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Simple Pareto-Improving Policies

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Pareto improvements that require no information about individual characteristics are analyzed. Initially, equilibrium must be production inefficient. After the policy change, consumer prices differ from producer prices, but allocations, although second-best, are Pareto superior and production efficient. Policy implementation is modeled as a dynamical system that governs aggregate consumer wealth, producer prices, and production levels. With knowledge of the maximum feasible level of consumer wealth and no other information, a policymaker can choose policy rules such that this system converges globally to the targeted equilibrium. With knowledge of only the aggregate income effect, local convergence is achievable. *Journal of Economic Literature* Classification Numbers: D51, D60, P11.

1. INTRODUCTION

Many traditional policy recommendations harm some agents substantially. Policies eliminating production inefficiencies furnish the most conspicuous examples: unrestricted use of new technologies, opening closed economies to trade, and the introduction of competitive markets into a planned economy all lead to gains for some agents and to losses for others. The apparent absence of Pareto improvements presents economists with a difficult choice: either remain agnostic on policy issues or adopt some system for making interpersonal comparisons of welfare. This paper reexamines this dilemma; we argue that if production is initially inefficient then Pareto improvements can be attained. Moreover, the information needed to design these policies is modest. This fact is paramount. The policies described here do not lead to first-best outcomes; if the set of available policies is unrestricted, second-best policies can only be justified on the basis of their comparatively small informational demands.

The losses that accompany typical policy changes can be seen as a consequence of uncompensated changes in the relative prices facing agents.

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A policy that raises the price of some good *i* slightly and leaves endowments and other prices unchanged will harm net purchasers of *i*, while a small decrease in price will harm net suppliers of *i*. Conversely, if relative prices do not change and the endowments of agents do not fall, no agent can be harmed by a policy change. This last fact suggests a simple way to construct Pareto improvements. Consider *price-stabilized* policies, where the relative prices of goods for which consumers can be either net purchasers or sellers are held fixed and the prices of goods for which consumers are known to be net suppliers are not permitted to decrease. Since each agent's ex ante consumption choice remains affordable under a price-stabilized policy, no individual is made worse off. Now suppose that production (or some other sector of the economy that trades with consumers) is inefficient ex ante. The welfare of some agents can then increase if the inefficiencies are removed and either agents receive an additional endowment from the government or the prices of the net-supplied goods rise. With agents experiencing a de facto increase in wealth, Pareto-improving allocations of goods are both affordable and, given the move to efficient production, feasible.

The dual requirements of efficient production and price stability for consumers necessitate that consumers and producers face different prices, which can be accomplished either by segregating consumers and producers to different markets or by commodity taxation. Using a general equilibrium model that allows consumer and producer prices to differ, we demonstrate the consistency of price-stabilized policies: agents receive an allocation that is Pareto improving relative to the status quo and there exist producer prices at which competitive firms produce this allocation. Since firms maximize profits, the allocation is production efficient. This last feature is similar to Diamond and Mirrlees [3].

Why might a policymaker institute a price-stabilized equilibrium rather than a first-best optimum? First, limitations on the set of policy tools may prevent compensation being paid to the agents harmed by a policy change. Second, the policymaker may not know the parameters of the model—in particular the identity of agents suffering losses and the magnitude of those losses—with certainty. Both answers justify price-stabilized policies, but the second rationale carries greater theoretical weight. Accordingly, the principal advantage of the present approach is that a policymaker who is ignorant of agent characteristics can infer that no agent is harmed solely from the fact that the consumer price vector is price stabilized. Still, price-stabilized policies have *some* informational requirements; the remainder of the paper therefore investigates whether these requirements are reasonably small.

Given that consumer prices essentially remain at their status quo level, the information issue reduces to two questions: how do producer prices come to be set at the appropriate levels and how much wealth should be distributed to consumers? We address these questions with a traditional dynamic model of the behavior of competitive producers and market prices. Producers change their scale of operation in proportion to profitability and producer prices follow a *tâtonnement*, increasing or decreasing as aggregate excess demand is positive or negative. The *tâtonnement* can describe either the unregulated behavior of a distinct producers' market or, given that consumer prices are fixed, the government's rule for adjusting commodity tax rates. In the latter case, policy adjustments at each point in time depend only on the current value of aggregate excess demand. If the policymaker knows the level of wealth that agents must have in order for equilibrium to be on the production possibilities frontier, it is easy to show, using one of the Arrow–Hurwicz theorems on gradient processes, that the above process converges globally: starting from an arbitrary point, producer prices and production levels converge to an equilibrium that is production efficient.

As for the second question, how should the government set policy parameters if it does not know the appropriate level of wealth to distribute? Our second convergence result assumes that the policymaker instead knows the aggregate income effect of consumers (specifically, the effect of increasing consumer wealth on aggregate consumer demand when prices are held at their ex ante values) and that the aggregate demand for each good is increasing in consumer wealth. We then show that a policymaker with this knowledge can dynamically adjust the distribution of wealth so that consumer demand, producer prices and production levels locally converge to values consistent with a Pareto-improving and production-efficient equilibrium. To sum up, informational requirements are indeed modest compared to what first-best policies presuppose: only knowledge of the aggregate income effect—and no information about individual characteristics—is (locally) necessary.

The fact that the *tâtonnement* described here converges is significant. Due to the bad behavior of consumer demand functions, *tâtonnements* frequently do not converge in standard general equilibrium models. We get local convergence by choosing an appropriate rule for adjusting consumer wealth and by fixing consumer prices. Technically, our convergence results rely on the Arrow–Hurwicz theory of gradient processes, but our underlying model differs markedly from the related Arrow–Hurwicz theory of planning [2]. Arrow and Hurwicz use a single utility function to represent the aggregate goals of society, and the planner adjusts output by comparing the marginal social utility of aggregate consumption with the marginal cost of consumption. In our model, outside of setting the institutional rule that producer prices are governed by a *tâtonnement*, the only role for the policymaker is to expand or contract the level of consumer wealth. The policymaker does not impose an objective function and budget-constrained agents make utility-maximizing consumption choices. The present paper thus models a decentralized, not a planned, economy. Analytically, the costs of these changes are that a Pareto optimum is not reached and convergence can be local, not global. But we hope to show that the Arrow–Hurwicz techniques can be applied fruitfully to non-planning situations.

The following section presents the model and gives conditions for the existence of a price-stabilized policy. Section 3 presents the dynamic adjustment model and provides the two convergence results. Section 4 discusses the related optimal taxation literature.

2. A MODEL OF POLICY CHANGE

There are $\ell > 0$ goods, ℓ_1 of which are *consumption goods* and $\ell_2 = \ell - \ell_1$ of which are *factors*. We assume that $0 < \ell_1 < \ell$. Each agent $i = 1, ..., n$, has an endowment of the two types of goods $\omega^i = (\omega_1^i, \omega_2^i) \in R_+^\ell$. The vector of all of the individuals' endowments is $\Omega = (\omega^1, ..., \omega^n)$. A consumption vector for agent i is denoted $x^i = (x_1^i, x_2^i)$.

The price vector facing agents for consumption goods and factors is given by $p = (p_1, p_2) \geqslant 0$. Each agent i has a continuous, locally non-satiated utility function $u^i(x^i)$, which is maximized subject to $p \cdot x^i \leqslant p \cdot \omega^i$, $x^i \geqslant 0$, yielding the demand function, $x^i(p, \omega^i) = (x_1^i(p, \omega^i), x_2^i(p, \omega^i))$. Aggregate demand is $x(p, \Omega) = (x_1(p, \Omega), x_2(p, \Omega)) = \sum_{i=1}^n x^i(p, \omega^i)$.

We assume that the preferences and endowments are such that $x(p, \Omega)$ is single-valued and continuous in both arguments at any $p \gg 0$.[1] The following can be kept in mind as the primary rationale. For each agent there may be a subset of goods that do not affect the utility of the agent. If, say, the kth factor is such a good for some agent i, then $x_2^i(k)(p, \omega^i) = 0$ when $p_2(k) > 0$. If, when the consumption levels of such goods are held fixed, each agent's utility is a strictly quasiconcave function of the remaining goods, then $x(p, \Omega)$ will be single-valued and continuous at any $p \gg 0$.

Assume, for all i, $p \gg 0$, and $\hat{\omega}^i \geqslant \omega^i$, that $x_2^i(p, \hat{\omega}^i) \leqslant \omega_2^i$. Agents, that is, are net suppliers of factors at endowments greater than their initial endowment. Of course, excess demand for consumption goods can be nonpositive as well. So far, the assumption that factors are net-supplied is virtually without content, since it may be that factors are neither owned nor give utility to any agent.

Technology is described by a closed production set $Y \subset R^\ell$, perhaps generated from a set of individual firm production sets. We assume (1) Y

[1] We use the notation: $x \gg y \Leftrightarrow x(k) > y(k)$ for all k; $x \geqslant y \Leftrightarrow x(k) \geqslant y(k)$ for all k; and $x > y \Leftrightarrow x \geqslant y$ and $x \neq y$.

is a convex cone with vertex 0, i.e., Y is convex, and if $y \in Y$, then for all $\phi \geqslant 0$, $\phi y \in Y$, (2) $-R^{\ell}_{+} \subset Y$ (free disposal), and (3) $Y \cap R^{\ell}_{+} = \{0\}$ (impossibility of free production). Define $y \in Y$ as *production efficient* if $y \in \partial Y$ (where ∂ is the boundary) and $y \in Y$ as *production inefficient* if $y \in \text{Int } Y$ (where "Int" denotes "the interior of").

Prior to any change in policies, the state may own a vector of endowments $\omega^s = (\omega^s_1, \omega^s_2) \geqslant 0$. We discuss interpretations of the government endowment immediately following the statement of Proposition 1. The *aggregate endowment* available to society is $\omega = (\omega_1, \omega_2) = \omega^s + \sum_{i=1}^{n} (\omega^i_1, \omega^i_2)$. Given ω and a production set Y, define the set of technically feasible aggregate consumption vectors as $F(\omega) = \{x \in R^{\ell}_{+} : \exists y \in Y \text{ such that } x = y + \omega\}$. Under our assumptions, $F(\omega)$ is compact. For $y \in Y$ and $\omega \in R^{\ell}_{+}$ such that $x = y + \omega \geqslant 0$, $x \in \partial F(\omega)$ if and only if $y \in \partial Y$ (and $x \in \text{Int } F(\omega)$ if and only if $y \in \text{Int } Y$).

A *status quo equilibrium* is a price vector $\bar{p} \gg 0$ such that $x(\bar{p}, \Omega) \in F(\omega)$. The *status quo allocation* is $(x^1(\bar{p}, \omega^1), ..., x^n(\bar{p}, \omega^n))$. Note that this definition makes no mention of how production is organized. Accordingly, production decisions can be made in a non-profit-maximizing manner (e.g., by the government) or with a restricted production set that excludes certain technologies or the opportunities for international trade. Say that a status quo equilibrium is production efficient (resp. inefficient) if $x(\bar{p}, \Omega) - \omega$ is production efficient (resp. inefficient).

If the status quo equilibrium is production inefficient, a policy distributing the state's endowment to the n agents and allowing profit-maximizing firms to organize production will, given the lack of externalities or other market failures, lead to a Pareto optimum. But since the resulting allocation is unlikely to be Pareto-ranked relative to the status quo, alternative policies deserve consideration. Broadly speaking, we analyze policies with the following three features: (1) some or all of the government's endowment vector is distributed to agents; (2) to ensure that no agent is made worse off, consumer prices are kept at their status quo levels; and (3) to achieve production efficiency, production decisions are made by profit-maximizing firms.

Formally, producers face a vector of producer prices $q > 0$ that may differ from the consumer prices p facing agents. The level of profits earned by firms facing prices q and choosing the production vector y is $q \cdot y$.

We assume the government transfers to each agent i a vector $\omega^i_g \in R^{\ell}$. A vector $\Omega_g \in \{(\omega^1_g, ..., \omega^n_g) \in R^{\ell n} : \sum_{i=1}^{n} \omega^i_g \leqslant \omega^s\}$ is called a *distribution of the state's endowment*. Given Ω_g, the *policy-induced endowment* of each agent i is $\omega^i_t = \omega^i + \omega^i_g$. The vector of the individuals' endowments is now $\Omega_t = (\omega^1_t, ..., \omega^n_t) = \Omega + \Omega_g$. The government's policy-induced endowment is $\omega^s_t = \omega^s - \sum_{i=1}^{n} \omega^i_g$. A *policy equilibrium* is a $(p^*, q > 0, \Omega_g)$ such that

$$x(p^*, \Omega_t) - \omega = y \in Y \tag{2.1}$$

$$q \cdot y' \leqslant q \cdot y \qquad \text{for all } y' \in Y, \tag{2.2}$$

where $\Omega_t = \Omega + \Omega_g$. The corresponding allocation is $(x^1(p^*, \omega_t^1), \ldots, x^n(p^*, \omega_t^n))$. A policy equilibrium is *Pareto improving* if, for all i, $u^i(x^i(p^*, \omega_t^i)) \geqslant u^i(x^i(\bar{p}, \omega^i))$, and strict inequality holds for some i.

The equilibrium can be interpreted in the two ways. First, imagine that there are two conceptually distinct marketplaces. At one market, the retail store, consumers face the price vector p^*. The other, producer marketplace occurs at the factory door and the price vector q rules there. Since generally $p^* \neq q$, agents must be prohibited from arbitraging the two markets. Alternatively, think of $\tau = p^* - q$ as a vector of enforced excise taxes and subsidies. Under either interpretation, the government can be seen as an intermediary between consumers and firms. The difference between the government's receipts and expenditures is then $(p^* - q) \cdot y + p^* \cdot \omega_t^s$. Since Walras' law (i.e., $p \cdot (x(p, \Omega_t) - \sum_{i=1}^n \omega_t^i) = 0$ for all p) holds in this model, (2.1) implies that $p^* \cdot (y + \omega_t^s) = 0$. Given (2.2) and the fact that $0 \in Y$, $q \cdot y = 0$. The government's budget therefore balances in equilibrium.

If $\bar{p}$ indicates the status quo equilibrium price vector, a policy (or the associated equilibrium) is *price stabilized* if there is a $\lambda > 0$ such that $p_1^* = \lambda \bar{p}_1$ and $p_2^* \geqslant \lambda \bar{p}_2$. The following proposition confirms that under very mild conditions if the status quo equilibrium is production inefficient, then a price-stabilized and Pareto-improving policy exists.

PROPOSITION 1. *If a status quo equilibrium* $\bar{p} \gg 0$ *is production inefficient and either* (1) $\omega_2 > 0$, *or* (2) *there exists a* $\bar{\Omega}_g \geqslant 0$ *such that* $x(\bar{p}, \Omega + \bar{\Omega}_g) \notin F(\omega)$, *then there exists a price-stabilized policy equilibrium that is Pareto improving and production efficient.*

There are two prominent rationales for why $\omega_2^s > 0$ and thus $\omega_2 > 0$. First, in applications such as the transition from planned to market economies, the state will be endowed with a variety of pure factors, such as factories and raw materials, that do not provide direct utility and which agents therefore will net supply. Second, we may think of one of the factors as being a type of money or other credit that the government can print (at zero or little cost). Agents will be net suppliers of such a factor and the government's endowment of the factor will be positive. Condition (2) can also be justified along these lines: if there is a money good, presumably the government can print enough of it to ensure that demand is infeasible.

Note in the proof below that if either $\omega_2^s > 0$ or condition (2) of the Proposition holds, then there exists a production-efficient equilibrium that is a strict Pareto improvement.

Proof of Proposition 1. First, note that for any $\tilde{\omega}^i \geqslant \omega^i$, $p_2 \geqslant \bar{p}_2$, we have, for each i, $(\bar{p}_1, p_2) \cdot x^i(\bar{p}, \omega^i) \leqslant (\bar{p}_1, p_2) \cdot \tilde{\omega}^i$. Thus, $u^i(x^i((\bar{p}_1, p_2), \tilde{\omega}^i)) \geqslant u^i(x^i(\bar{p}, \omega^i))$ for each i.

There are two cases under which condition (1) is satisfied: (i) $\sum_{i=1}^n \omega_2^i > 0$ and (ii) $\omega_2^s > 0$. For case (i), set $\Omega_g = 0$, i.e., $\Omega_t = \Omega$. We show that for $(\bar{p}_1, \mu\bar{p}_2)$ with μ sufficiently greater than 1, $x((\bar{p}_1, \mu\bar{p}_2), \Omega) \notin F(\omega)$. It is sufficient to show that for any sequence $\mu_n \to \infty$, the sequence $x^i((\bar{p}_1, \mu_n\bar{p}_2), \omega^i)$ is unbounded for some i. Suppose, to the contrary, that $x^i((\bar{p}_1, \mu_n\bar{p}_2), \omega^i)$ is bounded for all i. Then each $x^i((\bar{p}_1, \mu_n\bar{p}_2), \omega^i)$ has an accumulation point, say $\hat{x}^i$. Given that $\sum_{i=1}^n \omega_2^i > 0$, $\omega_2^i > 0$ for some i. For this i and any x^i s.t. $x_2^i \leqslant \omega_2^i$, there exists a sequence $x_n^i \to x^i$ such that $(\bar{p}_1, \mu_n\bar{p}_2) \cdot x_n^i \leqslant (\bar{p}_1, \mu_n\bar{p}_2) \cdot \omega^i$, or equivalently,

$$((1/\mu_n)\, \bar{p}_1, \bar{p}_2) \cdot x_n^i \leqslant ((1/\mu_n)\, \bar{p}_1, \bar{p}_2) \cdot \omega^i. \tag{2.3}$$

To see why, for each μ_n let $x_n^i = \lambda_n x^i$, where λ_n is maximized s.t. (2.3) and $\lambda_n \leqslant 1$; given $\omega_2^i > 0$, $\mu_n \to \infty$ implies $\lambda_n \to 1$. We conclude, by the continuity of u^i, that

$$u^i(\hat{x}^i) \geqslant u^i(x^i) \qquad \text{for any } x^i \text{ such that } x_2^i \leqslant \omega_2^i. \tag{2.4}$$

Due to nonsatiation, there is a $\tilde{x}^i$ arbitrarily near $\hat{x}^i$ such that $u^i(\tilde{x}^i) > u^i(\hat{x}^i)$. This, however, contradicts i being a net supplier of factors: for $\tilde{\omega}^i = \omega^i + (e, 0)$, where $e \in R_{++}^{\ell_1}$ is sufficiently large, $\bar{p} \cdot \tilde{x}^i \leqslant \bar{p} \cdot \tilde{\omega}^i$. Given (2.4), $x_2^i(\bar{p}, \tilde{\omega}^i) > \tilde{\omega}_2^i$.

Since for μ sufficiently large, $x((\bar{p}_1, \mu\bar{p}_2), \Omega) \notin F(\omega)$, we conclude, by the continuity of x, that there is a $p_2^* > \bar{p}_2$ such that $x((\bar{p}_1, p_2^*), \Omega) \in \partial F(\omega)$. In this case, set $p^* = (\bar{p}_1, p_2^*)$. We have already shown in the first paragraph that no agent is worse off; to see that some agent must be better off at p^*, suppose, for all i, that $u^i(x^i(p^*, \omega^i)) = u^i(x^i(\bar{p}, \omega^i))$. Since both $x^i(p^*, \omega^i)$ and $x^i(\bar{p}, \omega^i)$ are affordable at p^*, we would have, since demand is single-valued, that $x^i(p^*, \omega^i) = x^i(\bar{p}, \omega^i)$, for all i. This contradicts the fact that $x(\bar{p}, \Omega) \in \operatorname{Int} F(\omega)$ and $x(p^*, \Omega) \in \partial F(\omega)$.

Now consider case (ii): $\omega_2^s > 0$. Let $\omega_g^i = \phi\omega^s$ for each i. Given the continuity of x, for ϕ near 0, $x(\bar{p}, \Omega_t) \in F(\omega)$. Fix ϕ (and thus Ω_g and Ω_t) at such a value. Since $\omega_{t,2}^i > 0$ for some (in fact, all) i, we can repeat the argument used for the case when $\sum_{i=1}^n \omega_2^i > 0$ to conclude that there is a $p_2^* > \bar{p}_2$ such that $x((\bar{p}_1, p_2^*), \Omega_t) \in \partial F(\omega)$. Again set $p^* = (\bar{p}_1, p_2^*)$. Clearly, $u^i(x^i(p^*, \omega_t^i)) > u^i(x^i(\bar{p}, \omega^i))$ for all i. For condition (2), continuity directly implies that there exists a ϕ such that $x(\bar{p}, \Omega_t) \in \partial F(\omega)$ when $\omega_t^i = \omega^i + \phi\bar{\omega}_g^i$, for all i; clearly $u^i(x^i(\bar{p}, \omega_t^i)) > u^i(x^i(\bar{p}, \omega^i))$ for all i. In this case, set $p^* = \bar{p}$.

For both conditions (1) and (2), we have that $x(p^*, \Omega_t)$ is an element of $\partial F(\omega)$. Hence, given that $-R_+^\ell \subset Y$, there exists a supporting vector $q > 0$

such that $q \cdot y = 0$ for $y = x(p^*, \Omega_t) - \omega$ and such that $q \cdot y' \leqslant 0$ for all $y' \in Y$. ▌

3. INSTITUTING PRICE-STABILIZED POLICIES

We now consider the information necessary to institute a price-stabilized policy and how the government can acquire this information. Informally, keep the following procedure in mind. The policymaker first makes a small distribution of the state's endowment, Ω_g. Then the policymaker attempts to purchase the resulting vector $x(p^*, \Omega + \Omega_g) - \omega$ at the producer market, with the producer price vector q adjusting in proportion to excess demand to clear the market. If $x(p^*, \Omega + \Omega_g) - \omega$ can be purchased, then the government makes a larger distribution, continuing until further increases in Ω_g are infeasible.

We formalize this process with differential equations specifying the adjustment of production levels and producer prices through time: at each moment, producer prices change in proportion to excess demand (a *tâtonnement*) and production levels change in proportion to the effect that increases in production have on profitability. These relationships can be thought of as features of the economy not directly subject to government control. Alternatively, if $\tau = p - q$ is a policy variable, the equation governing q can be seen as a policy rule for setting τ. In the latter interpretation, changes in q at each point in time will only be a function of contemporaneous aggregate excess demand, not the demands of particular agents and firms; hence, no direct knowledge of the production set or agent utilities is utilized.

We present two convergence results. In the first, the government knows the maximum proportional distribution of its endowment consistent with feasibility. If the policymaker permanently fixes endowment distributions at this level, consumer demand does not change during the adjustment to equilibrium and consequently production levels and producer prices globally converge to an equilibrium that is Pareto improving and production efficient. The second result assumes that the government knows the effect of increasing consumer wealth on aggregate consumer demand, but not the maximum level of wealth that can be distributed. (The latter would require knowledge of the production set and aggregate endowment levels.) We then show that there is a dynamic rule for distributing the government endowment such that producer prices, production levels, and endowment distributions locally converge to a Pareto-improving and production-efficient equilibrium. Changes in endowments affect consumer demand; but despite the bad behavior of consumer demand functions, the government can choose a distribution rule consistent with local convergence.

We specialize the framework somewhat. Technology is described by a set of nonlinear activities with each activity $j=1, ..., r$, characterized by a vector of continuously differentiable output functions, $f_j : R \to R^{\ell_1}$ and a vector of continuously differentiable input functions, $g_j : R \to R^{\ell_1}$. We assume, for all activities j and goods $k=1, ..., \ell_1$, that $f_{j,k}$ is either strictly concave or identically zero and $g_{j,k}$ is either strictly convex or identically zero. The production set, $Y \subset R^{\ell_1}$, is $\{y \in R^{\ell_1} : \exists \gamma \in R^r_+ \text{ with } \sum_{j=1}^r (f_j(\gamma_j) - g_j(\gamma_j)) \geqslant y\}$. As in Section 2, we assume that $Y \cap R^{\ell_1}_+ = \{0\}$.

Since the production technology no longer displays constant returns to scale, we need to specify the ownership of the production processes. So suppose that there are r factors representing ownership in the r production processes. We assume, without loss of generality, that there are no other factors besides the ownership goods (i.e., $r=\ell_2$) and let the demand functions $x^i(p, \omega^i)$ and $x(p, \Omega)$ henceforth refer only to the ℓ_1 consumption goods. For $j=1, ..., r$, we require that $\omega^s_2(j) + \sum_{i=1}^n \omega^i_2(j) = 1$, and that, for all i and x^i, $u^i(x^i_1, x^i_2(1), ..., x^i_2(j-1), \cdot, x^i_2(j+1), ..., x^i_2(r))$ is a constant function.

A status quo equilibrium is a $\bar{p} \gg 0$ and $\gamma \geqslant 0$ such that $x(\bar{p}, \Omega) - \omega_1 \leqslant \sum_{j=1}^r (f_j(\gamma_j) - g_j(\gamma_j))$. A $(p \in R^{\ell}_+, q \in R^{\ell_1}_+ \backslash \{0\}, \gamma \geqslant 0, \Omega_g)$ is a policy equilibrium if

$$x(p, \Omega_t) - \omega_1 \leqslant \sum_{j=1}^{r} (f_j(\gamma_j) - g_j(\gamma_j)) \tag{3.1}$$

$$q \cdot (f_j(\gamma_j) - g_j(\gamma_j)) \geqslant q \cdot (f_j(\gamma'_j) - g_j(\gamma'_j)), \quad \text{for all } j, \text{ and } \gamma'_j \geqslant 0, \tag{3.2}$$

where $\Omega_t = \Omega + \Omega_g$. An equilibrium is production efficient if $(x(p, \Omega_t) - \omega_1) \in \partial Y$.

We suppose that a version of condition (2) of Proposition 1 holds: the government endowment is large enough that, if distributed in equal parts to all agents, aggregate demand would be infeasible. This assumption guarantees the existence of a price-stabilized equilibrium with consumer prices exactly equal to the status quo price vector $\bar{p}$, which will simplify notation. Policy-induced endowments will take the form $\Omega_t(m) = (\omega^i + m(1/n)\, \omega^s)_{i=1}^n$, for $m \in R_+$. For all i, define $x^i(m) = (x^i_1(\bar{p}, \Omega_t(m)), ..., x^i_{\ell_1}(\bar{p}, \Omega_t(m)))$. Let $x(m) = \sum_{i=1}^n x^i(m)$. We state our assumption formally as $(x(1) - \omega_1) \notin Y$. If one of the consumption goods is a type of money or credit (see Section 2), this condition is very weak. (Results similar to the propositions below can be proved if increasing p_2 must be used as a device for increasing consumer wealth.) Let m^* be an arbitrary element of $\{m \in R_+ : (x(m) - \omega_1) \in \partial Y\}$.

When $\omega_g^i = m(1/n)\,\omega^s$, for all i, we use (p, q, γ, m) as shorthand for (p, q, γ, Ω_g). If, at (p, q, γ, m), $p = \bar{p}$, then no agent is worse off relative to the status quo allocation.

We use $m(t)$, $\gamma(t)$, and $q(t)$, defined for $t \in [0, \infty)$, to indicate the time paths of endowment distributions, activity levels and producer prices that occur during the policy adjustment process. The differential equations governing γ_j, $j = 1, ..., r$, and q_k, $k = 1, ..., \ell_1$, are

$$\dot{\gamma}_j(t) = \delta_{\gamma_j} q(t) \cdot [Df_j(\gamma_j(t) - Dg_j(\gamma_j(t))] \tag{3.3}$$

$$\dot{q}_k(t) = \delta_{q_k}[x_k(m(t)) - \omega_1(k) - \sum_{j=1}^{r} f_{j,k}(\gamma_j(t)) + \sum_{j=1}^{r} g_{j,k}(\gamma(t))], \tag{3.4}$$

where $\delta_{\gamma_j} = 0$ if $\gamma_j(t) = 0$ and $q(t) \cdot [Df_j(\gamma_j(t) - Dg_j(\gamma_j(t))] < 0$, and a fixed positive constant otherwise, and where $\delta_{q_k} = 0$ if $q_k(t) = 0$ and $x_k(m(t)) - \omega_1(k) - \sum_{j=1}^{r} f_{j,k}(\gamma_j(t)) + \sum_{j=1}^{r} g_{j,k}(\gamma_j(t)) < 0$, and a fixed positive constant otherwise.

A *policy adjustment process* is a set of functions $m(t)$, $\gamma(t)$, and $q(t)$ satisfying (3.3) and (3.4). We assume that $(m(0), q(0), \gamma(0)) \geqslant 0$.

PROPOSITION 2. *If the status quo equilibrium $\bar{p}$ is production inefficient, $m(t) = m^*$ for all t, and $q(0) \gg 0$, then $(\gamma(t), q(t))$ converges to a (γ^*, q^*) such that $(\bar{p}, q^*, \gamma^*, m^*)$ is Pareto improving and production efficient.*

Proof. A saddle-point for $\mathscr{L}(\gamma, q) = q \cdot (\sum_{j=1}^{r} (f_j(\gamma_j) - g_j(\gamma_j)) - x(m^*) + \omega_1)$, i.e., a (γ^*, q^*) such that $\mathscr{L}(\cdot, q^*)$ reaches a maximum over $\gamma \geqslant 0$ at γ^* and $\mathscr{L}(\gamma^*, \cdot)$ reaches a minimum over $q \geqslant 0$ at q^*, constitute a policy equilibrium when $m = m^*$ and $p = \bar{p}$. Since saddle-points exist, our strict concavity and convexity assumptions on the f_j and the g_j that are not constant are then sufficient for convergence to a saddle-point. See Uzawa [4] and Arrow and Hurwicz [1, Theorem 6]. Conditions (3.1) and (3.2) then follow from the definition of a saddle-point, efficiency from the definition of m^*, and that the equilibrium is Pareto improving from the fact that $m^* > 0$.

To show that $q^* \neq 0$, note that due to the Kuhn–Tucker theorem there is some $(\gamma^*, q' > 0)$ that is a saddle-point. Clearly, for any $\lambda > 0$, $(\gamma^*, \lambda q')$ is also a saddle-point. Since the Euclidean distance d between $(\gamma(t), q(t))$ and any saddle-point is nonincreasing in t (see the above references), $d((\gamma^*, \lambda q'), (\gamma(0), q(0)))^2 \geqslant d((\gamma^*, \lambda q'), (\gamma^*, q^*))^2$. If $q^* = 0$,

$$(\lambda q' - q(0)) \cdot (\lambda q' - q(0)) + (\gamma^* - \gamma(0)) \cdot (\gamma^* - \gamma(0)) \geqslant \lambda q' \cdot \lambda q',$$

$$q(0) \cdot q(0) + (\gamma^* - \gamma(0)) \cdot (\gamma^* - \gamma(0)) - 2(q(0) \cdot \lambda q') \geqslant 0.$$

Since $q(0) \gg 0$, the last inequality yields a contradiction for λ sufficiently large. ∎

Suppose now that the government does not know the value of a m^*. It is clear that if the government instead knows the value of any $\hat{m} > 0$ such that $(x(\hat{m}) - \omega_1) \in Y$, then the same procedure as above will allow $(\gamma(t), q(t))$ to converge to a (γ^*, q^*) such that $(\bar{p}, q^*, \gamma^*, \hat{m})$ is a policy equilibrium. Thus global convergence to a Pareto improvement is still possible. If we wish to reach m^* and achieve full production efficiency, the policymaker must know more. Our assumption will be that the policymaker knows the function $x(m)$—the aggregate income effect—but not the particular value of a m^*. Formally, say that $m(t)$ is *accessible* if, for all $\bar{t}$, $\dot{m}(\bar{t})$ is a function only of the mapping $x(m)$ and of values of $m(t)$ and $q(t)$ in $t \in [0, \bar{t}]$. We are thus effectively supposing either that $q(t)$ is directly observable (when q are market prices) or that $m(t)$ is calculated simultaneously with $q(t)$ (when q or τ are policy parameters). A *policy adjustment process is accessible* if $m(t)$ is accessible and $q(t)$ and $\gamma(t)$ obey (3.3) and (3.4). Finally, we say that $x(m)$ is *monotonic* if $x(m)$ is continuously differentiable and if, for $k = 1, ..., \ell_1$, either $Dx_k(m) > 0$ for all m or $x_k(m)$ is a constant function. The latter possibility lets one of the consumption goods be a money or credit.

The $x(m)$ function is meant to capture the effect of macro expansion. If the government distributes only a money good (rather than all goods it owns), $x(m)$ can be seen as the effect on aggregate demand of a tax cut (funded by helicopter drop of money rather than debt). Furthermore, since $x(m)$ reports the effect of m on demand at a single price vector, accessibility presupposes only the type of information that governments routinely assemble.

PROPOSITION 3. *Suppose the status quo equilibrium $\bar{p}$ is production inefficient and that $x(m)$ is monotonic, and let $(\bar{p}, q', \gamma', m^*)$ be a policy equilibrium. Then there is an accessible policy adjustment process such that if $(m(0), \gamma(0), q(0))$ is sufficiently close to (m^*, γ', q'), $(m(t), \gamma(t), q(t))$ converges to a Pareto-improving, production-efficient equilibrium.*

Proof. Define, for $k = 1, ..., \ell_1$, $\chi_k : R \times R_+ \to R_+$ by $\chi_k(\mu, m(0)) = x_k(m(0) + \mu^2 + c\mu)$. Let $\chi: R \times R_+ \to R_+^{\ell_1}$ be defined by $(\chi_1(\mu, m(0)), ..., \chi_{\ell_1}(\mu, m(0)))$. We apply the gradient method to the problem of maximizing

$$\kappa\mu \text{ s.t. } \chi(\mu, m(0)) - \omega_1 - \sum_{j=1}^{r} (f_j(\gamma_j) - g_j(\gamma_j)) \leqslant 0, \qquad \gamma \geqslant 0, \tag{3.5}$$

with respect to μ and γ, where $c, \kappa \in R_{++}$. If the constraint set of (3.5) is nonempty, the solution occurs at a point (μ^*, γ) where μ^* is the largest of the (two) possible values of μ satisfying $m(0) + \mu^2 + c\mu = m^*$. (Note that under our monotonicity assumption, m^* can only take one value.) When μ^* is well-defined, let $\mu^*(m(0))$ indicate the dependence of μ^* on $m(0)$.

When $m(0)$ is sufficiently close to m^*, the constraint set of (3.5) is nonempty. To see why, note that for all k such that $x_k(m)$ is not a constant function, $D_\mu \chi_k(0, m(0)) = cDx_k(m(0)) > 0$. Hence, when $m(0)$ is near m^*, there will exist a $\mu < 0$ such that $\chi(\mu, m(0)) - \omega_1 - \sum_{j=1}^r (f_j(\gamma'_j) - g_j(\gamma'_j)) \leqslant 0$. Furthermore, at some sequence $m(0)_n \to m^*$ where for all $m(0)_n$ the constraint set is nonempty, $\mu^*(m(0)_n) \to 0$.

Next, we establish the local convexity of the $\chi_k(\cdot, m(0))$ at $\mu = 0$. By the monotonicity assumption, if $x_k(m)$ is not a constant function, then

$$D^2_\mu \chi_k(0, m(0)) = c^2 D^2 x_k(m(0)) + 2Dx_k(m(0)) > 0$$

when $c \in [0, \varepsilon_k]$ for some $\varepsilon_k > 0$. Letting c henceforth be an element $(0, \min\{\varepsilon_1, ..., \varepsilon_{\ell_1}\}]$, we conclude that there is a open set containing $\mu = 0$ such that, for all k, χ_k is either a constant or strictly convex function of μ.

Let $(m(0), q(0))_n$ be an arbitrary sequence converging to (m^*, q') and set $\kappa_n = q(0)_n \cdot D_\mu \chi(0, m(0)_n)$. Given the concavity/convexity of the f_j and g_j and the convexity of $\chi_k(\cdot, m(0))$ in a neighborhood of 0 when $m(0)$ is near m^*, it follows that for all n sufficiently large, there is a unique $\lambda_n > 0$ such that $(\mu^*(m(0)_n), \gamma', \lambda_n q')$ is a local saddle-point of

$$\mathscr{L}_n(\mu, \gamma; q) = \kappa_n \mu + q \cdot \left(\sum_{j=1}^r (f_j(\gamma_j) - g_j(\gamma_j)) - \chi(\mu, m(0)_n) + \omega_1 \right).$$

Since $\mu^*(m(0)_n) \to 0$, it is easy to confirm that $\lambda_n \to 1$. Letting $d(x, y)$ denote Euclidean distance between x and y, define $\delta(m(0), \gamma(0), q(0)) = d((\mu(0) = 0, \gamma(0), q(0))_n, (\mu^*(m(0)_n), \gamma', \lambda_n q'))$. Hence, if $(m(0)_n, \gamma(0)_n, q(0)_n) \to (m^*, \gamma', q')$, $\delta(m(0)_n, \gamma(0)_n, q(0)_n) \to 0$.

To simplify notation, we henceforth let the dependence of κ, the χ_k, and μ^* on $m(0)$ and $q(0)$ be implicit. Set $(m(0), \gamma(0), q(0))$ sufficiently near to (m^*, γ', q') so that the non-constant χ_k are all strictly convex at any μ within $2\delta(m(0), \gamma(0), q(0))$ of 0 and so that $q(0) > 0$. Given the strict concavity/convexity of the non-constant $f_{j,k}$ and $g_{j,k}$, the gradient process defined by $\mu(0) = 0$, (3.3) and

$$\dot{q}_k(t) = \delta_{q_k}[\chi_k(\mu(t)) - \omega_1(k) - \sum_{j=1}^r f_{j,k}(\gamma_j(t)) + \sum_{j=1}^r g_{j,k}(\gamma_j(t))], \quad k = 1, ..., \ell_1 \tag{3.6}$$

$$\dot{\mu}(t) = \kappa - q(t) \cdot D\chi(\mu(t)) \tag{3.7}$$

converges to some local saddle-point (μ^*, γ^*, q^*) of

$$\mathscr{L}(\mu, \gamma; q) = \kappa\mu + q \cdot \left(\sum_{j=1}^{r} (f_j(\gamma_j) - g_j(\gamma_j)) - \chi(\mu) + \omega_1 \right). \tag{3.8}$$

Since the Euclidean distance to any local saddle-point is nonincreasing, our choice of $(m(0), \gamma(0), q(0))$ implies that $\mu(t)$ will not leave the domain on which the non-constant χ_k are strictly convex. Note that $(\mu(t), \gamma(t), q(t))$ need not converge to (μ^*, γ', q'), but any local saddle-point of (3.8) solves problem (3.5) and therefore $(\bar{p}, q^*, \gamma^*, m^*)$ is efficient and Pareto improving.

Set $m(t) = m(0) + \mu(t)^2 + c\mu(t)$. The convergence of $(\mu(t), \gamma(t), q(t))$ to (μ^*, γ^*, q^*) guarantees the convergence of $(m(t), \gamma(t), q(t))$ to (m^*, γ^*, q^*). To confirm that the $m(t)$ defined in this way is accessible, note that

$$\dot{m}(t) = \kappa(2\mu(t) + c) - (2\mu(t) + c)^2\, q(t) \cdot Dx(m(0) + \mu(t)^2 + c\mu(t))$$

is a function only of the mapping $x(m)$ and the values of $\mu(t)$, $q(t)$, c, and κ. Furthermore, c was determined only by the properties of the mapping $x(m)$, κ only by $m(0)$, $q(0)$, and c; and, since locally $\mu(t)$ is a 1-to-1 function of $m(t)$, $\mu(t)$ can be calculated from the time path of $m(t)$ from 0 to t. Finally, by setting (3.7) equal to 0 it is clear, given the definition of κ and that $q(0) > 0$, that $q^* > 0$. ∎

A global extension of Proposition 3 faces a difficulty: it may be impossible to define a variable, comparable to μ in the proof, in terms of which commodity demands are globally convex. And even if commodity demands can be made convex functions of some constructed variable on arbitrarily large sets, the policymaker must additionally know upper bounds of the equilibrium values of m and γ and of $(m(0), q(0), \gamma(0))$.

4. DISCUSSION

It is illuminating to compare the current model to the classical theory of optimal taxation and with Diamond and Mirrlees [3] in particular. Similarly to this paper, Diamond and Mirrlees use consumer and producer price vectors as policy tools and derive the policies generated by the maximization of a social welfare function subject to feasibility constraints. The key difference is not that we study Pareto improvements rather than social welfare maxima; one can find Pareto improvements in Diamond and Mirrlees by maximizing an appropriately weighted Leontief social welfare function. The distinctive feature of price-stabilized policies is that Pareto improvements are attained with little knowledge of agent characteristics.

Diamond and Mirrlees, in contrast, use knowledge of the aggregate excess demand function and the production set to determine the set of feasible policies, and knowledge of individual utility functions to calculate social welfare as a function of consumer prices.

In special cases there can be combinations of producer and consumer prices that Pareto dominate all price-stabilized policies; in such cases, of course, Diamond and Mirrlees cannot recommend price-stabilized policies. The reason that this paper does not recommend the Pareto-superior policies is that they would involve relative price changes that *might* harm some individual; we do not presuppose that the policymaker has the information necessary to distinguish cases in which a change in the relative consumption good prices harms some agent from cases in which no agent is harmed.

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