent some $\succeq' \neq \succeq$).

A property P is simply a set of functions into the real line, where the domains of the functions can differ. A utility function u: $A \rightarrow \mathbb{R}$ *satisfies property* P if and only if $u \in P$.

Definition 2.2 A $\boldsymbol{U}$ *maximally satisfies* property P if and only if for each $u \in \boldsymbol{U}$, u satisfies P, and there does not exists a psychology $\boldsymbol{V} \supsetneq \boldsymbol{U}$ such that each $v \in \boldsymbol{V}$ satisfies P.

So a $\boldsymbol{U}$ maximally satisfies P if it is largest among those psychologies whose constituent utility functions each satisfy P. We say that property P *intersects* property Q if and only if $P \cap Q \neq \varnothing$.

Definition 2.3:

- Property P is *no stronger than* property Q, or $P \succeq_{NS} Q$, if and only if for all $\boldsymbol{U}$ that maximally satisfy P and all $\boldsymbol{V}$ that maximally satisfy Q, $\boldsymbol{U} \cap \boldsymbol{V} \neq \varnothing$ implies $\boldsymbol{U} \supset \boldsymbol{V}$.
- Property P is *weaker than* property Q, or $P \succeq_{W} Q$, if and only if $P \succeq_{NS} Q$ and not $Q \succeq_{NS} P$.
- Property P is *strictly weaker* than property Q, or $P \succeq_{SW} Q$, if and only if (1) P intersects Q and (2) whenever a complete $\boldsymbol{U}$ maximally satisfies P, a complete $\boldsymbol{V}$ maximally satisfies Q, and $\boldsymbol{U} \cap \boldsymbol{V} \neq \varnothing$, $\boldsymbol{U}$ is weaker than $\boldsymbol{V}$.

To see the difference between P being weaker than Q and being strictly weaker than Q, observe that P is weaker than Q if it is merely the case that P is no stronger than Q and there is *some* $\boldsymbol{U}$ that maximally satisfies P and *some* $\boldsymbol{V}$ that maximally satisfies Q such that $\boldsymbol{U}$ is weaker than $\boldsymbol{V}$.

The above binary relations on properties need not be transitive or complete. Since our ordering of psychologies itself is not complete, the incompleteness is to be expected. The intransitivity may be more of a surprise and is the subject of this note.

Definition 2.4 (Ordinality) The functions u and v *agree* on A if and only if, for all $x, y \in A$, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$. A psychology $\boldsymbol{U}$ with domain X is *ordinal* if and only if $u \in \boldsymbol{U}$ implies that if $v \in \mathbb{R}^X$ and u and v agree on X then $v \in \boldsymbol{U}$. Equivalently, a psychology $\boldsymbol{U}$ with domain X is ordinal if and only if $u \in \boldsymbol{U} \Rightarrow (v \in \boldsymbol{U} \Leftrightarrow v \in \mathbb{R}^X$ and there exists an increasing transformation g such that $v = g \circ u)$.

Definition 2.5 (Cardinality) A function $g: E \to \mathbb{R}$, where $E \subset \mathbb{R}$, is an *increasing affine transformation* if and only if there exist $a > 0$ and b such that, for all $x \in E$, $g(x) = ax + b$. A psychology $\boldsymbol{U}$ with domain X is *cardinal* if and only if $u \in \boldsymbol{U} \Rightarrow (v \in \boldsymbol{U} \Leftrightarrow v \in \mathbb{R}^X$ and there exists an increasing affine transformation g such that $v = g \circ u)$.

Definition 2.6 A property P is *ordinal* (resp. *cardinal*) if and only if any $\boldsymbol{U}$ that maximally satisfies P is ordinal (resp. cardinal).

3. Acyclic and transitive domains for properties

An example of a set of properties on which the "strictly weaker than" relation $\succeq_{SW}$ (see Definition 3.3) cycles will illustrate the intransitivity problem.

Example 5.1 Let X be a nonempty open convex subset of $\mathbb{R}^n$ and let $\succeq_1$, $\succeq_2$, and $\succeq_3$ be distinct complete binary relations on X. Suppose the relations $\succeq_1$ and $\succeq_2$ each have concave and nonconstant utility representations, u_1 for $\succeq_1$ and u_2 for $\succeq_2$, and suppose $\succeq_3$ has the utility representation u_3. Define the properties α, β, and γ as follows:

$\alpha = \{u \in \mathbb{R}^X$: u is a concave representation of $\succeq_1$ or a concave representation of $\succeq_2\}$,

$\beta = \{u \in \mathbb{R}^X$: u is a continuous representation of $\succeq_2$ or $u = u_3\}$,

$\gamma = \{u \in \mathbb{R}^X$: u is an increasing linear transformation of u_1 or $u_3\}$.

It is immediate that β is strictly weaker than α, γ is strictly weaker than β, and α is strictly weaker than γ. ■

From the vantage point of trying to specify a well-behaved compromise between ordinality and cardinality, Example 5.1 depicts a worst-case intransitivity. Property α is strictly weaker than any cardinal property δ such that $\alpha \cap \delta \neq \varnothing$, and any cardinal property δ such that $\gamma \cap \delta \neq \varnothing$ is strictly weaker than γ (note that "linear" rather than "affine" appears in the definition of γ). Yet, one can move via $\succeq_{SW}$ from α to γ.

Still, the cycle here hinges on the fact that, since a ranking of properties P and Q depends only on the utility functions in $P \cap Q$, P can be weaker than Q even though Q may contain a comparatively large set of utilities for preferences not represented by any of the utilities in $P \cap Q$. To construct an acyclic domain of properties, therefore, properties must include only sets of utility representations that somehow treat different preference relations symmetrically. One way to proceed is to employ sets of utility transformations, similarly but not identical to the way they are used in measurement theory (see Krantz et al. (1971)).

Definition 3.1 The psychology $\boldsymbol{U}$ has a *generator with respect to a set of transformations* $F \subset \mathbb{R}^{\mathbb{R}}$ if and only if there is a $u \in \boldsymbol{U}$ such that: $v \in \boldsymbol{U} \Leftrightarrow$ there is a $f \in F$ such that $v = f \circ u$.

Definition 3.2 A property P is *transformational* if and only if there exists a set of transformations F_P such that, for all psychologies $\boldsymbol{U}$ that maximally satisfy P, $\boldsymbol{U}$ has a generator with respect to F_P. The set F_P is called a set of P *transformations*. $\mathcal{P}_T$ will denote the set of transformational properties.

By associating sets of utility transformations with properties, we are taking a step towards the traditional measurement theory. But note that a transformational property P is distinctive in that (1) it is a set of utility functions rather than transformations and, more importantly, (2) the transformations in F_P *must* be applied to the generator of a psychology that maximally satisfies P rather than an arbitrary utility function – otherwise the utility functions generated need not satisfy P or one might not generate all of the functions that satisfy P. As an example, consider the property P_{CC} consisting of the concave and continuous functions on some convex set X. The set $F_{CV} \subset \mathbb{R}^{\mathbb{R}}$ of all increasing concave transformations is a set of P_{CC} transformations. Given a $\boldsymbol{U}$ maximally satisfying P_{CC}, any of the *least concave* utility representations of $\succeq_{\boldsymbol{U}}$ (see Debreu (1976)) may serve as a generator with respect to F_{CV}. If we apply any $f \in F_{CV}$ to a function u satisfying P_{CC} we generate another function satisfying P_{CC}. But in order to generate all of the functions in P_{CC} that agree with u, we must apply the $f \in F_{CV}$ to a least concave utility. Of course, if we apply F_{CV} to a nonconcave utility then some of the functions generated will not satisfy P_{CC}. This example thus illustrates that transformations by themselves cannot specify properties, which must be seen as sets of utility functions.

Although the “strictly weaker than” relation can cycle on some sets of transformational properties, transformational properties that are “comparable” to some cardinal property cannot circle around cardinal properties in the manner of Example 5.1.

Definition 3.3 A set of properties $\mathcal{P}$ is *acyclic with respect to cardinality* if there does not exist a finite set of properties $\{P_1, ..., P_n\} \subset \mathcal{P}$ such that P_1 is weaker than some cardinal property, some cardinal property is weaker than P_n, and, for $1 < i \leq n$, P_i is weaker than P_{i-1}.

Definition 3.4 Property P is *comparable* to property Q if there exists some $\boldsymbol{U}$ maximally satisfying P and some $\boldsymbol{V}$ maximally satisfying Q such that $\boldsymbol{U} \cap \boldsymbol{V} \neq \varnothing$ and either $\boldsymbol{U} \subset \boldsymbol{V}$ or $\boldsymbol{U} \supset \boldsymbol{V}$.

Comparability is relatively weak: P and Q can be comparable even if it is neither the case that P is no stronger than Q nor the case that Q is no stronger than P.

Theorem 3.1 Any set of properties $\mathcal{P}_C \subset \mathcal{P}_T$ such that each $P \in \mathcal{P}_C$ is comparable to some cardinal property is acyclic with respect to cardinality.

Every property discussed in Mandler (2003) is comparable to some cardinal property.

Proof of Theorem 3.1 Suppose there is a $\{P_1, ..., P_n\} \subset \mathcal{P}_C$ such that P_1 is weaker than some cardinal property, some cardinal property is weaker than P_n, and, for $1 < i \leq n$, P_i is weaker than P_{i-1}. Let P_k be the element of $\{P_2, ..., P_n\}$ with the smallest index such that there exist $\boldsymbol{U}_{Q_k}$ and $\boldsymbol{U}_{P_k}$ meeting the conditions (1) $\boldsymbol{U}_{Q_k}$ maximally satisfies a cardinal property Q_k, (2) $\boldsymbol{U}_{P_k}$ maximally satisfies P_k, and (3) $\boldsymbol{U}_{Q_k} \supset \boldsymbol{U}_{P_k}$. Given the comparability assumption, there exist $\boldsymbol{U}_{Q_{k-1}}$ and $\boldsymbol{U}_{P_{k-1}}$ such that (a) $\boldsymbol{U}_{Q_{k-1}}$ maximally satisfies some cardinal property Q_{k-1}, (b) $\boldsymbol{U}_{P_{k-1}}$ maximally satisfies P_{k-1}, and (c) $\boldsymbol{U}_{Q_{k-1}} \subset \boldsymbol{U}_{P_{k-1}}$. (If $P_k = P_2$, this conclusion follows from our supposition on P_2 and in the other cases from the fact that P_k has the smallest index.) Let F_{P_k} be a set of P_k transformations and let $u_{P_k} \in \boldsymbol{U}_{P_k}$ be a generator for $\boldsymbol{U}_{\boldsymbol{P_k}}$ with respect to F_{P_k}. Since Q_k is cardinal, the set of Q_k transformations is $F_{IA} = \{ f \in \mathbb{R}^{\mathbb{R}}$: f is an increasing affine transformation$\}$ and each $u \in \boldsymbol{U}_{Q_k}$ is a generator for $\boldsymbol{U}_{Q_k}$ with respect to F_{IA}. Hence u_{P_k} is a generator for $\boldsymbol{U}_{Q_k}$ with respect to F_{IA}. Since for each $f \in F_{P_k}$, $f \circ u_{P_k} \in \boldsymbol{U}_{P_k}$ and hence $f \circ u_{P_k} \in \boldsymbol{U}_{Q_k}$, $F_{P_k} \subset F_{IA}$. Using similar reasoning, we may also infer that $F_{P_{k-1}} \supset F_{IA}$.

On the other hand, since P_k is weaker than P_{k-1}, there exists a $\hat{\boldsymbol{U}}_{P_k}$ that maximally satisfies P_k and a $\hat{\boldsymbol{U}}_{P_{k-1}}$ that maximally satisfies P_{k-1} such that $\hat{\boldsymbol{U}}_{P_k} \supsetneq \hat{\boldsymbol{U}}_{P_{k-1}}$. Let $\hat{u}_{P_k}$ be a generator for $\hat{\boldsymbol{U}}_{P_k}$ with respect to F_{P_k} and let $\hat{u}_{P_{k-1}}$ be a generator for $\hat{\boldsymbol{U}}_{P_{k-1}}$ with respect to $F_{P_{k-1}}$. For each $u_{P_k} \in \hat{\boldsymbol{U}}_{P_k}$, there exists $f_{u_{P_k}} \in F_{P_k}$ such that $f_{u_{P_k}} \circ \hat{u}_{P_k} = u_{P_k}$. Since $F_{IA} \supset F_{P_k}$, each $f_{u_{P_k}}$ is increasing and affine and hence so is $f^{-1}_{u_{P_k}}$. Since $\hat{\boldsymbol{U}}_{P_k} \supsetneq \hat{\boldsymbol{U}}_{P_{k-1}}$, $\hat{u}_{P_{k-1}} \in \hat{\boldsymbol{U}}_{P_k}$ and so there is a $f_{\hat{u}_{P_{k-1}}} \in F_{P_k}$ such that $f_{\hat{u}_{P_{k-1}}} \circ \hat{u}_{P_k} = \hat{u}_{P_{k-1}}$. Hence $\hat{u}_{P_k} = f^{-1}_{u_{P_{k-1}}} \circ \hat{u}_{P_{k-1}}$. We therefore have, for any

$u_{P_k} \in \hat{\boldsymbol{U}}_{P_k}$, $u_{P_k} = f_{u_{P_k}} \circ f^{-1}_{\hat{u}_{P_k}} \circ \hat{u}_{p_{P-1}}$. Since $f_{u_{P_k}} \circ f^{-1}_{\hat{u}_{P_k}}$ is increasing and affine and $F_{P_{k-1}} \supset F_{IA}$, each u_{P_k} is an element of $\boldsymbol{U}_{P_{k-1}}$, which contradicts $\hat{\boldsymbol{U}}_{P_k} \supsetneq \hat{\boldsymbol{U}}_{P_{k-1}}$. ■

The assumptions of Theorem 3.1 eliminate the most disturbing cases where the "weaker than"or the "strictly weaker than" relations cycle. Moreover, by taking the transitive closure of the "weaker than" ordering $\succeq_W$, we may use Theorem 3.1 to define a transitive ordering of properties that preserves the compromise status of properties that are weaker than cardinal properties but not as weak as ordinal properties.

Specifically, fix an arbitrary set of properties $\mathcal{P}_C \subset \mathcal{P}_T$ such that each $P \in \mathcal{P}_C$ is comparable to some cardinal property, and define $\overline{\succeq}_W \subset \mathcal{P}_C \times \mathcal{P}_C$ by $P \ \overline{\succeq}_W \ Q$ if and only if there exists a finite set of properties $\{S_1, ..., S_n\}$ in $\mathcal{P}_C$ such that

$$P \succeq_W S_1 \succeq_W \cdots \succeq_W S_n \succeq_W Q.$$

Then, if P is weaker than some cardinal property Q_1, $P \succeq_W Q_1$, it cannot be that $Q_2 \ \overline{\succeq}_W \ P$ for any cardinal property Q_2. To see why, suppose to the contrary that there is some cardinal Q_2 such that $Q_2 \ \overline{\succeq}_W \ P$. Then there are $\{S_1, ..., S_n\} \subset \mathcal{P}_C$ such that

$$Q_2 \succeq_W S_1 \succeq_W \cdots \succeq_W S_n \succeq_W P \succeq_W Q_1,$$

which violates Theorem 3.1 (S_i here takes the role of P_{n-i+1} in Definition 3.3). We may also conclude that if some ordinal property Q_1 is weaker than P, $Q_1 \succeq_W P$, then not $P \ \overline{\succeq}_W \ Q_2$ for any ordinal property Q_2. For if not, there would exist $\{S_1, ..., S_n\} \subset \mathcal{P}_C$ such that

$$P \succeq_W S_1 \succeq_W \cdots \succeq_W S_n \succeq_W Q_2,$$

which is impossible since no property is weaker than any ordinal property. Summing up, we have,

Theorem 3.2 For all $P, Q, T \in \mathcal{P}_C$, (1) if $P \succeq_W Q$ and Q and T are cardinal, then not $T \ \overline{\succeq}_W \ P$, and (2) if $Q \succeq_W P$ and Q and T are ordinal, then not $P \ \overline{\succeq}_W \ T$.

So in particular the transitive ordering $\overline{\succeq}_W$ will not reverse the ordering of a property by $\succeq_W$ as weaker than some cardinal property or stronger than some ordinal property: if $P \succeq_W Q$ and Q is cardinal then $P \ \overline{\succeq}_W \ Q$ and not $Q \ \overline{\succeq}_W \ P$, and if $Q \succeq_W P$ and Q is ordinal then $Q \ \overline{\succeq}_W \ P$ and not $P \ \overline{\succeq}_W \ Q$.

These results are sufficient for the purpose of establishing a robust compromise ground between cardinality and ordinality. Still, it is illuminating to investigate conditions under which some transitive ordering can provide a sufficiently fine ranking of properties within that compromise ground. A couple hurdles stand in the way. First, a property P will vacuously be no stronger than property Q if P and Q do not intersect. Hence, one cannot expect $\succeq_{NS}$ to be transitive even on a well-behaved set of properties such as $\mathcal{P}_T$. To illustrate, let $X = \mathbb{R}$ and suppose P is the transformational property of mapping a set $A \subset \mathbb{R}$ onto the interval [0, 1] and Q is the transformational property of mapping A onto [2, 3]. Vacuously, P is no stronger than Q (and Q is no stronger than P). To generate an intransitivity, let S be the property of mapping $B \subset \mathbb{R}$ onto [2, 3], where $A \cap B = \varnothing$. Once again, vacuously, Q is no stronger than S, but obviously it is not the case that P is no stronger than S. No reasonable domain restriction can eliminate such intransitivities.

Second, although the relations $\succeq_W$ or $\succeq_{SW}$ do not suffer from exactly the same vacuity that afflicts $\succeq_{NS}$, similar problems appear. For example, when $P \succeq_W Q$ and $Q \succeq_W S$ hold, in which case P is comparable to Q and Q is comparable to S, P can nevertheless not be comparable to S, implying that $P \succeq_W S$ cannot hold. One might at least hope for the acyclicity of $\succeq_W$ on a well-behaved domain. The following example shows, however, that $\succeq_W$ or $\succeq_{SW}$ can cycle on $\mathcal{P}_T$.

Example 5.2 For some nonempty open set $X \subset \mathbb{R}^n_+$, let $\succeq_1$, $\succeq_2$, and $\succeq_3$ be distinct complete binary relations on X, each of which has a concave utility representation. For $i = 1, 2, 3$, let $\underline{u}_i$ denote one such representation. Let $F_{CV} \subset \mathbb{R}^{\mathbb{R}}$ be the set of increasing concave transformations. Given some $f' \in F_{CV}$ that is strictly concave on the range of each $\underline{u}_i$, define $\bar{u}_i = f' \circ \underline{u}_i$. Define the properties α, β, and γ as follows:

$$\alpha = \{u \in \mathbb{R}^X : u = f \circ \bar{u}_1 \text{ or } u = f \circ \underline{u}_3 \text{ for some } f \in F_{P_{CC}}\},$$
$$\beta = \{u \in \mathbb{R}^X : u = f \circ \bar{u}_2 \text{ or } u = f \circ \underline{u}_1 \text{ for some } f \in F_{P_{CC}}\},$$
$$\gamma = \{u \in \mathbb{R}^X : u = f \circ \bar{u}_3 \text{ or } u = f \circ \underline{u}_2 \text{ for some } f \in F_{P_{CC}}\}.$$

Each of these properties is transformational: for all three, F_{CV} may serve as the set of transformations, $\bar{u}_1$ and $\underline{u}_3$ are generators for α, $\bar{u}_2$ and $\underline{u}_1$ for β, and $\bar{u}_3$ and $\underline{u}_2$ for γ. Yet we have $\beta \succeq_{SW} \alpha$, $\gamma \succeq_{SW} \beta$, and $\alpha \succeq_{SW} \gamma$. ■

The key to Example 5.2 is that while each property is ranked relative to the other two, no pair of psychologies that maximally satisfy distinct properties have a generator in common. Thus one property may be weaker than another even though they share the same set of transformations. One way to proceed, therefore, is to declare that when a pair of properties *never* have a generator in common they are not ranked.

Definition 3.5 A property P is *uniquely transformational* if and only if there exists one and only one set of transformations F_P, called the *unique P-transformations*, such that, for all psychologies $\boldsymbol{U}$ that maximally satisfy P, $\boldsymbol{U}$ has a generator with respect to F_P. Let $\mathcal{P}_{UT} \subset \mathcal{P}_T$ denote the set of uniquely transformational properties.

Definition 3.6 The relation $\succeq^*_{NS} \subset \mathcal{P}_{UT} \times \mathcal{P}_{UT}$ is defined by $P \succeq^*_{NS} Q \Leftrightarrow$ whenever $\boldsymbol{U}$ maximally satisfies P, $\boldsymbol{V}$ maximally satisfies Q, and there exists a $w \in \boldsymbol{U} \cap \boldsymbol{V}$ that is a generator for $\boldsymbol{U}$ with respect to the unique P-transformations and a generator for $\boldsymbol{V}$ with respect to the unique Q-transformations, then $\boldsymbol{U} \supset \boldsymbol{V}$. Let $\succeq^*_W \subset \mathcal{P}_{UT} \times \mathcal{P}_{UT}$ be defined by $P \succeq^*_W Q \Leftrightarrow P \succeq^*_{NS} Q$ and not $Q \succeq^*_{NS} P$.

Theorem 3.3 The relation $\succeq^*_W$ is acyclic.

Proof Suppose to the contrary that there exists a finite set $\{P_1, ..., P_n\} \subset \mathcal{P}_{UT}$ such that $P_1 \succeq^*_W P_n$ and, for $1 < i \le n$, $P_i \succeq^*_W P_{i-1}$. For $i \in \{2, ..., n\}$, there is a $\boldsymbol{U}_{P_i}$ maximally satisfying P_i and a $\boldsymbol{U}_{P_{i-1}}$ maximally satisfying P_{i-1} such that (i) $\boldsymbol{U}_{P_i} \supsetneq \boldsymbol{U}_{P_{i-1}}$ and (ii) there exists a $w \in \boldsymbol{U}_{P_i} \cap \boldsymbol{U}_{P_{i-1}}$ that is a generator for $\boldsymbol{U}_{P_i}$ with respect to the set of unique P_i-transformations, say F_{P_i}, and a generator for $\boldsymbol{U}_{P_{i-1}}$ with respect to the set of unique P_{i-1}-transformations, say $F_{P_{i-1}}$. For each $f \in F_{P_{i-1}}$, there exists $u_{P_{i-1}} \in \boldsymbol{U}_{P_{i-1}}$ such that $f \circ w = u_{P_{i-1}}$. Since $\boldsymbol{U}_{P_i} \supset \boldsymbol{U}_{P_{i-1}}$, $f \in F_{p_i}$. Since $\boldsymbol{U}_{P_i} \supsetneq \boldsymbol{U}_{P_{i-1}}$, there exists a $f' \in F_{P_i}$ such that $f' \circ w \notin \boldsymbol{U}_{P_{i-1}}$. So $f' \notin F_{P_{i-1}}$ and therefore $F_{P_i} \supsetneq F_{P_{i-1}}$. Repeating this argument for $\boldsymbol{U}_{P_1}$ and $\boldsymbol{U}_{P_n}$, we have $F_{P_1} \supsetneq F_{P_n}$. So $F_{P_n} \supsetneq F_{P_n}$, a contradiction. ■

Given Theorem 3.3, we may as before use the transitive closure of $\succeq^*_W$, say $\overline{\succeq}^*_W$, to define a transitive ordering. That is, let $\succeq^*_{W-cl} \subset \mathcal{P}_{UT} \times \mathcal{P}_{UT}$ be defined by $P \overline{\succeq}^*_W Q$ if and only if there exists a finite set of uniquely transformational properties $\{S_1, ..., S_n\}$ such that $P \succeq^*_W S_1 \succeq^*_W \cdots \succeq^*_W S_n \succeq^*_W Q$. The acyclicity of $\succeq^*_W$ ensures that $\overline{\succeq}^*_W$ is asymmetric, and so $\overline{\succeq}^*_W$ does not reverse any of the orderings in $\succeq^*_W$.

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