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Sequential regularity in smooth production economies

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Abstract

In general equilibrium models where agents trade sequentially in multiple periods, the equilibria that endogenously occur in later periods can be robustly indeterminate if production sets are not everywhere differentiable (as, for example, with linear activities). The present paper proves that if technology is smooth, then equilibria are *sequentially regular*. That is, the equilibria of the endogenously generated economies occurring in later periods that confirm the unanimous expectations formed by agents in earlier periods are regular (and thus, for example, isolated from other equilibria). The paper also proves and utilizes the standard result that the overall intertemporal equilibria are regular. © 1997 Elsevier Science B.V.

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1. Introduction

The standard approach to studying the determinacy of the general equilibrium model recognizes that although indeterminacy can occur for certain parameters, equilibria are at least generically locally unique (see, for example, Mas-Colell, 1985, for an overview). Among the parameters of these models, however, are factor endowments, variables that are generated endogenously by past processes of

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capital accumulation rather than determined exogenously. As a consequence, theorems stating that the set of parameters exhibiting indeterminacy has measure zero leave open the possibility that certain endowment points, despite being contained in a measure-zero set, arise systematically. To address this complication, we consider a multiple-period general equilibrium model and interpret the intertemporal equilibria as occurring through a sequence of trading at different dates. In a two-period model, for example, a sequential interpretation of equilibrium has agents in the first period unanimously anticipating a second-period price vector and, on the basis of these expectations, making first-period consumption choices and contracting for certain goods ('real assets') to be delivered in the second period. In the second period, agents deliver or receive the goods they contracted for in the first period and trade again to arrive at their second-period consumption choices.

Under this interpretation, first-period trading generates a second-period economy, thus permitting a formal study of the determinacy question with endowments endogenously determined. It is natural to speculate that, generically in the parameters of the multiple-period economy, the economies that arise in the second-period are regular and, in particular, have a finite number of equilibria. This conclusion is not correct. With linear activities technologies – the reference model for this literature (see Kehoe, 1980, and Mas-Colell, 1985) – and inelastically supplied factors of production, Mandler (1995) showed that there are non-empty open sets of parameters such that the resulting second-period economies typically have indeterminate equilibria. Indeed, it is precisely the equilibria of the second-period economies that 'continue' the intertemporal equilibrium (i.e. that confirm the prices unanimously foreseen in the first period) that can be arbitrarily close to other equilibria. We call this phenomenon *sequential indeterminacy*. In contrast, a model is *sequentially regular* if almost all of the second-period economies generated by the intertemporal equilibria of the model are themselves regular. We show that if the technology of an economy is described by a sufficiently smooth transformation function, then sequential indeterminacy does not arise robustly. More precisely, the set of economies that are both regular and sequentially regular is open and dense in our parameter space. Naturally, we allow the model to be rich enough to incorporate the inelastically supplied factors that are crucial for the appearance of sequential indeterminacy in the linear activities case.

What is the significance of the fact that sequential determinacy occurs only with certain types of technology? A central triumph of the Arrow–Debreu approach to equilibrium theory is its generality. A remarkable diversity of models can be incorporated into the framework; in particular, restrictions on the nature of technology are minimal. Earlier neoclassical descriptions of technology using differentiable production functions have always been criticized on the grounds that marginal products were ill-defined and that some form of fixed coefficients were more accurate empirically (see Pareto, 1897, sections 714 and 717, for an early example). The only important restriction on technology in the Arrow–Debreu theory, however, is that production sets are convex. Production sets can therefore be of the linear activity analysis variety (as in Koopmans, 1951), which many regard as a basic description of technology. Subsequently, the theory of regular economies also incorporated the activity analysis model of production. The Arrow–Debreu criticism of earlier versions of equilibrium analysis therefore appears complete: differentiable technologies are not only superfluous for existence and optimality but for determinacy as well. The earlier paper (Mandler, 1995) showed that endowments associated with indeterminacy can in fact arise routinely when the economy is placed in an intertemporal setting and there are linear activities. The present paper in contrast shows that with differentiable technologies later-period indeterminacy essentially disappears. To have a sensible model of production, therefore, we have to make just those painful assumptions about technology that the set-theory approach was, in part, designed to avoid. More positively, our results imply that classical marginal productivity theory retains its relevance for economic theory, a position that many outside the Arrow–Debreu tradition never relinquished.

In the next section we introduce a two-period model with a smooth production technology and prove that generically the equilibria are locally unique. Our approach to regularity is distinctive and simple in that the first-order conditions of firms are used directly in the definition of regularity; the proof of generic regularity is correspondingly straightforward. Naturally, the restriction to smooth technologies is a limitation compared with the most general models of regular production economies (see, for example, Mas-Colell, 1975). However, even given our focus on smooth technologies the set-up of the present model does not strive for maximum generality. For notational convenience, we require that the inelastically supplied 'factors of production' are the only second-period goods that can be contracted for in the first period. And to keep the proofs shorter, we assume that goods are either produced and have a zero endowment or are not produced and have a strictly positive endowment; this supposition, along with standard boundary conditions, will guarantee the convenience that equilibria always occur in the same coordinate subspace. Given the potential link between inelastically supplied factors and indeterminacy, however, we take some care to allow for such factors.

In Section 3 we introduce the second-period economies generated by a two-period equilibrium and prove that generically the two-period economies are sequentially regular. The proof will use the earlier result on generic regularity.

The concluding discussion in Section 4 begins by addressing a potential paradox. A smooth technology may be closely approximated by a linear activities technology. The smooth economy nevertheless generically has determinate second-period equilibria while the 'nearby' linear activities economy can be robustly indeterminate. We use an example to explain how these results are consistent. We then indicate how the results of the two-period model can be extended to a finite number of periods.

2. A two-period model of general equilibrium

2.1. Commodities

There are two time periods, $i = 1, 2$. In each period i, commodities are partitioned into a set of L_i consumption goods and M_i factors of production. The latter are goods that are used in production but do not give any agent utility directly. A price vector for commodities is $(p = (p_1, p_2), w = (w_1, w_2)) \in R_+^{L_1+L_2} \times R_+^{M_1+M_2}$. We use w_2^m to denote the first m coordinates of w_2.

2.2. Production

Production sets will be defined using two transformation functions, g_1: $R^{L_1} \times R^{M_1} \times (R^m \times \{0\}^{M_2-m}) \to R$ and g_2: $R^{L_2} \times R^{M_2} \to R$. $Y_1 = \{(z_1, f_1, f_p) \in R^{L_1} \times R^{M_1} \times (R^m \times \{0\}^{M_2-m})$: $g_1(z_1, f_1, f_p) \leq 0\}$ describes the production processes of the first period, giving the set of possible productions among first-period consumption goods, z_1, first-period factors, f_1, and an m-dimensional coordinate subspace of second-period factors, f_p. We assume that $m > 0$. $Y_2 = \{(z_2, f_2) \in R^{L_2} \times R^{M_2}$: $g_2(z_2, f_2) \leq 0\}$ gives the productions that are possible among second-period consumption goods, z_2, and second-period factors, f_2. Factors are the only second-period goods produced by production processes using first-period inputs. This assumption is only for simplicity and could be generalized to incorporate second-period consumption goods being produced directly by first-period inputs; alternatively, although some of our assumptions are not natural for this interpretation, we may think of second-period consumption goods as being produced indirectly through the creation of second-period factors which in turn produce second-period consumption goods. Although we do not need an explicit assumption, it is natural to suppose that if $f_p^*(j) < f_p(j) \leq 0$ for any $j \in \{1, \ldots, m\}$ and if $g_1(z_1, f_1, f_p) = 0$, then $g_1(z_1, f_1, f_p^*) \geq 0$, i.e. second-period factors, which can be produced in first-period production processes, cannot be used productively as inputs in those processes.

We use the following assumptions on the g_i.

Assumption A1. The functions g_1 and g_2 are convex, and have the following properties: (i) If the sequence $(z_1, f_1, f_p)_n$ is such that $z_1(j)_n \to \infty$ for some $j \in \{1, \ldots, L_1\}$ or $f_p(j)_n \to \infty$ for some $j \in \{1, \ldots, m\}$, then $g_1((z_1, f_1, f_p)_n) > 0$ for all n sufficiently large; if $(z_2, f_2)_n$ is such that $z_2(j)_n \to \infty$ for some $j \in \{1, \ldots, L_2\}$, then $g_2((z_2, f_2)_n) > 0$ for all n sufficiently large. (ii) For all $\lambda \in R_+$, $g_1(z_1, f_1, f_p) = 0 \Rightarrow g_1(\lambda z_1, \lambda f_1, \lambda f_p) = 0$ and $g_2(z_2, f_2) = 0 \Rightarrow g_2(\lambda z_2, \lambda f_2) = 0$.

Assumption A2. If (z_1, f_1, f_p) is such that $f_1(j) \geq 0$, for any $j \in \{1, \dots, M_1\}$, then either $g_1(z_1, f_1, f_p) > 0$ or $(z_1, f_p) = 0$. Similarly, if (z_2, f_2) is such that $f_2(j) \geq 0$, for any $j \in \{1, \dots, M_2\}$, then either $g_2(z_2, f_2) > 0$ or $z_2 = 0$.

Assumption A3. There exists a (z_1, f_1, f_p) with $(z_1, f_p) \gg 0$ such that $g_1(z_1, f_1, f_p) \leq 0$, and a (z_2, f_2) with $z_2 \gg 0$ such that $g_2(z_2, f_2) \leq 0$.

Assumption A4. At points $(z_1 \gg 0, f_1 \ll 0, f_p \gg 0)$ and $(z_2 \gg 0, f_2 \ll 0)$ in their domains: (i) g_1 and g_2 are C^2. (ii) If S_1 is an arbitrary $n_1 < L_1 + M_1 + m$ dimensional coordinate subspace of $R^{L_1} \times R^{M_1} \times (R^m \times \{0\}^{M_2-m})$ and $y \in S_1 \setminus \{0\}$, then $y \cdot Dg_1 = 0$ implies $y' D^2 g_1 y > 0$; if S_2 is an arbitrary $n_2 < L_2 + M_2$ dimensional coordinate subspace of $R^{L_2+M_2}$ and $y \in S_2 \setminus \{0\}$, then $y \cdot Dg_2 = 0$ implies $y' D^2 g_2 y > 0$. (iii) $Dg_1 \gg 0$ and $Dg_2 \gg 0$.

Assumption A1 is standard. Assumption A2 says that in order for positive outputs to be produced all inputs are required. Along with our other assumptions, A2 guarantees that all equilibria occur in the same coordinate subspace. Assumption A3 says that all consumption goods and produced factors may simultaneously be produced in positive amounts. The second and third parts of Assumption A4, respectively, require that the transformation functions are differentiably strictly quasi-convex and differentiably monotone. There is no technological barrier to any consumption good in either period being an input; we will, however, momentarily impose requirements that, in equilibrium, preclude consumption goods from being used in the aggregate as inputs.

2.3. Agents

Each agent $j \in \{1, \dots, J\}$ has an endowment of first-period and non-produced second-period factors, $\omega^j = (\omega_1^j, \omega_2^j) \in R_{++}^{M_1} \times (\{0\}^m \times R_{++}^{M_2-m})$. A consumption plan for agent j is a vector $x^j = (x_1^j, x_2^j) \in R_+^{L_1+L_2}$. Agent j maximizes a utility function $u^j : R_+^{L_1+L_2} \to R$ over the budget set $B^j(p, w) = \{x^j \in R_+^{L_1+L_2} : p \cdot x^j \leq w \cdot \omega^j\}$. The following two assumptions on preferences are standard.

Assumption A5. For all j and at all $x^j \in R_{++}^{L_1+L_2}$, u^j is C^2. For $x^j \in R_{++}^{L_1+L_2}$, $Du^j(x^j) \gg 0$ and $D^2 u^j(x^j)$ is negative definite.

Assumption A6: For all j and all $\hat{x}^j \in R_{++}^{L_1+L_2}$, $\{x^j \in R_+^{L_1+L_2} : u^j(x^j) \geq u^j(\hat{x}^j)\} \subset R_{++}^{L_1+L_2}$.

2.4. Equilibrium

Let $z^j(p, w) = (z_1^j(p, w), z_2^j(p, w)) = \text{argmax } u^j(x^j)$ s.t. $x^j \in B^j(p, w)$. Given that interior indifference curves do not intersect the coordinate axes and the negative definiteness in Assumption A5, $z^j(p, w)$ is C^1 when $(p, w) \gg 0$. The aggregate demand function is $z(p, w) = \sum_{j=1}^{J} z^j(p, w)$. An agent j's excess demand for factors is $-(\omega_1^j, \omega_2^j)$. Aggregate excess demand for factors is $-\omega = -(\omega_1, \omega_2) = -\sum_{j=1}^{J}(\omega_1^j, \omega_2^j)$.

Definition. Let $(z, f) = (z_1, z_2, f_p, f_1, f_2)$. An *equilibrium* for a two-period economy is a (p, w, z, f) such that

$$(z_1, f_1, f_p) \in \text{argmax } p_1 \cdot z_1 + w_2 \cdot f_p + w_1 \cdot f_1 \text{ s.t. } g_1(z_1, f_1, f_p) \le 0,$$

$$(z_2, f_2) \in \text{argmax } p_2 \cdot z_2 + w_2 \cdot f_2 \text{ s.t. } g_2(z_2, f_2) \le 0. \tag{1}$$

$$z(p, w) - z \le 0, \quad -\omega_1 - f_1 \le 0, \quad -(f_p + \omega_2) - f_2 \le 0. \tag{2}$$

Walras' law, namely $p \cdot z(p, w) - w \cdot \omega = 0$, holds for any economy. Along with the fact that in equilibrium $p_1 \cdot z_1 + w_2 \cdot f_p + w_1 \cdot f_1 = 0$ and $p_2 \cdot z_2 + w_2 \cdot f_2 = 0$, Walras' law implies that if an inequality in (2) is strict, then the price of that commodity equals zero. However, owing to Assumption A6, the fact that factors are required in production (Assumption A2), and our monotonicity assumptions (Assumption A4(iii)), we have $(p, w) \gg 0$. Therefore, all of the inequalities in (2) hold with equality. Note also that for any agent j, the equilibrium value of x^j must provide as much utility as any element of $X^j = \{x^j \in R_+^{L_1+L_2} : \exists f_p \text{ s.t. } g_1(x_1^j, -\omega_1^j, f_p) \le 0, g_2(x_2^j, (-f_p - \omega_2^j)) \le 0\}$. Since X^j has a non-empty interior (see Assumption A3), and given Assumption A6 and the boundedness of the set of feasible z (see Assumption A1(i)), there is a $z' \gg 0$ such that any equilibrium value for z satisfies $z \ge z'$. Owing to Assumption A2 and the continuity of the g_i there are vectors $(f_1', f_2') \ll 0$ and $f_p' \gg 0$ such that any equilibrium value for f satisfies $(f_1, f_2, -f_p) \le (f_1', f_2', -f_p')$. We conclude that in equilibrium no coordinate of z or f can equal 0. Finally, owing to Assumption A4(iii), $g_1(z_1, f_1, f_p) = 0$ and $g_2(z_2, f_2) = 0$.

As is standard, given that all prices are positive, the price of one positively priced commodity can be set equal to one without changing the set of equilibrium allocations or equilibrium relative prices. We set the first coordinate of p_2 equal to one. Owing to Walras' law, the fact that prices are positive, and the zero profitability of production, one inequality in (2) is redundant for the determination of equilibrium: we omit the inequality in (2) corresponding to the first coordinate in z_2. We adjust our notation by letting $\bar{p}_2$ and $\bar{p}$ indicate that the first coordinate of p_2 has been set equal to one, and letting $\bar{z}_2$ and $\bar{z}$ indicate that the first coordinate of z_2 has been omitted. Let $\bar{z}_2(\cdot)$ and $\bar{z}(\cdot)$ indicate the corresponding demand functions.

We can now describe the equilibria of an economy as the solution to a system of equations. Given an equilibrium (p, w, z, f), another consequence of our monotonicity assumptions is that there is a (unique) $\lambda_1 \in R_{++}$ such that

$\lambda_1 Dg_1(z_1, f_1, f_p) - (p_1, w_1, w_2^m) = 0$ and a (unique) $\lambda_2 \in R_{++}$ such that $\lambda_2 Dg_2(z_2, f_2) - (\bar{p}_2, w_2) = 0$. We consider the following equations:

$$g_1(z_1, f_1, f_p) = 0 \qquad (1), \tag{3}$$

$$\lambda_1 Dg_1(z_1, f_1, f_p) - (p_1, w_1, w_2^m) = 0 \qquad (L_1 + M_1 + m), \tag{4}$$

$$g_2(z_2, f_2) = 0 \qquad (1), \tag{5}$$

$$\lambda_2 Dg_2(z_2, f_2) - (\bar{p}_2, w_2) = 0 \qquad (L_2 + M_2), \tag{6}$$

$$z(\bar{p}, w) - \bar{z} = 0 \qquad (L_1 + L_2 - 1), \tag{7}$$

$$-\omega_1 - f_1 = 0 \qquad (M_1), \tag{8}$$

$$-(f_p + \omega_2) - f_2 = 0 \qquad (M_2). \tag{9}$$

The number in parentheses on the right indicates the number of equations, while the 'endogenous' variables are $\bar{p}$, w, z, f, and $\lambda \equiv (\lambda_1, \lambda_2)$. The left-hand sides of (3)–(9) therefore define a function, say F, from $R_{++}^{L_1+(L_2-1)} \times R_{++}^{M_1+M_2} \times R_{++}^{L_1+L_2} \times (R_{++}^m \times -R_{++}^{M_1+M_2}) \times R_{++}^2$ to $R^{2L_1+2L_2+2M_1+2M_2+m+1}$. Corresponding to any zero of F is a normalized equilibrium, $(\bar{p}, w, z, f)$, and, conversely, corresponding to any equilibrium (p, w, z, f) is a zero of F, ascertained by normalizing prices and using (4) and (6) to calculate λ. We therefore refer to zeros of F as equilibria. Also, since F is differentiable at a zero, we can say that an equilibrium (p, w, z, f) *induces* a particular value for the square matrix, $DF(\bar{p}, w, z, f, \lambda)$.

Definition. An equilibrium (p, w, z, f) for a (two-period) economy is *regular* if the induced DF is non-singular. An economy is regular if all of its equilibria are regular.

Applying the inverse function theorem to F at a regular equilibrium $(\bar{p}, w, z, f, \lambda)$ indicates that, locally, F has a unique solution at $(\bar{p}, w, z, f, \lambda)$; consequently, equilibrium allocations and relative prices are also locally unique. To see that a regular economy can have only a finite number of normalized equilibria, we recall first that we have already remarked that there is a $z' \gg 0$ such that any equilibrium value for z satisfies $z \geq z'$ and also $(f_1', f_2') \ll 0$ and $f_p' \gg 0$ such that any equilibrium value for f satisfies $(f_1, f_2, -f_p) \leq (f_1', f_2', -f_p')$. Given Assumption A1(i), there is a compact set $C = C_{z_1} \times C_{z_2} \times C_{f_p} \times C_{f_1} \times C_{f_2} \subset R_{++}^{L_1+L_2+m} \times (-R_{++}^{M_1+M_2})$ that contains the set of equilibrium values of $(z_1, z_2, f_p, f_1, f_2)$. Consequently, given Assumption A4(iii), we have that $D_{z_2(1)} g_2(C_{z_2} \times C_{f_2})$ is a compact subset of R_{++}. Hence, the set $L_2 = \{\lambda_2 \in R_+ : \exists (z_2, f_2) \in C_{z_2} \times C_{f_2}$ s.t. $\lambda_2 D_{z_2(1)} g_2(z_2, f_2) = 1\}$ is compact. Furthermore, if λ_2 is restricted to L_2 and (z_2, f_2) is restricted to $C_{z_2} \times C_{f_2}$, then the set of

$(\bar{p}_2, w_2)$ that solves (6), say P_2, is compact and a subset of $R_{++}^{L_2-1+M_1}$. We can now reason similarly for the variables λ_1 and (p_1, w_1): letting P_2^m indicate the projection of P_2 onto the m coordinates corresponding to w_2^m, the sets $L_1 = \{\lambda_1 \in R_+ : \exists(z_1, f_1, f_p) \in C_{z_1} \times C_{f_1} \times C_{f_p}$ and $w_2^m \in P_2^m$ s.t. $\lambda_1 D_{f_p} g_1(z_1, f_1, f_p) = w_2^m\}$ and $P_1 = \{(p_1, w_1) \in R_+^{L_1+M_1} : \exists \lambda_1 \in L_1$ and $(z_2, f_2, f_p) \in C_{z_1} \times C_{f_1} \times C_{f_p}$ s.t. $\lambda_1 Dg_1(z_1, f_1, f_p) - (p_1, w_1, w_2^m) = 0\}$ are compact and $P_1 \subset R_{++}^{L_1+M_1}$. Since F is continuous on the compact set $P_1 \times P_2 \times C \times L_1 \times L_2$, a regular economy has a finite number of normalized equilibria.

2.5. *The set of perturbations*

Technology in the first period is parameterized by specifying a set of admissible first-period transformation functions. Given a fixed g_1 that meets our assumptions, we suppose, for each α in some open, bounded set $A \subset R_{++}^m$, that $g_1^\alpha(z_1, f_1, f_p) = g_1(z_1, f_1, \alpha_1 f_p(1), \ldots, \alpha_m f_p(m))$ is admissible.

To parameterize preferences we construct a set U^i of acceptable utilities for some $i \in \{1, \ldots, J\}$. First, given a symmetric matrix $H \in R^{(L_1+L_2)(L_1+L_2+1)/2}$ and an $h \in R^{L_1+L_2}$, let $h \cdot x^i + (x^i)' H x^i$ be denoted $f(H, h)$. Giving (H, h) the Euclidean norm, let $\mathscr{F}$ be the set of all $f(H, h)$ such that $\|(H, h)\| < b$, for some fixed $b \in R_{++}$. Given a fixed utility function $\bar{u}^i$ that meets the assumptions imposed above, we define U^i by the requirement that $u^i \in U^i$ if and only if there is some $f(H, h) \in \mathscr{F}$ such that $u^i = \bar{u}^i + f(H, h)$. As necessary, we will indicate the functional dependence of i's demand and aggregate demand on u^i.

The utilities for i introduced by these perturbations may violate Assumption A5 or A6. If the perturbations are in sufficiently small sets, however, then we can assume, using arguments similar to the reasoning used above to conclude that the unperturbed model has a finite number of equilibria, that, for all perturbations, all equilibrium values of (z, f), $(\bar{p}, w)$, and λ are contained in compact sets that do not intersect the coordinate axes. We label these sets K, P, and Λ, respectively. It is then not difficult to show, again requiring the sets of perturbations to be in smaller sets if necessary, that, for all $(\bar{p}, w) \in P$ and $u^i \in U^i$, $z^i(p, w, u^i)$ and therefore $z(p, w, u^i)$ are C^1. In the following definition the set of perturbations is to be understood as obeying these restrictions.

Definition. The set of perturbations is $\mathscr{E} = A \times U^i$, and a (two-period) economy is a $(\alpha, u^i) \in \mathscr{E}$.

We observe that our earlier argument that regular economies have a finite number of equilibria can be extended to any regular economy in the set of perturbations by restricting $(\bar{p}, w, z, f, \lambda)$ to the compact set $P \times K \times \Lambda$.

Theorem 2.1. The set of regular two-period economies is open and dense in the set of perturbations.

Proof. Given the compactness of $P \times K \times \Lambda$, openness is standard. To prove density, let G: $P \times K \times \Lambda \times \mathscr{E} \to R^{2L_1+2L_2+2M_1+3M_2+1}$ be defined as $F(\bar{p}, w, z, f, \lambda)$ for the economy $(\alpha, u^i) \in \mathscr{E}$. By the transversality theorem (see, for example, Guillemin and Pollack, 1974), to show that there is a dense set of parameters such that $DF(\bar{p}, w, z, f, \lambda)$ is non-singular if $F(\bar{p}, w, z, f, \lambda) = 0$, it is sufficient to show that $DG(\bar{p}, w, z, f, \lambda, \alpha, u^i)$ has full row rank if $G(\bar{p}, w, z, f, \lambda, \alpha, h) = 0$. (Recall that h is the linear term in the perturbations of u^i.)

We observe first, given that $(\lambda_1, f_p) \gg 0$ at equilibrium and that $\alpha \gg 0$, that Assumption A4(ii) and (iii) imply that the matrix $B =$

$$\begin{bmatrix} D_{z_1} g_1' & f_p(1) D_{f_p(1)} g_1 & \cdot & f_p(m) D_{f_p(m)} g_1 & 0 \\ \lambda_1 D^2_{z_1 z_1} g_1 & \lambda_1 f_p(1) D^2_{f_p(1) z_1} g_1 & \cdot & \lambda_1 f_p(m) D^2_{f_p(m) z_1} g_1 & D_{z_1} g_1 \\ \lambda_1 \alpha_1 D^2_{z_1 f_p(1)} g_1 & \lambda_1 \left(D_{f_p(1)} g_1 + \alpha_1 f_p(1) D^2_{f_p(1) f_p(1)} g_1 \right) & & \lambda_1 \alpha_1 f_p(m) D^2_{f_p(m) f_p(1)} g_1 & \alpha_1 D_{f_p(1)} g_1 \\ \vdots & & \ddots & & \vdots \\ \lambda_1 \alpha_m D^2_{z_1 f_p(m)} g_1 & \lambda_1 \alpha_m f_p(1) D^2_{f_p(1) f_p(m)} g_1 & & \lambda_1 \left(D_{f_p(m)} g_1 + \alpha_m f_p(m) D^2_{f_p(m) f_p(m)} g_1 \right) & \alpha_m D_{f_p(m)} g_1 \end{bmatrix}$$

is nonsingular. If we represent G as $(\bar{z}(\bar{p}, w, h) - \bar{z}, -\omega_1 - f_1, g_1^{\alpha}(z_1, f_1, f_p), \lambda_1 D_{z_1} g_1^{\alpha}(z_1, f_1, f_p) - p_1, \lambda_1 D_{f_p(j)} g_1^{\alpha}(z_1, f_1, f_p)\alpha_j - w_2^m(j), j = 1, \ldots, m, \lambda_1 D_{f_1} g_1^{\alpha}(z_1, f_1, f_p) - w_1, -(f_p + \omega_2) - f_2, g_2(z_2, f_2), \lambda_2 D_{z_2} g_2(z_2, f_2) - \bar{p}_2, \lambda_2 D_{f_2} g_2(z_2, f_2) - w_2)$, then B is the matrix of derivatives of the third, fourth, and fifth blocks of entries in G with respect to (z_1, α, λ_1). Consequently, the following subset of the columns of DG,

$$\begin{array}{cccccccc} h & f_1 & (z_1, \alpha, \lambda_1) & w_1 & f_2 & z_2 & \lambda_2 & w_2 \end{array}$$
$$\begin{bmatrix} D_h \bar{z} & 0 & \cdot & \cdot & 0 & \cdot & 0 & \cdot \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & B & 0 & 0 & 0 & 0 & \cdot \\ 0 & \cdot & \cdot & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & D_{z_2} g_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \lambda_2 D^2_{z_2 z_2} g_2 & D_{z_2} g_2 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & -I \end{bmatrix},$$

has full row rank when $G(\cdot) = 0$, and the proof is complete, if $D_h \bar{z}(\bar{p}, w, h)$ has rank $L_1 + L_2 - 1$ when $(\bar{p}, w)$ is an equilibrium price vector. To see that this is the case, an application of the implicit function theorem to the first-order conditions of agent i's optimization problem indicates that $D_h z^i(\bar{p}, w, h)$ is equal to the upper left $L_1 + L_2$ rows and columns of

$$\begin{bmatrix} D^2 u^i & \bar{p} \\ \bar{p}' & 0 \end{bmatrix}^{-1}.$$

It therefore follows from standard results in consumer theory (see, for example, Barten and Böhm, 1982), and easy to confirm directly, that $D_h z^i(\bar{p}, w, h)$ (a multiple of the Slutsky substitution matrix) is such that $D_h z^i(\bar{p}, w, h)\ \bar{p} = 0$ and has rank $L_1 + L_2 - 1$. However, since $\bar{p} \gg 0$ at any $(\bar{p}, w) \in P$, it follows that for any $y \in R^{L_1 + L_2 - 1} \setminus \{0\}$, y cannot be in the null space of $D_h \bar{z}^i(\bar{p}, w, h)$; in other words, $D_h \bar{z}^i(\bar{p}, w, h)\ \ y \neq 0$ for all $y \in R^{L_1 + L_2 - 1} \setminus \{0\}$. Hence $D_h \bar{z}^i(\bar{p}, w, h)$ and therefore $D_h \bar{z}(\bar{p}, w, h)$ have rank $L_1 + L_2 - 1$, as claimed. □

3. Sequential regularity

We now define the second-period economies generated by a given equilibrium, $(\bar{p}^*, w^*, z^*, f^*)$, of one of the two-period economies described in Section 2. We first describe how $(\bar{p}^*, w^*, z^*, f^*)$ can be interpreted as occurring through a sequence of trading in consecutive periods. For a more detailed discussion, see Mandler (1995). Each agent j has the same endowments and preferences as described in the previous section; now, however, each j transfers wealth across the two periods through the purchase or sale of a vector of produced factors, $k^j \in R^m \times \{0\}^{M_2 - m}$, to be delivered at the beginning of the second period. For simplicity (and in accord with a measure of realism), we allow agents to contract only for the delivery of producible factors and not for consumption goods and non-produced factors. Each agent faces a pair of budget constraints $p_1^* \cdot x_1^j + q \cdot k^j \leq w_1^* \cdot \omega_1^j$ in the first period and $\bar{p}_2^* \cdot x_2^j \leq w_2^* \cdot (k^j + \omega_2^j)$ in the second period, where $q \in R^{M_2}$ are the first-period prices for the delivery of second-period factors. In the first period, $\bar{p}_2^*$ and w_2^* are to be thought of as (unanimously) anticipated prices.

Setting q equal w_2^*, the two budget constraints that an agent now faces reduce to the single budget constraint of the previous section. Consequently the consumption choices of any agent will be identical to the choices made in the two-period economy's equilibrium. Furthermore, k^j will be chosen so that $\bar{p}_2^* \cdot z_2^j(\bar{p}, w) = w_2^* \cdot (k^j + \omega_2^j)$: each j saves or borrows in an amount such that the value of j's second-period endowment will equal the value of the second-period consumption bundle specified by the two-period equilibrium. In addition, in order for $(\bar{p}_2^*, w_2^*, z_2^*, f_2^*)$ to be an equilibrium for the economy that appears in the second period, we must have $\sum_{j=1}^J k^j = f_p^*$. Note that at the beginning of the second period all the characteristics of a well-defined economy are specified. Each agent j owns an endowment equal to $k^j + \omega_2^j$ and a utility function, u^j: $R_+^{L_2} \rightarrow R$, given by $u^j(x_1^j, \cdot)$, where $x_1^j = z_1^j(\bar{p}^*, w^*)$. The production set is Y_2.

The above discussion indicates how an arbitrary two-period equilibrium can be instituted through trading during each period and how a second-period economy is generated. There is a technical difficulty, however, owing to the fact that agents care only about the value of their endowment in the second period, not the particular vector of factors that comprise their endowment. Concretely, a unique $k=(k^j)_{j=1}^J$ need not be specified by the requirements

$$\bar{p}_2^* \cdot z_2^j(\bar{p}^*, w^*) = w_2^* \cdot (k^j + \omega_2^j), \quad j \in \{1, \ldots, J\},$$
$$\sum_{j=1}^{J} k^j = f_p^*. \tag{10}$$

In fact, considering that $p_2^* \cdot z_2^* = w_2^* \cdot f_2^*$, it is easy to confirm that the set of k that satisfies (10), which we label $\mathscr{E}_2$, is a $(J-1)(m-1)$-dimensional affine subspace in R^{Jm}. Thus, unless there is either one individual or one produced factor, a single two-period equilibrium will generate an infinite number of possible economies in the second period. We summarize the above discussion with the following definition.

Definition. An equilibrium $(\bar{p}^*, w^*, z^*, f^*)$ of a two-period economy, $(\alpha, u^i) \in \mathscr{E}$, *generates* a $(J-1)(m-1)$-dimensional set of *second-period* economies, each identified with a *distribution of endowments*, $k \in \mathscr{E}_2$.

When $(J-1)(m-1)>0$, we give the set of second-period economies the Lebesgue measure; when $(J-1)(m-1)=0$, we define the singleton set of second-period economies to have full measure and the empty set to have zero measure.

Given any second-period economy k, each j maximizes $u^j(x_1^j, x_2^j)$ subject to the constraints $p_2 \cdot x_2^j \le w_2 \cdot (k^j + \omega_2^j)$ and $x_1^j = z_1^j(\bar{p}_2^*, w^*)$. In some open set $P^2 \subset R_{++}^{L_2-1+M_2}$ that contains $(\bar{p}_2^*, w^*)$, the solution to each consumer's problem, $z_2^j(p_2, w_2, k^j)$, and the aggregate demand function, $z_2(p_2, w_2, k) = \sum_{j=1}^J z_2^j(p_2, w_2, k^j)$, are C^1 functions of $(\bar{p}_2, w_2)$. Since we are only interested in the regularity and determinacy of equilibria with price vectors near $(\bar{p}_2^*, w^*)$, the discussion that follows should be understood as referring only to $(\bar{p}_2, w_2) \in P^2$.

Definition. A (normalized) equilibrium for a second-period economy is a $(\bar{p}_2, w_2, z_2, f_2)$ such that

$$(z_2, f_2) \in \operatorname{argmax}\ p_2 \cdot z_2 + w_2 \cdot f_2 \text{ s.t. } g_2(z_2, f_2) \le 0. \tag{11}$$

$$\bar{z}_2(\bar{p}_2, w_2, k) - \bar{z}_2 \le 0. \tag{12}$$

$$-\omega_2 - (f_p^* + f_2) \le 0. \tag{13}$$

If $(\bar{p}^*, w^*, z^*, f^*)$ is an equilibrium of the underlying two-period economy, then it is straightforward to confirm that $(\bar{p}_2^*, w_2^*, z_2^*, f_2^*)$ is an equilibrium for the generated second-period economy. We use the term *continuation equilibrium* to refer to $(\bar{p}_2^*, w_2^*, z_2^*, f_2^*)$.

We now turn to regularity of the continuation equilibria of the set of generated second-period economies. Given an equilibrium $(\bar{p}_2, w_2, z_2, f_2)$ of a second-period economy k with $(\bar{p}_2, w_2) \in P_2$, we can describe the nearby equilibria using the following system of equations:

$$g_2(z_2, f_2) = 0 \qquad (1), \tag{14}$$

$$\lambda_2 Dg_2(z_2, f_2) - (\bar{p}_2, w_2) = 0 \qquad (L_2 + M_2), \tag{15}$$

$$-\omega_2 - (f_p^* + f_2) = 0 \qquad (M_2), \tag{16}$$

$$\bar{z}_2(\bar{p}_2, w_2, k) - \bar{z}_2 = 0 \qquad (L_2 - 1). \tag{17}$$

Here the endogenous variables are $(\bar{p}, w_2)$, z_2, f_2, and λ_2; the left-hand side of (14)–(17) therefore defines a C^1 function F_2^k: $P^2 \times R_{++}^{L_2} \times -R_{++}^{M_2} \times R_{++} \to R^{2L_2+2M_2}$. Since, corresponding to any equilibrium $(\bar{p}_2, w_2, z_2, f_2)$, there is unique λ_2 that satisfies (15), we can say that an equilibrium $(\bar{p}_2, w_2, z_2, f_2)$ *induces* a value for the matrix $DF_2^k(\bar{p}_2, w_2, z_2, f_2, \lambda_2)$.

Definition. An equilibrium of a second-period economy is *regular* if the induced $DF_2^k(\bar{p}_2, w_2, z_2, f_2, \lambda_2)$ is non-singular. A two-period equilibrium is *sequentially regular* if the continuation equilibrium of almost every generated second-period economy is regular. A two-period economy is *sequentially regular* if each two-period equilibrium of the economy is sequentially regular.

Theorem 3.1. The set of two-period economies that are both regular and sequentially regular is open and dense in the set of perturbations.

Proof. The proof proceeds in two steps. In Step 1 we show that if the continuation equilibrium of one of the second-period economies generated by a two-period equilibrium is regular, then the continuation equilibrium of almost every second-period economy generated is regular. Step 1 will also verify the openness claim. To prove density, we show in Step 2 that the set of two-period economies, such that each two-period equilibrium generates at least one second-period economy whose continuation equilibrium is regular, is dense in the set of perturbations.
Step 1. Let $\hat{k}$ denote the second-period economy whose continuation equilibrium is regular. If the set of second-period economies contains one element, then our claim is trivially satisfied. If the set of second-period economies contains more than one element, then we parameterize the set by letting each element of the set, except $\hat{k}$, be represented as a $(\eta, \mu) \in \sigma^{(J-1)(m-1)-1} \times R_{++}$, where $\sigma^{(J-1)(m-1)-1}$ is the $(J-1)(m-1)-1$ dimensional sphere with radius 1 and center $\hat{k}$. Each k is identified with the $\eta \in \sigma^{(J-1)(m-1)-1}$ that is an element of the line that contains $\hat{k}$ and k and the $\mu \in R_{++}$ that equals the Euclidean distance between $\hat{k}$ and k. Let k^j: $\sigma^{(J-1)(m-1)-1} \times R_{++} \to R^m$ be the function that indicates the k^j identified by (η, μ).

It is sufficient to show that for every η, $DF_2^k(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular for almost every μ, since then Fubini's theorem implies that, for almost every (η, μ), $DF_2^k(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular. Of course, it then follows that $DF_2^k(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular for almost every k.

First, we express F_2^k as $(-\omega_2 - (f_p^* + f_2), g_2(z_2, f_2), \lambda_2 D_{z_2} g_2(z_2, f_2) - \bar{p}_2, \lambda_2 D_{f_2} g_2(z_2, f_2) - w_2, \bar{z}_2(\bar{p}_2, w_2, k) - z_2)$. We then have,

$$DF_2^k = \begin{array}{c} \begin{array}{ccccc} f_2 & z_2 & \lambda_2 & w_2 & \bar{p}_2 \end{array} \\ \begin{bmatrix} -I & 0 & 0 & 0 & 0 \\ \cdot & D_{z_2} g_2' & 0 & 0 & 0 \\ \cdot & \lambda_2 D^2_{z_2, z_2} g_2 & D_{z_2} g_2 & 0 & -I \\ \cdot & \cdot & \cdot & -I & 0 \\ 0 & -I & 0 & D_{w_2} \bar{z}_2 & D_{\bar{p}_2} \bar{z}_2 \end{bmatrix} \end{array}.$$

Given the assumptions on the utilities and the standard Slutsky decomposition, we have

$$DF_2^k = \begin{bmatrix} -I & 0 & 0 & 0 & 0 \\ \cdot & D_{z_2} g_2' & 0 & 0 & 0 \\ \cdot & \lambda_2 D^2_{z_2, z_2} g_2 & D_{z_2} g_2 & 0 & -I \\ \cdot & \cdot & \cdot & -I & 0 \\ 0 & -I & 0 & \sum_{j=1}^{J} \nu^j \left(k^j + \omega_2^j\right)' & \sum_{j=1}^{J} \left(S^j - \nu^j \bar{z}_2^{j\prime}\right) \end{bmatrix}.$$

where, for each j, S^j is the Slutsky substitution matrix and ν^j is the column vector of income effects. Fixing η, we can view the above matrix as a function, say γ, of μ. We then have $\gamma(\mu) =$

$$\begin{bmatrix} -I & 0 & 0 & 0 & 0 \\ \cdot & D_{z_2} g_2' & 0 & 0 & 0 \\ \cdot & \lambda_2 D^2_{z_2, z_2} g_2 & D_{z_2} g_2 & 0 & -I \\ \cdot & \cdot & \cdot & -I & 0 \\ 0 & -I & 0 & \sum_{j=1}^{J} \nu^j \left(\mu k^j(\eta, 1) + (1-\mu)\hat{k}^j + \omega_2^j\right)' & \sum_{j=1}^{J} \left(S^j - \nu^j \bar{z}_2^{j\prime}\right) \end{bmatrix},$$

and letting $c = \sum_{j=1}^{J} \nu^j (k^j(\eta, 1) - k^j(\eta, 1))'$,

$$\gamma(\mu) = \gamma(0) + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 \end{bmatrix}.$$

We have assumed that $\gamma(0)$ is non-singular. The following lemma shows that $\gamma(\mu)$ is non-singular for almost every μ, which proves Step 1.

Lemma 3.1. Let A and B be $n \times n$ matrices. If there exists a $\mu \in R$ such that $A + \mu B$ is non-singular, then for almost every $\mu \in R$, $A + \mu B$ is non-singular.

Proof of Lemma 3.1. If the function h: $R \to R$ defined by $h(\mu) = \det(A + \mu B)$ is constant, then $h(\mu) \neq 0$ for all μ. If $h(\mu)$ is non-constant, then it is a polynomial of some order between 1 and n, and so for at most n values of μ, $\det(A + \mu B) = 0$. For the remaining values of μ, $A + \mu B$ is non-singular.

To see that the openness conclusion now follows, we consider a regular and sequentially regular economy $(\alpha, u^i)^I$ and an open neighborhood $\mathscr{E}'$ of $(\alpha, u^i)^I$ such that all $(\alpha, u^i) \in \mathscr{E}'$ are regular and where, consequently, the (finite) number of two-period equilibria is constant. For each two-period equilibrium $(\bar{p}^*, w^*, z^*, f^*)$ of $(\alpha, u^i)^I$, $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular for some $\hat{k}$ (indeed for almost every k). By fixing $(J-1)(m-1)$ coordinates of $\hat{k}$, we may vary $\hat{k}$ continuously as a function of $(\alpha, u^i) \in \mathscr{E}'$ if we vary the two-period equilibrium continuously as a function of the parameters. Hence, we can assume that $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ can also be varied continuously in the parameters and therefore there is an open subset $\mathscr{E}'' \subset \mathscr{E}'$ that contains $(\alpha, u^i)^I$ such that $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ remains non-singular. Hence, for all $(\alpha, u^i) \in \mathscr{E}''$, $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular for almost every second-period economy generated by $(\bar{p}^*, w^*, z^*, f^*)$. Since we may construct a $\mathscr{E}''$ in this way for each of the finite number of two-period equilibria, the intersection of the $\mathscr{E}''$ can serve as an open neighborhood within which all equilibria are sequentially regular.

Step 2. As for density, we observe first that by Assumption A4(ii), the upper left 4×4 block of $DF_2^{\hat{k}}$ is non-singular. The following lemma shows that if we can perturb the main diagonal elements of $D_{\bar{p}_2} z_2$, then $DF_2^{\hat{k}}$ is non-singular.

Lemma 3.2. Given a matrix,

$$C = \begin{bmatrix} A & b \\ c & B \end{bmatrix},$$

with B square and A square and non-singular, C is non-singular for almost every choice of the main diagonal elements $B_1, \ldots, B_L$, of B.

Proof of Lemma 3.2. We define L matrices consisting of the first $N+i$, $i = 1, \ldots, L$, rows and columns of C. We consider d_1: $R \to R$ defined by $d_1(B_1) = \det(\text{matrix } i = 1)$. Since $\det(A) \neq 0$, 0 is a regular value of d_1. Hence, by the implicit function theorem, $d_1^{-1}(0)$ is a set of measure 0. We can now proceed by induction. Let d_2: $R \times R \setminus d_1^{-1}(0)$ be defined by $f_2(B_2, B_1) = \det(\text{matrix } i = 2)$. Since 0 is again a regular value of d_2, $d_2^{-1}(0)$ is a set of measure 0. Given that at each stage i only sets of measure 0 are removed from R^i, we are left with a set of full measure in R^L such that $\det(C) \neq 0$.

We now show that the main diagonal elements of $D_{\bar{p}_2} z_2(\bar{p}_2^*, w_2^*, \hat{k})$ can in fact be perturbed without altering the non-diagonal elements or the other entries of $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$. Specifically, given an arbitrary small perturbation of the main diagonal entries in $D_{\bar{p}_2} z_2(\bar{p}_2^*, w_2^*, \hat{k})$, we will show that there is a $\hat{u}^i \in U^i$ that will achieve this perturbation and leave $z^i(\bar{p}^*, w^*)$ unchanged. Moreover, beginning with u^i, as the perturbation of demand becomes small, $\hat{u}^i \to u^i$. With $z^i(\bar{p}^*, w^*)$ unchanged, $(\bar{p}^*, w^*, z^*, f^*)$ remains a two-period equilibrium and $\hat{k}$ remains a second-period economy for $(\bar{p}^*, w^*, z^*, f^*)$, and consequently the other entries of $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ are unchanged. To confirm that such a $\hat{u}^i$ exists, let $\hat{x}^i = z^i(\bar{p}^*, w)$ and recall from consumer theory that

$$\begin{bmatrix} D_{p_2} z_2^i(\bar{p}_2^*, w_2^*, \hat{k}) \\ D_{p_2} \mu(\bar{p}_2^*, w_2^*, \hat{k}) \end{bmatrix} = \begin{bmatrix} Z & \nu \\ \nu' & \phi \end{bmatrix} \begin{bmatrix} \mu I \\ -\hat{x}_2^{i\prime} \end{bmatrix}. \tag{18}$$

where μ is the Lagrange multiplier in i's second-period maximization problem, Z is $L_2 \times L_2$, and

$$\begin{bmatrix} Z & \nu \\ \nu' & \phi \end{bmatrix} = \begin{bmatrix} D^2_{x_2, x_2} u^i & p_2 \\ p_2' & 0 \end{bmatrix}^{-1}. \tag{19}$$

We can therefore arrive at the desired perturbation of $D_{\bar{p}_2} z_2(\bar{p}_2^*, w_2^*, \hat{k})$ if we can perturb $(Z_{22}, \ldots, Z_{L_2 L_2})$ while leaving $z^i(\bar{p}^*, w^*)$, μ, ν, and Z_{ij} (for $i, j = 2, \ldots, L_2$, $i \neq j$) unchanged. To find a $\hat{u}^i$ that meets these requirements, we first complete the construction of a perturbation of Z, say $\hat{Z}$. We already have values for $\hat{Z}_{ij}(i, j = 2, \ldots, L_2)$; we use these values, the equation $\hat{Z}p_2 = 0$, and the requirement that $\hat{Z}$ is symmetric to attain (unique) new values for $\hat{Z}_{11}, \ldots, \hat{Z}_{1L_2}, \hat{Z}_{21}, \ldots, \hat{Z}_{L_2 1}$. Next, we define

$$\begin{bmatrix} V & \pi \\ \pi' & \alpha \end{bmatrix} = \begin{bmatrix} \hat{Z} & \nu \\ \nu' & \phi \end{bmatrix}^{-1}.$$

where V is $L_2 \times L_2$ and where ν and ϕ are kept at the values specified in (18). We have $V\hat{Z} + \pi\nu' = I$ and therefore $V\hat{Z}p_2 + \pi\nu' p_2 = p_2$. Given that we have required $\hat{Z}p_2 = 0$ and that, from standard consumer theory, $\nu' p_2 = 1$, we have $\pi = p_2$. Consequently, $\pi'\hat{Z} + \alpha\nu' = \alpha\nu' = 0$, and therefore $\alpha = 0$. Thus, given (18) and (19), to make the prescribed perturbations it is sufficient that $D^2_{x_2, x_2} \hat{u}^i(\hat{x}^i) = V$ and $D\hat{u}^i(\hat{x}^i) = Du^i(\hat{x}^i)$ hold. The latter equality guarantees that $z^i(\bar{p}^*, w^*)$ remains unchanged. Defining $W = V - D^2_{x_2, x_2} u(\hat{x}^i)$, a $\hat{u}^i$ that will meet these conditions is $u^i(x^i) + (1/2)x_2^{i\prime} W x_2^i - \hat{x}_2^{i\prime} W x_2^i$. (Note that, given the symmetry of

$\hat{Z}$, V and thus W are symmetric.) Taking sufficiently small demand perturbations, W can be chosen to be arbitrarily near the 0 matrix and therefore $\hat{u}^i$ arbitrarily near u^i. For the construction of a similar perturbation argument, see Geanakoplos and Polemarchakis (1986).

The density conclusion is now straightforward. If we wish to find a regular and sequentially regular economy within ε of some arbitrary (α, u^i), we choose an open set $\mathscr{E}'$ that contains only regular economies with precisely n equilibria such that each element of $\mathscr{E}'$ is within ε of (α, u^i). Given an equilibrium $(\bar{p}^*, w^*, z^*, f^*)$ of an arbitrary $(\alpha, u^i)' \in \mathscr{E}'$, the preceding argument implies that we can find a non-empty open set $\mathscr{E}^1 \subset \mathscr{E}'$ such that $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular at all $(\alpha, u^i) \in \mathscr{E}^1$. (As in the argument concluding Step 1, we vary the equilibrium $(\bar{p}^*, w^*, z^*, f^*)$ and $\hat{k}$ continuously in the parameters.) Proceeding sequentially through the other $n-1$ equilibria in $\mathscr{E}'$, we construct a series of non-empty open sets, $\mathscr{E}^2, \ldots, \mathscr{E}^n$, with $\mathscr{E}^{i+1} \subset \mathscr{E}^i$, $i = 1, \ldots, n-1$, where for at least i of the equilibria in $\mathscr{E}^i$, $DF_2^{\hat{k}}(\bar{p}_2^*, w_2^*, z_2^*, f_2^*, \lambda_2^*)$ is non-singular. Given Step 1, any economy in $\mathscr{E}^n$ is sequentially regular. □

4. Discussion

There may seem to be a paradox in that a smooth production technology can be closely approximated by a linear activities technology. Theorem 3.1 shows that, generically, the second-period continuation equilibria in the smooth case are regular, while Mandler (1995) shows that in the linear activities case second-period continuation equilibria can be robustly indeterminate. Thus, an arbitrarily small change in technology can eliminate second-period local uniqueness.

As the following example should make clear, there is no contradiction in this 'discontinuity'. Suppose, in the second period of a two-period model, there is one consumption good and two factors of production. Let $\omega \in R^2_{++}$ indicate the aggregate endowment of the two factors and let the production set be given by $Y = \{(z, f) \in R^3 \colon g(f) + z \leq 0\}$, where $g\colon -R^2_+ \to R$ is convex. The smooth case is described by assuming that g is C^1 on the interior of its domain. Setting the price of the single second-period consumption good equal to 1, equilibrium prices for the factors $w \in R^2_+$ must equal $Dg(-\omega)$. In Fig. 1 the smooth curve labeled S is the isoquant set $\{f \in -R^2_+\colon g(f) = g(-\omega)\}$. The uniqueness of the equilibrium values for w then corresponds to the uniqueness of the hyperplane supporting $-\omega$. In the linear activities case, the transformation function will lead to a piecewise-linear isoquant; the set L in Fig. 1 is an example. Since L has a kink at $-\omega$, there is a multiplicity of supporting hyperplanes at $-\omega$, or, in other words, an indeterminacy of equilibrium factor prices. Note that for *generic* endowments $-\omega$ will not occur at a kink; however, Mandler (1995) shows that when endowments are determined endogenously, second-period endowments at kinks arise robustly.

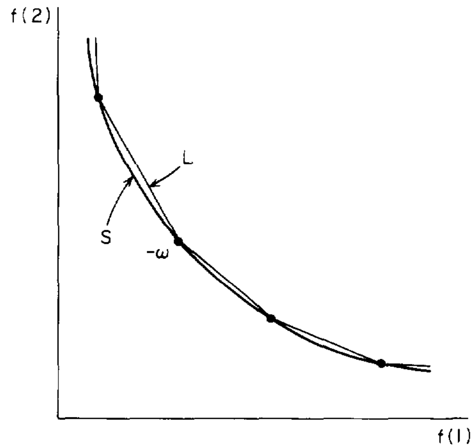

Fig. 1.

In what sense can we think of a sequence linear activities production sets converging to a smooth production set? Since both smooth and piecewise linear transformation functions are elements of the set of continuous functions defined on $-R^2_+$, it is natural to define a sequence of production sets as converging if there exist transformation functions that describe the production sets that converge with the topology of C^0 uniform convergence on compacta. In terms of this topology, a smooth element of the function space may be approximated by a piecewise-linear element. The convergence of a sequence of piecewise-linear g_n to a smooth g can be seen graphically in terms of isoquants. Let each L_n have kinks only at elements of S; then simultaneously increase the number of kinks and let the maximum distance between kinks be well-defined and decrease to 0 as $n \to \infty$. As long as a kink remains at $-\omega$, a one-dimensional set of equilibrium factor prices will occur at each element of the sequence g_n. The fact that a g_n arbitrarily close to a smooth g still need not be everywhere differentiable thus allows indeterminacy to persist. It is clear from Fig. 1, however, that the set of equilibrium values for w along the sequence g_n will converge to $Dg(-\omega)$; in fact, such a convergence result holds in more general contexts.

The two-period results of this paper extend to models with an arbitrary finite number of periods T. To mimic the two-period model of Section 2, suppose that agents in periods $t = 1, \dots, T-1$ can contract for produced factors that are delivered and then used in production in period $t+1$. If parameters that perturb the production of factors in periods before T are included in the model, then a proof of the generic regularity of the T-period equilibria can proceed analogously to the proof of Theorem 2.1. As for sequential regularity, we can associate with each T-period equilibrium the sets of tth-period economies occurring at $t = 2, \dots, T$, and the *continuation* equilibria (which now last $T+1-t$ periods) that confirm the expectations formed in the T-period equilibrium. Similarly to the two-period model, the presence of multiple agents and produced factors of production in some period $t = 2, \ldots, T$ will lead the set of tth-period economies to be multi-dimensional. [1] A T-period equilibrium is sequentially regular if, for each $t = 2, \ldots, T$, the continuation equilibrium of almost every tth-period economy is regular. The proof of generic sequential regularity iterates the argument given for the two-period case. For each $t = 2, \ldots, T$, the existence of a tth-period economy whose continuation equilibrium is regular implies that the continuation equilibrium of almost every tth-period economy is regular. The existence of a tth-period economy (for $t = 2, \ldots, T$) whose continuation equilibrium is regular for a dense set of T-period economies is guaranteed by the perturbation argument. The only substantial difference relative to Step 2 of the proof of Theorem 3.1 is that, for $\tau = t, \ldots, T-1$, a perturbation of the second derivatives of the τth-period transformation function (leaving the first derivatives unchanged) is necessary.

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[1] We are considering tth-period economies to be parameterized only by agents' asset holdings of the produced factors appearing in period t and not by the planned asset holdings in periods $t+1, \ldots, T$. That is, we are considering the intertemporal equilibria of an economy beginning in period t, just as in Section 2 we dealt with the two-period equilibria and did not parameterize by k. The determinacy of equilibrium prices and aggregate quantities of a t-period equilibrium do not depend on the multiplicity of equilibrium asset holdings in periods $t+1, \ldots, T$.