

Rational agents are the quickest*

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Abstract

We consider agents who choose by proceeding through a checklist of criteria (for any pair of alternatives the first criterion that ranks the pair determines the agent's choice). Regardless of the discriminating capacity of the criteria in a checklist, choices that maximize complete and transitive preferences can always be the outcome of a 'quick' checklist that uses a small number of criteria. For any irrational preference on the other hand there is always a discriminatory capacity for criteria such that the preference is not the outcome of a quick checklist. Moreover, if we fix the discriminatory capacity then, as the size of the domain increases, the proportion of irrational preferences that can be the outcome of a quick checklist goes to 0 at a super-exponential rate. We also show that checklists that can be assembled at lowest cost always lead to rational preferences.

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1 Introduction

Consider an agent who makes decisions by proceeding through a sequence of criteria. For any pair of alternatives, the agent checks whether the first criterion C_1 recommends one over the other; if not, the agent proceeds to the second criterion C_2 ; and so on. If no criterion ranks the pair, the agent decides that either one is acceptable. For example, the field of alternatives might be a set of possible meals, C_1 might categorize meals by their meat content (vegetarian, red meat, white meat, all other cases), C_2 by their cuisine (French, Italian, Chinese, all other countries), C_3 by flavor (predominantly salty, sweet, or none of the above). Each criterion will order some of the categories it defines but a criterion need not be transitive – it might cycle for example – and need not be complete. We call a sequence of criteria a *checklist*.

A checklist C will always implicitly maximize a preference relation \succ that we call the *outcome of C* . Conversely, given any preference \succ , there will under mild restrictions be a checklist that leads to \succ . Which preferences can be the outcome of the *quick* checklists that employ the theoretical minimum number of criteria? The agents with these preferences can make choice discriminations with great dispatch: for quick checklists, the ratio of the number of criteria to the number of preference discriminations n rapidly converges to 0 as n increases.

To ensure a fair test, we compare preferences with the same number of indifference classes and give agents access to criteria that share the same capacity to discriminate among alternatives. We show that if \succ is complete and transitive – the preference is rational – there is always a quick checklist that leads to \succ . In contrast, for *any* irrational preference there is a discriminatory capacity for criteria such that the preference cannot be the outcome of a quick checklist: each irrational preference fails to be the outcome of some of the efficient decision-making procedures. So the set of efficient decision-makers always includes the rational agents but, as we vary the discriminatory capacity, each irrational agent will drop out of the set at some point. Even preferences whose only irrationality is incompleteness, which might seem to require fewer decision-making discriminations, will fail to be the outcome of some quick checklist. We arrive at similar results if we judge

the speed of decision-making by the expected number of criteria an agent must examine before making a choice rather than by the total number of criteria in a checklist.

The ‘symmetry’ concept that underlies quick sequential decision-making is as important as the specific results. An agent’s first criterion discriminates among alternatives and thus partitions the domain of alternatives into a set of equivalence classes; the number of equivalence classes in fact defines the discriminatory capacity of a criterion. The equivalence classes of the agent’s second criterion then partitions each of the first criterion’s equivalence classes into finer subsets, and so on. For decision-making to be quick, each criterion must fully exploit its discriminatory capacity, partitioning *every* set of not-yet-discriminated alternatives as finely as possible. A criterion can accomplish this only if it orders every set of not-yet-discriminated alternatives in the ‘same way’, that is, order-isomorphically.

As we will see, there are irrational preferences that happen to have the required order-isomorphic subsets. But these examples are unusual: the proportion of irrational preferences that are the outcome of quick checklists shrinks to 0 at a super-exponential rate as the size of the domain of the alternatives increases. And the examples are fragile: there is no irrational preference that is the outcome of a quick checklist regardless of the discriminatory capacities of the decision-making criteria.

Rational preferences in contrast can always be the outcome of a quick checklist for the simple reason that rational preferences are endowed with the richest possible supply of order-isomorphic subsets: for a rational preference, any two sets of indifference classes of the same cardinality share the same linear ordering. No irrational preference has this flexibility.

This study pursues an agenda that Mandler, Manzini, and Mariotti (2008) (henceforth MMM) lays out but could not fully address. MMM considered checklists with binary criteria that partition alternatives into just two equivalence classes. Any preference that is the outcome of a binary-criteria checklist must be rational and conversely any rational preference will be the outcome of some quick checklist that uses binary criteria. But because binary criteria can never lead to irrational preferences, MMM could not stage a race between rational and irrational agents. Notice that each criterion in our opening

‘meals’ example divides meals into at least three categories, not two; that example can therefore admit irrational preferences.

Our results contrast with Herbert Simon’s (1990) view that utility-maximization places ‘a heavy (often unbearable) computational burden on the decision maker’. If we measure computational burden by the number of checklist criteria, we come to the conclusion that it is the irrational agents who bear the heavier burden. The difficulty with Simon’s and kindred psychological views is that they take utility to exist independently of an agent’s choices rather than being a mere representation of choice behavior; decision-making is difficult because true utility can be hard to discover. If instead we take an agent’s checklist as primitive then rational preferences can simply be the outcome of criteria that are themselves rational. Indeed the most crude but maybe also the most procedurally plausible criteria are the binary ones that divide the domain of alternatives in two, ranking one half better than the other, and binary criteria necessarily generate rational preferences. One of the least burdensome choice procedures imaginable necessarily leads to rationality.

Binary criteria in fact play a special role in the theory of sequential decision-making. Suppose we judge checklists not by the number of criteria they employ but by the number of rankings the criteria make, that is, by the number of decisions an agent has to make to build the checklist. We will see that binary criteria are always the cheapest by this standard: the easiest way to use criteria to make any given number of preference discriminations is to always divide each not-yet-discriminated set of alternatives into equal halves. Since binary criteria must lead to rational preferences, rational preferences are vindicated by this measure of decision-making burden as well. Binary distinctions are a common tool of everyday decision-making; our analysis suggests that they are a sign of efficiency, not mental weakness.

This paper simply assumes that choice behavior arises from a set of criteria, a framework that has recently been the subject of intensive research. See Kalai, Rubinstein, and Spiegel (2002) (henceforth KRS), Apesteguia and Ballester (2009a, 2009b) (henceforth AB), and Manzini and Mariotti (2007). Besides MMM (2008), KRS and AB (2009b) are closest to the present paper in that they assess a set of criteria by the number of criteria in the set (what we call ‘length’). But KRS and AB pursue a different agenda; they seek

the most concise explanations of an agent’s decisions, not the minimum speed with which an agent comes to a decision. The number of equivalence classes in criteria is therefore irrelevant, whereas for our aim of comparing the decision-making efficiency of different agents it is imperative to give every agent access to criteria of equivalent discriminatory power. Also, agents in our model do not use an unordered set of criteria; they proceed lexicographically through a fixed sequence of criteria, where it is the first criterion that ranks x and y that determines the checklist’s overall ranking of x and y , as in Manzini and Mariotti (2007), MMM (2008), AB (2009a), and much earlier the lexicographic utility theory of Chipman (1960, 1971) (see also Fishburn (1974)). Chipman’s framework in fact bears close resemblance to ours, but he let criteria make a continuum of discriminations; our measures of decision-making efficiency then do not apply.

Outside of the sequential-criteria setting of this paper, others have found that rationality has appealing efficiency properties. Tversky and Simonson (1993) adopt a view somewhere between Simon’s and ours; they recognize the dexterity of rational decision-making and are thus forced to find it paradoxical that agents often depart from rationality when they attempt to reduce decision-making complexity. Salant (2003) provides computer-science grounds for the efficiency of rational choice.

2 Criteria and Checklists

Fix a domain of alternatives X . An agent makes decisions by proceeding through a sequence of *criteria* C_1, \dots, C_T , where each criterion C_i is an asymmetric binary relation on X . If there is a criterion in the sequence that ranks a pair in X then the first criterion to do so determines the agent’s choice from the pair. These binary choices define a *preference* \succ which is also an asymmetric binary relation \succ on X . Criteria will typically divide X into a smaller number of equivalence classes (defined precisely in section 3) than preferences do. For a variation on the example in the introduction, X could be the set of all movies and an agent might choose between movie downloads by going through a list of criteria: one criterion might categorize movies by the languages the agent speaks (movies in French, movies in English, all others), a second by the type of movie (thriller, comedy,

science fiction, all others), and so on. Criteria need not be transitive and a criterion C_i need not rank every pair of the equivalence classes that C_i implicitly defines.

The simplest criteria partition X into two equivalence classes, one ranked better than the other, and are called *binary*, as in Example 1 below.

Definition 1 *A checklist is a finite sequence of criteria $C = (C_1, \dots, C_T)$. The preference \succ is the outcome of C if and only if*

- (1) *each C_i is an asymmetric relation on X , and*
- (2) *$x \succ y \Leftrightarrow \exists i$ with $1 \leq i \leq T$ such that $x C_i y$ and not $y C_j x$ for all $j < i$.*

The integer T (the maximum amount of time the checklist can take to rank a pair in X) is the length of C .

We will also say that the checklist ‘ C leads to the preference \succ ’ if \succ is the outcome of C . Definition 1 amounts to a mild generalization of the Chipman (1960) lexicographic utility representation, and of related concepts in set theory, e.g., Cuesta Dutari (1943, 1947). We discuss a closely related concept for choice functions momentarily.

We will remain agnostic whether \succ or C is primitive in Definition 1. In particular, an agent with a checklist C need not think about the preferences he or she is implicitly maximizing though of course we may still deduce \succ from C using (2) above.

Example 1 Some seemingly crude decision procedures while clearly sequential may not at first appear to qualify as checklists. Suppose an agent chooses between two cars x and y by proceeding through a list of ‘properties,’ P_1, P_2, P_3, \dots , where each P_i is a subset of the domain of all cars X . The agent first checks if x and y have property 1 (i.e., if $x \in P_1$ and $y \in P_1$). If only one car has property 1 then the agent chooses that car. Otherwise the agent proceeds to property 2, and so on. Here is a sample of possible properties:

$$\begin{aligned}
 P_1 &= \{\text{all cars that cost less than \$15,000}\} \\
 P_2 &= \{\text{all four-door cars}\} \\
 P_3 &= \{\text{all Japanese cars}\} \\
 &\vdots
 \end{aligned}$$

So this agent first sifts by a price cut-off, then sifts by whether a car is a four-door, then by whether the car is made in Japan, and so on. This list of properties, an example of the

model in MMM (2008), qualifies as a checklist of criteria since the agent could make the same decisions by using a binary criterion C_1 that ranks any car in P_1 as strictly superior to any car in $X \setminus P_1$, a binary C_2 that ranks any car in P_2 as strictly superior to any car in $X \setminus P_2$, and so on. ■

We will see in section 7 that binary-criteria examples such as this have unique efficiency properties.

A checklist C can have a small number of criteria relative to the number of indifference classes in the \succ to which C leads; the checklist then makes choice discriminations quickly. Our program is to see which \succ 's with a given number of indifference classes are the outcome of the quickest checklists.

Checklists can evidently be seen as a way to choose from sets A that consist of two elements. Given $A = \{x, y\}$, the agent proceeds through the checklist until he comes to the first criterion C_i that ranks x and y and then eliminates the C_i -inferior item; if no criterion ranks x and y then the agent deems both to be acceptable. But checklists can also be applied to larger choice sets: if we feed an arbitrary choice set A to the checklist $C = (C_1, \dots, C_T)$ we eliminate in each round i any element of A that has survived so far and that C_i ranks as inferior to some other survivor. We thus define ‘survivor sets’ $S_i(A)$ recursively by

$$S_0(A) = A$$

$$S_i(A) = \{x \in S_{i-1}(A) : \nexists y \in S_{i-1}(A) \text{ with } y C_i x\} \text{ for } i = 1, \dots, T,$$

as in Manzini and Mariotti (2007), AB (2009a), MMM (2008). Suppose c is a choice function on a domain \mathcal{D} of nonempty subsets of X that includes the two-element sets; for every $A \in \mathcal{D}$, $c(A)$ is a nonempty subset of A . We can then define a choice function c to ‘be the outcome of the checklist C ’ if $S_T(A) = c(A)$ for all $A \in \mathcal{D}$. While this setting is the principal interpretation we have in mind for checklists, the extra generality of A 's with more than two elements would be pointless for our purposes. As we will see, a rational preference \succ can be the outcome of the shortest possible checklist; and it will be clear that a choice function defined on arbitrary sets that maximizes \succ can be the outcome

of exactly the same checklist. Thus rational choice functions can be the outcome of the shortest possible checklists. We will point out in section 5 that the converse that holds for preferences also holds for choice functions: *only* the choice functions that maximize a rational preference can be the outcome of the shortest possible checklist regardless of the discriminatory capacity of the criteria.

3 Indifference and equivalence classes

Our first measure of the speed or efficiency of a checklist will be the number of criteria it uses, that is, the checklist's length. The minimum length of a checklist that leads to some \succ is determined in part by the number of indifference classes in \succ and in part by the number of discriminations that checklist criteria make. We need a definition of equivalence classes for asymmetric relations that gives an accurate count for both purposes.

Definition 2 *Given an asymmetric relation $>$ on X , the binary relation \approx on X is defined by*

$$x \approx y \Leftrightarrow \{z \in X : z > x\} = \{z \in X : z > y\} \text{ and } \{z \in X : x > z\} = \{z \in X : y > z\}$$

and is called the equivalence (or indifference) relation of $>$.

It is easy to confirm that \approx is in fact an equivalence relation and thus furnishes an appropriate definition of indifference for preferences. For a textbook discussion of \approx , specialized to the case where \succ is transitive, see Fishburn (1970).

Definition 3 *Given an asymmetric relation $>$ and its equivalence relation \approx , a $>$ -equivalence class is a nonempty $I \subset X$ such that (1) $x, y \in I \Rightarrow x \approx y$ and (2) $(x \in I \text{ and } x \approx y) \Rightarrow y \in I$.*

Given an asymmetric relation $>$ on X , the $>$ -equivalence classes form a partition of X .

When we apply Definitions 2 and 3 to a preference \succ we use \sim to denote the equivalence relation of \succ and call it an *indifference relation*. To keep our terminology clean,

we reserve *equivalence* for equivalence relations of criteria. We assume throughout the remainder of the paper and without further remark that any preference \succ has finitely many indifference classes.¹

To see how the count of criterion equivalence classes works, consider again the ‘meals’ example at the beginning of the paper. In that example, two of the criteria partitioned meals into four categories and one partitioned meals into three categories. For example, for the criterion C_2 that uses the ‘cuisine’ categories – French, Italian, Chinese, and a remainder of all other countries – to form equivalence classes, if x and y are, say, two French meals then $x C_2 y$ must *not* obtain and further C_2 must rank x and y the same way relative to any non-French meal z . The criterion C_2 will then have no more than four equivalence classes (it may have fewer if two categories have identical upper and lower contour sets but this will not occur when C_2 actually ranks each pair of cuisine categories).

In Example 1, each criterion specifies just two equivalence classes, those items in a property and those items excluded, and is therefore binary. Notice that any binary criterion is transitive.

Turning to preferences, a word of explanation is needed on why we use a strict preference \succ to define indifference classes rather than using a weak preference \succeq and letting the symmetric part of \succeq be our definition of indifference. The difficulty with the latter tactic is that a \succeq might generate two or more indifference classes that are unranked by \succeq but that have identical upper and lower contour sets; that is, we might have x and y such that neither $x \succeq y$ nor $y \succeq x$ holds but where

$$\{z \in X : z > x\} = \{z \in X : z > y\} \text{ and } \{z \in X : x > z\} = \{z \in X : y > z\}$$

where $>$ denotes the strict part of \succeq . In such a case, x and y are fully interchangeable from a behavioral point of view. In particular, any checklist that leads to \succeq would have to place x and y in the same $>$ -indifference class. So if we nevertheless labeled x

¹Our framework extends to preferences with an arbitrary number of indifference classes and to checklists that consist of an infinite sequence of criteria but both Observation 2 and Theorem 2 would no longer obtain.

and y as being in separate indifference classes, we would give a checklist that leads to \succsim unwarranted credit for tackling more indifference classes than show up in the agent's behavior. Conversely, \succsim might classify two elements as indifferent when their upper and lower contour sets do differ and hence would have to be distinguished by some criterion.

Definition 4 A discriminatory capacity is a sequence of integers $d = (d_1, d_2, \dots)$ such that each $d_i \geq 2$. The checklist (C_1, \dots, C_T) is a d -checklist if and only if each C_i has d_i or fewer C_i -equivalence classes.

When we say ' C is a (d_1, \dots, d_t) -checklist' we mean that C is a checklist of length t where each C_i has d_i or fewer equivalence classes. We exclude criteria C_i with $d_i = 1$ since they make no discriminations (if $d_i = 1$ then $C_i = \emptyset$). We use special labels for the following checklists: a $(2, 2, \dots)$ -checklist is a *binary checklist* and a $(3, 3, \dots)$ -checklist is a $\vec{3}$ -*checklist*.

4 Checklists that lead to rational and irrational preferences

We use a definition of completeness that applies to asymmetric relations.

Definition 5 An asymmetric relation $>$ on X with equivalence relation \approx is complete if and only if for every $x, y \in X$ either $x (> \cup \approx) y$ or $y (> \cup \approx) x$ or both.

Equivalently, $>$ is complete if and only if for all $>$ -equivalence classes $I \neq J$, either $I > J$ or $J > I$.²

We define an asymmetric relation \succ to be *rational* if and only if \succ is complete and transitive. It is easy to confirm that an asymmetric \succ is complete and transitive if and only if it is negatively transitive (where $x \not\succeq y \not\succeq z$ implies $x \not\succeq z$), the traditional definition of rationality for asymmetric relations.

One way that a rational \succ can arise is when each criterion C_i in a d -checklist $C = (C_1, \dots, C_T)$ is itself rational (complete and transitive) on X . To see this, define $\bar{d} =$

²For any binary relation R on X , any subsets $A, B \subset X$, and any $c \in X$, we use $A R B$ to mean $a R b$ for all $a \in A$ and $b \in B$, $c R A$ to mean $c R a$ for all $a \in A$, and $A R c$ to mean $a R c$ for all $a \in A$.

$\max\{d_1, \dots, d_T\}$ and, for any $x \in X$ and criterion i , let x_i be the number of C_i -equivalence classes J for which $x C_i J$. We can then identify x with the integer whose expansion in base \bar{d} has x_i as its i th digit: call this integer $\text{Int}(x)$. If C is a checklist that leads to \succ , then $x \succ y$ if and only if the first criterion C_i that ranks x and y has $x C_i y$, i.e., the first digit i where $\text{Int}(x)$ and $\text{Int}(y)$ differ in base \bar{d} (reading from the left) has $x_i > y_i$ and so $\text{Int}(x) > \text{Int}(y)$. It is also easy to confirm that $x \sim y$ if and only if $\text{Int}(x) = \text{Int}(y)$. Since $\text{Int}(x)$ may therefore serve as a utility number for any $x \in X$, \succ has a utility representation and is therefore complete and transitive:

Observation 1 *If each criterion in a checklist C is complete and transitive and C leads to \succ , then \succ is complete and transitive.*

So, if we think of the checklist rather than \succ as primitive, we come to a conclusion that differs from Herbert Simon's. It is not difficult for an agent to be rational; rational preferences can merely indicate that an agent uses a checklist with complete and transitive criteria. Indeed, the crudest criteria – the binary criteria that arise from properties where the discriminatory capacity d is $(2, 2, \dots)$ – are automatically complete and transitive. Hence the preferences that are the outcome of binary checklists are necessarily rational; the properties in Example 1 may look primitive, but they lead to choice behavior that Simon took to be demanding.

Since only rational preferences can be the outcome of binary checklists, a checklist that leads to an irrational preference must have a discriminatory capacity d with some coordinates $d_i > 2$. The following observation shows that for an arbitrary irrational \succ we need only go to a d where $d_i \geq 3$ occurs sufficiently often.

Observation 2 *Any preference \succ with n indifference classes can be the outcome of some d -checklist if at least n coordinates of d have $d_i \geq 3$.*

If there is no restriction on the number of d_i 's at least as big as 3, a simple way to build a d -checklist that leads to \succ is just to list the preference rankings in \succ : for any two \succ -indifference classes I and J with $I \succ J$, define a criterion $C_{I,J}$ by $x C_{I,J} y$ if and only if $x \in I$ and $y \in J$, and then form a d -checklist C that assigns the $C_{I,J}$ in any order to the

C_i with $d_i \geq 3$. For the C_j with $d_j = 2$ (when they come before the final $C_{I,J}$ in C), set $C_j = \emptyset$. Such a C evidently leads to \succ and C is a d -checklist since each $C_{I,J}$ has three equivalence classes: I , J , and $X \setminus (I \cup J)$. An alternative and usually shorter construction is to define for each \succ -indifference class I a criterion C_I by $y C_I x$ if and only if $x \in I$ and $y \succ x$, and form C by assigning the C_I to the C_i with $d_i \geq 3$, again arranged in any order. Now we get a d -checklist that leads to \succ where the number of nonnull criteria equals the number of \succ -indifference classes n , thus establishing Observation 2. If \succ is complete, our first construction in contrast uses $\binom{n}{2} = \frac{n(n-1)}{2}$ nonnull criteria. The worst case – where the quickest d -checklist has n nonnull criteria – can in fact occur (see Example 3).

5 The comparison of decision-making speed

It may well be that an irrational agent, overwhelmed by the difficulty of making decisions, makes only a few preference discriminations; such behavior, not surprisingly, can be the outcome of a very short checklist. So, to compare like with like we consider preferences that share the same number of preference indifference classes and let the length of the shortest checklist that can lead to those preferences be the measure of decision-making speed.

To determine this length, it is useful to generalize slightly the standard logarithm: for any discriminatory capacity d , let $\lceil \log_d n \rceil$ denote the smallest integer t such $d_1 d_2 \cdots d_t \geq n$. When each d_i equals the same integer – as with binary checklists – $\lceil \log_d n \rceil$ is indeed the ceiling of $\log_{d_i} n$.³

Theorem 1 *If a d -checklist C leads to a preference with n indifference classes then C has at least $\lceil \log_d n \rceil$ criteria.*

Proof. If a preference with n indifference classes is the outcome of a d -checklist of length T , the Singleton Indifference Class Lemma in the appendix implies there is a d -checklist

³If we define $\log_d n = s + r$ where (s, r) solves $d_1 \cdots d_s d_{s+1}^r = n$ (for $n > 0$, $s \in \{0, 1, \dots\}$, $r \in [0, 1]$, and $d \gg (1, 1, \dots)$), then throughout the paper we can use the standard definitions of $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$. That is, for a real number a , $\lceil a \rceil$ is the smallest integer b such that $b \geq a$ and $\lfloor a \rfloor$ is the largest integer b such that $b \leq a$.

C of length T that leads to a \succ with n singleton indifference classes. We show that $T \geq \lceil \log_d n \rceil$.

Given a d -checklist C' of length k , define a C' -undecided set to be a $A \subset X$ such that $x, y \in A \Rightarrow$ (for each $i = 1, \dots, k$, \exists a C'_i -equivalence class I such that $\{x, y\} \subset I$), define a *maximal C' -undecided set* to be a C' -undecided set A such that no C' -undecided set B has $|B| > |A|$, and finally define the *maximal C' -undecided cardinality* to equal $|A|$ where A is a maximal C' -undecided set.

We show, by induction on k , that if the C that leads to \succ has length k then the maximal C -undecided cardinality is greater than or equal to $(d_1 \cdots d_k)^{-1}n$. This claim holds for $k = 1$ since if every C_i -equivalence class has strictly fewer than $d_1^{-1}n$ elements we would have $|X| < n$. Assuming then that the claim holds for arbitrary k , let $C = (C_1, \dots, C_k, C_{k+1})$ be a d -checklist of length $k + 1$. So there is a (C_1, \dots, C_k) -undecided set A with $|A| \geq (d_1 \cdots d_k)^{-1}n$. Since the d_{k+1} C_{k+1} -equivalence classes, $I_1, \dots, I_{d_{k+1}}$, form a partition of X , the sets $I_1 \cap A, \dots, I_{d_{k+1}} \cap A$ partition A ; hence one of these sets, say $I_j \cap A$, must have at least $(d_1 \cdots d_{k+1})^{-1}n$ elements. And since $I_j \cap A$ is a $(C_1, \dots, C_k, C_{k+1})$ -undecided set, the maximal $(C_1, \dots, C_k, C_{k+1})$ -undecided cardinality is at least $(d_1 \cdots d_{k+1})^{-1}n$.

Since C leads to \succ , the maximal C -undecided cardinality must equal 1. If not there would be distinct $x, y \in X$ and a C_i -equivalence class I_i for each $i = 1, \dots, T$ such that $\{x, y\} \subset I_i$ for $i = 1, \dots, T$; and then, $\forall z \in X$, $(x C_i z \Leftrightarrow y C_i z)$ and $(z C_i x \Leftrightarrow z C_i y)$ which implies that x and y are in the same \succ -indifference class. Hence the previous paragraph gives $1 \geq (d_1 \cdots d_T)^{-1}n$, that is, $d_1 \cdots d_T \geq n$. So $T \geq \lceil \log_d n \rceil$. ■

Theorem 1 leads to the following definition of a quick checklist.

Definition 6 *Given a preference \succ with n indifference classes, define a d -checklist that leads to \succ to be quick if it has $\lceil \log_d n \rceil$ criteria.*

The label ‘quick’ is well-deserved: if we fix d , then, as the number of \succ -indifference classes n increases, $\lceil \log_d n \rceil$ increases at a less-than-polynomial rate. Since the ratio of $\lceil \log_d n \rceil$ to n converges to 0 rapidly as n increases, quick checklists are highly efficient discriminators.

We use Definition 6 to run the contest between rational and irrational agents: which class of preferences is more often the outcome of a quick checklist? This way of posing the question judges preferences by their *potential* decision-making speed; we pin down the

preferences that can be the outcome of the quick decision-making procedures. While any preference can be the outcome of a checklist with arbitrarily many criteria – criteria can make few or no discriminations and a checklist can even pointlessly use the same criterion more than once – the potential discrimination speed of a preference is a clear-cut marker (see also section 8).

Theorem 2 will show that, for any complete and transitive \succ and any discriminatory capacity d , \succ is the outcome of a quick d -checklist: rational agents can potentially make choice discriminations at least as quickly as any other type of agent. The proof of Theorem 2 will establish this formally (and applies to the binary checklists not covered in the statement of the theorem), but an example will be instructive.

Example 2 For $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let \succ be the usual ordering of the integers. The unique quick $\vec{3}$ -checklist (where $\vec{3} = (3, 3, \dots)$) that leads to \succ is then the (C_1, C_2) given by

$$\begin{aligned} &\{9, 8, 7\} C_1 \{6, 5, 4\} C_1 \{3, 2, 1\} \text{ and } \{9, 8, 7\} C_1 \{3, 2, 1\} \\ &\{9, 6, 3\} C_2 \{8, 5, 2\} C_2 \{7, 4, 1\} \text{ and } \{9, 6, 3\} C_2 \{7, 4, 1\}, \end{aligned}$$

which is pictured in Figure 1. Criterion C_1 divides X into a top third, a middle third, and a bottom third, and then ranks these thirds as \succ does, while C_2 groups together the top items from the three C_1 -equivalence classes, the middle items from the three C_1 -equivalence classes, and the bottom three items of the three C_1 -equivalence classes, and then linearly ranks the tops over the middles over the bottoms. ■

Example 2 reverse engineers the construction used for Observation 1 and provides a model, for any rational \succ , of how to build a quick checklist that leads to \succ . One begins by partitioning X into d_1 C_1 -equivalence classes that are ordered by \succ . Then at each subsequent stage i , for each maximal $A \subset X$ that has not yet been divided into distinct criterion equivalence classes, partition A into d_i cells so that these cells are ordered by \succ and group together all the cells across A 's that have the same rank according to \succ to form the C_i -equivalence classes. When the number of \succ -indifference classes exactly equals $d_1 \cdots d_T$ for some integer T , the cells at each stage must contain the same number

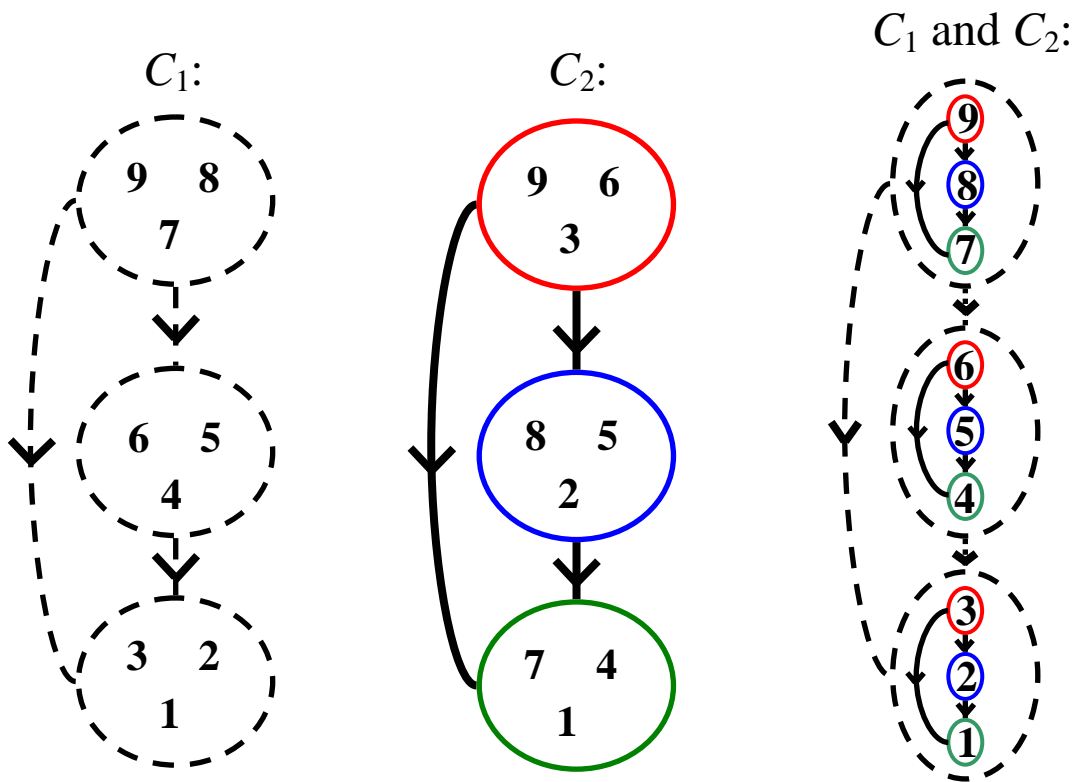


Figure 1: A quick $\bar{3}$ -checklist for a rational γ

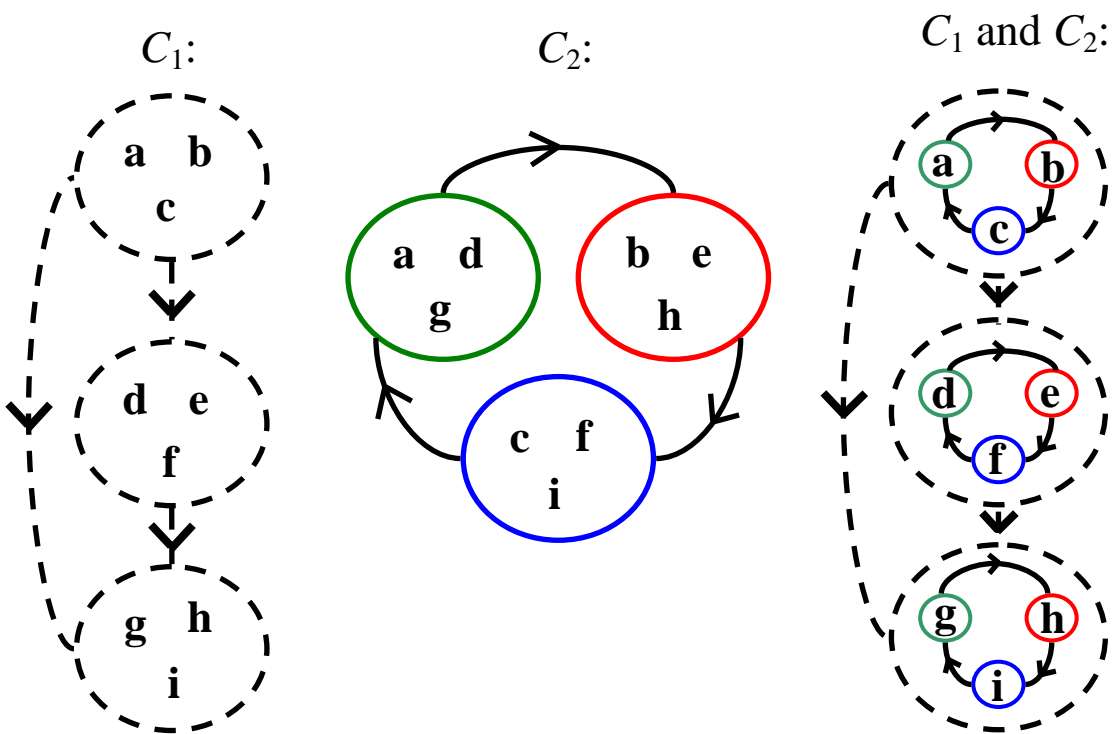


Figure 2: A quick $\bar{3}$ -checklist for an irrational γ

of \succ -indifference classes; in all other cases, there will be some latitude in how to apportion the \succ -indifference classes but one may always partition each A into cells where the number of \succ -indifference classes differs by at most one. See the proofs of Theorems 2 and 6.

The checklists built by this method do not have to be seen as decision-making procedures for agents with preexisting preferences. If there are features of the alternatives that map into a one-dimensional scale, then an agent could just proceed through the features in some sequence, at each stage i dividing X into d_i categories as ordered by feature i 's scale. If X were a domain of cars, as in Example 1, examples of such features could be price, horsepower, prestige, and so forth. An agent who follows this method will build a quick checklist and the preference that is the outcome will turn out to be rational.

The critical feature of any of the preferences \succ that the above checklists lead to, and the key to decision-making speed, is that each subset of not-yet-discriminated alternatives is ‘ordered the same way’ by \succ . Rational preferences always display this ‘symmetry’: for a rational \succ any two sets of \succ -indifference classes with the same cardinality will be order isomorphic.⁴ Consequently a criterion can order any family of sets of \succ -indifference classes in a way that respects \succ . In fact, only rational preferences have such a rich supply of order-isomorphic subsets; the above order-isomorphism property fully characterizes the rational preferences with at least four indifference classes (see Mandler (2009)).

We have seen that any complete and transitive \succ is the outcome of a quick checklist that consists of complete and transitive criteria and, conversely, Observation 1 reported that any sequence of complete and transitive criteria will lead to a \succ that is complete and transitive. To consider the decision-making efficiency of agents who do not end up maximizing a rational preference, we begin with an example of a preference that cannot be the outcome of any quick $\vec{3}$ -checklist (and hence cannot be rational).

Example 3 For $X = \{w, x, y, z\}$, let \succ be defined by $w \succ x \succ y \succ z$ (so \succ is both incomplete and intransitive). Set $d = \vec{3}$. If there were a quick $\vec{3}$ -checklist that leads to \succ , it must have length $2 = \lceil \log_3 4 \rceil$. So then either C_1 makes two out of the three \succ -rankings – that is, C_1 has $w C_1 x C_1 y$ or $x C_1 y C_1 z$ or $(w C_1 x \text{ and } y C_1 z)$ – or C_1 makes one \succ -ranking and C_2 makes the other two. If the criterion C_i that makes two rankings

⁴Two sets, where each is endowed with a binary relation, (A, \geq_A) and (B, \geq_B) , are order-isomorphic if and only if there is a bijection $f : A \rightarrow B$ such that, for all $x, y \in A$, $x \geq_A y \Leftrightarrow f(x) \geq_B f(y)$.

has $w C_i x C_i y$ then the omitted item z must be placed in the same C_i -equivalence class as one of $\{w, x, y\}$ and thus be ranked by C_i vis-à-vis x if joined with w or y or ranked vis-à-vis w if joined with x , contrary to the rankings given by \succ . The C_i that makes two rankings similarly cannot have $x C_i y C_i z$. And finally if C_i has ($w C_i x$ and $y C_i z$) then (since $d_i = 3$) either $\{w, y\} C_i \{x, z\}$ or $w C_i \{x, y\} C_i \{z\}$, which again imposes rankings not given by \succ .

The shortest-length $\overrightarrow{3}$ -checklist that leads to \succ has length 3. Here are two samples:

C defined by $w C_1 x, x C_2 y, y C_3 z$

C' defined by $w C'_1 x, x C'_2 y, y C'_3 z, y C'_3 x$.

For C'_3 , the three C'_3 -equivalence classes are $\{w\}$, $\{x, z\}$, and $\{y\}$.

The above \succ is not quite the worst case for $\overrightarrow{3}$ -checklists. The \succ 's that come in last in the race are the \succ 's with n indifference classes that require a $\overrightarrow{3}$ -checklist of length n . The smallest n where such a \succ arises is 5: let \succ on $\{a, b, c, d, e\}$ be the cycle $a \succ b \succ c \succ d \succ e \succ a$. The argument that there is no $\overrightarrow{3}$ -checklist shorter than length 5 that leads to this \succ is similar to the case above and we leave the details to the reader (see also Mandler (2009)). ■

The moral of Example 3 is that while it may seem that an incomplete \succ should have a speed advantage (the criteria that lead to \succ need to make fewer rankings), incompleteness forces any \succ -unranked pair to be unranked by *every* criterion. This restriction burns up criterion equivalence classes and hence slows down the quickest checklist. In sequential decision-making, the conventional description of incomplete preferences as ‘indecisive’ turns out to be apt.

In some carefully constructed cases, however, irrational preferences can be the outcome of a quick checklist. As we have seen, if a checklist is quick then after the application of the first k criteria each set of alternatives A that has not yet been discriminated by criteria 1 through k must be \succ -order-isomorphic.⁵ While an irrational preference does

⁵More precisely, if C_1, \dots, C_k are the first k criteria then these not-yet-discriminated sets A are the cells of the coarsest common refinement of the partitions $\mathcal{C}_1, \dots, \mathcal{C}_k$ where each \mathcal{C}_i is formed by the C_i -equivalence classes. Furthermore, a quick checklist requires only that each of these cells can be \succ -order-isomorphically embedded in the cell with the largest number of \succ -equivalence classes.

not enjoy as rich a supply of order-isomorphic subsets as a rational preference does, it still might have enough – given a particular discriminatory capacity d – to form a quick d -checklist.

Example 4 Let $X = \{a, b, c, d, e, f, g, h, i\}$ and define \succ by

$$\begin{aligned} \{a, b, c\} \succ \{d, e, f\} \succ \{g, h, i\} \text{ and } \{a, b, c\} \succ \{g, h, i\} \\ a \succ b \succ c \succ a \text{ and } d \succ e \succ f \succ d \text{ and } g \succ h \succ i \succ g. \end{aligned}$$

Thus there are three triples that \succ orders linearly but within each triple \succ orders the items as a cycle. The $\vec{3}$ -checklist (C_1, C_2) defined below and pictured in Figure 2 both leads to \succ and is quick:

$$\begin{aligned} \{a, b, c\} C_1 \{d, e, f\} C_1 \{g, h, i\} \text{ and } \{a, b, c\} C_1 \{g, h, i\} \\ \{a, d, g\} C_2 \{b, e, h\} C_2 \{c, f, i\} C_2 \{a, d, g\}. \quad \blacksquare \end{aligned}$$

The only important difference relative to Example 2 is that C_2 in Example 4 is a cycle rather a line. Both cases share the indispensable feature that the C_1 -equivalence classes are \succ -order isomorphic. We can in fact mimic Examples 2 and 4 to generate all of the \succ 's with nine singleton indifference classes that are the outcome of a quick $\vec{3}$ -checklist. In any such (C_1, C_2) , each C_i must have three C_i -equivalence classes and each C_i -equivalence class must consist of three \succ -indifference classes. Moreover, we must put the three \succ -indifference classes in each C_1 -equivalence class into separate C_2 -equivalence classes. A $\vec{3}$ -checklist that does not meet these requirements would have to put some nonindifferent pair x, y in *both* the same C_1 -equivalence class *and* the same C_2 -equivalence class, and then since x and y would be ranked the same way relative to every other $z \in X$ we would have $x \sim y$ (see the proof of Theorem 1). Now up to an order isomorphism there are only four asymmetric orderings R of three equivalence classes. Labeling the equivalence classes a, b , and c , these are defined by: (1) $a R b R c$ and $a R c$, (2) $a R b R c R a$, (3) $a R b$ and $b R c$, and (4) $a R b$.⁶ By letting C_1 order the C_1 -equivalence classes according to one of the orderings (1) through (4) and letting C_2 order the items in each C_1 -equivalence class according to one of the same orderings (the same ordering for every C_1 -equivalence class),

⁶The orderings defined by $(a R b, c R b)$, $(a R b, a R c)$, and the empty ordering do not qualify since they have 2, 2, and 1 equivalence classes respectively.

we can generate, up to an order isomorphism, all of the preferences \succ with nine singleton indifference classes that are the outcome of quick $\overrightarrow{3}$ -checklists. Evidently there are 16 of which we have seen two: the \succ in Example 4 is the outcome of criterion orderings (1) and (2) and the \succ in Example 2 is the outcome of two copies of ordering (1). Since 15 of these 16 preferences are not rational and since, up to an order isomorphism, there are approximately 400 billion irrational \succ 's with nine singleton indifference classes it is apparently exceedingly unusual event for a \succ in this class to be the outcome of a quick $\overrightarrow{3}$ -checklist.

We further pursue the count of how many irrational \succ 's are the outcome of quick checklists in section 6. Here we pin down the unique flexibility of rational preferences by showing that for any irrational \succ there is always a discriminatory capacity d such that \succ is not the outcome of a quick d -checklist. Since any rational preference can be the outcome of a quick d -checklist for any d , the property of being the outcome of a quick d -checklist for all d 's in fact characterizes rationality. One proviso is necessary however: since only rational preferences can be the outcome of binary checklists, the characterization should not use a binary checklist to show that an irrational preference cannot be quick – otherwise we would just be repeating the substance of Observation 1.

The delicacy of the checklist in Example 4 illustrates the logic of our main theorem. To achieve quickness, the items in each of the three C_1 -equivalence classes must be ordered by \succ in the same way. But the very fact that \succ has three \succ -order-isomorphic subsets can make it impossible to partition X into a different number of \succ -order-isomorphic subsets, as a d -checklist with $d_1 \neq 3$ will require; an irrational preference may have a particular combination of order-isomorphic subsets but it cannot have all the combinations that quick checklists can require. Indeed the \succ in Example 4 is not the outcome of either a (5, 2)-checklist (a checklist (C_1, C_2) with 5 and 2 equivalence classes respectively) or a (2, 5)-checklist. A preference with a smaller number of indifference classes gives a simpler example of the same point.

Example 5 To see that a preference can be the outcome of a quick checklist for some but not all discriminatory capacities, let \succ on $X = \{a, b, c, x, y, z\}$ be defined by $a \succ b \succ c \succ a$ and $x \succ y \succ z \succ x$ and $\{a, b, c\} \succ \{x, y, z\}$. Then \succ can be the outcome of a (2, 3)-

checklist, e.g., where C_1 is $\{a, b, c\} C_1 \{x, y, z\}$ and C_2 is $\{a, x\} C_2 \{b, y\} C_2 \{c, z\} C_2 \{a, x\}$. Note the characteristic \succ -order isomorphism of the two C_1 -equivalence classes. But this \succ cannot be the outcome of a (3, 2)-checklist. Since \succ is complete both criteria must be complete; so, given \succ , the first criterion of a (3, 2)-checklist must cycle. The only viable candidates for the first criterion must therefore include the rankings $a C_1 b C_1 c C_1 a$ and $x C_1 y C_1 z C_1 x$; but since only 3 C_1 -equivalence classes are allowed C_1 would have to rank some element of $\{x, y, z\}$ superior to some element of $\{a, b, c\}$, contrary to \succ . ■

Given n , we say that the discriminatory capacity d is *admissible* if and only if $d_i > 2$ for some $i \in \{1, \dots, \lceil \log_d n \rceil\}$. As always, any discriminatory capacity d , whether admissible or not, satisfies $d_i \geq 2$ for all i .

Theorem 2 *A preference \succ with $n \geq 5$ indifference classes is rational if and only if, for any admissible d , \succ is the outcome of a quick d -checklist.*

Extension to choice functions. We saw in section 2 that any checklist defines a choice function c that selects from arbitrary choice sets. If c maximizes a rational preference \succ then c can be the outcome of the quick checklist that the proof of Theorem 2 uses. For a converse, fix the domain of choice functions to include all of the two-element subsets of X . Let us say that c ‘can always be the outcome of a quick checklist’ if, for any admissible d , c can be the outcome of a d -checklist with $\lceil \log_d n \rceil$ criteria, where n is the number of indifference classes in the base relation R of c .⁷ Theorem 2 then implies that c can always be the outcome of a quick checklist if and only if its base relation is rational (and $n \geq 5$). It is easy to see that any choice function c that is the outcome of a checklist has a rational base relation if and only if c maximizes a rational preference. A full equivalence therefore obtains between choice functions that maximize rational preferences and choice functions that can always be the outcome of a quick checklist.

We give the proof of the ‘only if’ half of Theorem 2 here and relegate the lengthier proof of ‘if’ to the appendix. That part specifies a set of three discriminatory capacities such that any irrational \succ with $n \geq 5$ will fail to be the outcome of a quick d -checklist

⁷ R is defined by $a R b \Leftrightarrow (c(\{a, b\}) = \{a\})$.

for at least one d in this set.⁸

Half of proof of Theorem 2. Let \succ be a preference on X with n indifference classes that is complete and transitive. Fix some d which in this half of the proof we can allow to be not admissible. For any $x \in X$, let $[x]$ be the \succ -indifference class that contains x . For any \succ -indifference class $I \subset X$, define

$$r(I) = |\{J \subset X : J \text{ is a } \succ\text{-indifference class such that } I \succ J\}|,$$

which can be viewed as the common utility number of the elements in I . We can identify each integer r in $[0, n - 1]$ with its expansion in the mixed-radix number system (see Knuth (1997b)) with bases d_1, \dots, d_T , where $T = \lceil \log_d n \rceil$. That is, for each r there is a unique sequence of T integers, $k(r, 1), \dots, k(r, T)$, where $0 \leq k(r, j) < d_j$, such that $r = k(r, T) + \sum_{j=1}^{T-1} k(r, j)(d_T d_{T-1} \cdots d_{j+1})$. For $j = 1, \dots, T$, define C_j by $x C_j y$ if and only if $k(r([x]), j) > k(r([y]), j)$. For any $x, y \in X$, suppose there is a smallest integer i in $[1, T]$ such that $k(r([x]), i) \neq k(r([y]), i)$. If say $k(r([x]), i) > k(r([y]), i)$ then $r[x] > r[y]$ and so $x \succ y$. Moreover, $x C_j y$ and not $y C_j x$ for all integers $j < i$, as desired. If on the other hand $k(r([x]), i) = k(r([y]), i)$ for all integers i in $[1, T]$, then $r([x]) = r([y])$ and so $x \sim y$. And then not $x C_i y$ and not $y C_i x$ for all i in $[1, T]$. ■

6 The proportion of irrational preferences that are the outcome of quick checklists

We consider the proportion of preferences that are the outcome of a quick d -checklist as the size of the domain increases. Throughout this section we fix the discriminatory capacity at a d such that there is a \bar{d} with $d_i \leq \bar{d}$ for all $i \in \mathbb{N}$. We first deal with the proportion of all preferences that are the outcome of a quick checklist – it is a more

⁸An earlier version of this paper provided a variant of Theorem 2: for any irrational \succ with $n \geq 10$ there is always a subdomain of X on which \succ is not the outcome of a quick $\bar{3}$ -checklist, while a rational \succ is the outcome of a quick d -checklist on any subdomain and for any d . See the Conclusion for a brief discussion of subdomains. This variant has the advantage that it establishes a speed failure of irrational \succ 's using only one type of checklist, but with the cost of resorting to subdomains and an extremely lengthy proof. For an outline, see the extended abstract, Mandler (2009).

straightforward calculation – and then turn to our real concern, irrational preferences.

For any n , let X_n denote a generic domain of n objects, $\{1, \dots, n\}$, let $\succ(n)$ denote the set of all preferences on X_n ,

$$\succ(n) = \{\succ \subset X_n \times X_n : \succ \text{ is asymmetric}\},$$

and let $|\succ(n)|$ denote the cardinality of $\succ(n)$. As for quick checklists, let $q(n)$ denote the set of preferences on X_n that are the outcome of a quick checklist,

$$q(n) = \{\succ \in \succ(n) : \succ \text{ is the outcome of a quick } d\text{-checklist}\},$$

and let $|q(n)|$ denote the cardinality of $q(n)$.

Theorem 3 *The proportion of preferences on X_n that are the outcome of a quick d -checklist converges to 0 at a super-exponential rate as $n \rightarrow \infty$. That is, for any $k > 0$, $e^{kn} \frac{|q(n)|}{|\succ(n)|}$ converges to 0.*

For all but the smallest values of n , the set of possible preferences on X_n is vast. Only a comparative handful of these are rational; when n is reasonably large, most preferences on X_n have n indifference classes, and yet there are only $n!$ preference relations on X_n with n indifference classes that are complete and transitive (and up to an order isomorphism there is only one rational preference on X_n with n indifference classes). It is therefore no surprise that the conclusion of Theorem 3 continues to hold if we narrow our focus to irrational preferences.

Let $\succ_{\text{ir}}(n)$, $|\succ_{\text{ir}}(n)|$, $q_{\text{ir}}(n)$, and $|q_{\text{ir}}(n)|$ denote the restriction of $\succ(n)$, $|\succ(n)|$, $q(n)$, and $|q(n)|$ to preferences that are not complete and transitive (e.g., $\succ_{\text{ir}}(n) = \{\succ \in \succ(n) : \succ \text{ is not complete and transitive}\}$ and $q_{\text{ir}}(n) = \{\succ \in \succ_{\text{ir}}(n) : \succ \text{ is the outcome of a quick } d\text{-checklist}\}$).

Theorem 4 *The proportion of irrational preferences on X_n that are the outcome of a quick d -checklist converges to 0 at a super-exponential rate as $n \rightarrow \infty$. That is, for any $k > 0$, $e^{kn} \frac{|q_{\text{ir}}(n)|}{|\succ_{\text{ir}}(n)|}$ converges to 0.*

The size of the domain n need not be at all large for the proportion of irrational preferences that are the outcome of a quick checklist to be extremely small. To illustrate, the following table reports for $3 \leq n \leq 9$ the number of preference relations on X_n that fail to be complete and transitive, the number of those preferences that are the outcome of a quick $\vec{3}$ -checklist, and the ratio of these two numbers, i.e., the proportion of irrational preferences on X_n that are the outcome of quick $\vec{3}$ -checklists. To keep the count under control, we give only the number of preferences – both for the total and for preferences that are the outcome of a quick checklist – that are unique up to an order isomorphism. The count here does not suppose that indifference classes are singletons, unlike the discussion in section 5.⁹

n	# irrational preferences	# with quick $\vec{3}$ -checklists	ratio
3	3	3	1
4	34	31	.91176471
5	566	226	.39929329
6	21,448	1,084	.05054084
7	2,142,224	3,847	.00179580
8	575,016,091	11,143	.00001938
9	415,939,242,776	217,902	.00000052

7 Rational preferences are the easiest to build

So far we have taken the discriminatory capacity d to be exogenous; we had no choice since a fair comparison of the decision-making speed of different preferences requires that each preference can be the outcome of criteria of the same power. Here we reverse perspective and ask: for an agent who uses criteria to build his preferences, what is the best d to use? Specifically, which d lets preference relations be constructed at the

⁹Since there are only five relevant C_i orderings of C_i -indifference classes when $d = \vec{3}$, it is feasible to enumerate the irrational preferences on X_n that are outcomes of quick $\vec{3}$ -checklists and then weed out the preferences that are order isomorphic. I am indebted to Claire Blackman for writing code that performs this enumeration (it is available on request). The total number of preferences on X_n up to order isomorphism is given in Harary (1969); for the number of irrational preferences we subtract the number of rational preferences, which equals 2^{n-1} , from the Harary total.

lowest cost as measured by the number of criterion rankings the agent has to make? To evaluate the d 's on a level playing field, we compare checklists that distinguish a common number of preference indifference classes. The answer is that the minimum cost is achieved only by the checklists with binary criteria (where $d = (2, 2, \dots)$). So any agent who uses the minimum number of binary criteria – minimum for the number of preference discriminations the agent ends up distinguishing – is making preference discriminations as cheaply as possible. Since binary criteria necessarily lead to rational preferences, we have another advantage of rationality: only those agents who construct rational preferences can make the smallest decision-making effort.

We do not measure the cost of building a checklist that makes n preference discriminations by its length; by that standard it would always be cheapest to set $d_1 = n$ since then an agent could make do with a single criterion. Length moreover does not take into account the cost of making the judgments that define a checklist's criteria. So instead we measure the cost of a criterion C_i by the number of pairwise comparisons of C_i 's equivalence classes: for any two C_i -equivalence classes the agent must decide whether one or the other is superior or whether they are unranked. The cost of a criterion with d_i equivalence classes is therefore given by the binomial coefficient $\binom{d_i}{2} = \frac{d_i(d_i-1)}{2}$ and the cost of a checklist as a whole is given by the sum of the costs of its criteria.

Increasing the discriminatory capacities d_i seems to present a trade-off: while criteria become more expensive to form one can make do with fewer of them. The first effect turns out to dominate the second: the criterion-building cost of larger d_i 's outweighs the advantage of using fewer criteria. As a consequence the binary criteria that arose in Example 1, which at first glance appeared very crude, end up being the cheapest. To illustrate suppose there are $n = 9$ indifference classes. A quick binary checklist will require $\lceil \log_2 9 \rceil = 4$ criteria and hence $4\binom{2}{2} = 4$ criterion decisions whereas a quick $\overrightarrow{3}$ -checklist, seemingly a better fit with $n = 3^2$, requires $\lceil \log_3 9 \rceil = 2$ criteria and hence $2\binom{3}{2} = 6$ decisions.

Letting $e(C_i)$ denote the number of equivalence classes of a criterion C_i , we define the *cost of a checklist* $C = (C_1, \dots, C_T)$ to be $\sum_{i=1}^T \binom{e(C_i)}{2}$. A *more-than-binary checklist* is a (C_1, \dots, C_T) such that some C_j , $j \in \{1, \dots, T\}$, has strictly more than 2 equivalence classes.

Theorem 5 *Any more-than-binary checklist that leads to a preference with n indifference classes has a cost that is strictly larger than the cost of a quick binary checklist that leads to n indifference classes.*

Proof. A quick binary checklist that leads to a \succ with n indifference classes uses $\lceil \log_2 n \rceil$ criteria and by the proof of Theorem 2 such a checklist exists for rational \succ . So the cost of a quick binary checklist that leads to n indifference classes is $\sum_{i=1}^{\lceil \log_2 n \rceil} \binom{2}{2} = \lceil \log_2 n \rceil$.

Let $c : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ be defined by $c(r, n) = \frac{r(r-1)}{2} \log_r n$. Then $\frac{\partial}{\partial r} c(r, n) = (\frac{2r-1}{2} - \frac{(r-1)}{2 \ln r}) \log_r n$ which is strictly positive for any $(r, n) \in \mathbb{R}^2$ with $2 \leq r \leq n$. Thus for integers r and n such that $2 < r \leq n$, we have $c(r, n) > c(2, n)$. In addition, it is readily confirmed that $c(3, 3) - c(2, 3) > 1$ and that $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $\delta(n) = c(3, n) - c(2, n)$ is an increasing function. Hence $c(r, n) - c(2, n) > 1$ when $n \geq r \geq 3$. Therefore, for integers r, n , and t with $2 < r \leq n$ and $t > 0$, we have $\binom{r}{2} \log_r n > \log_2 n + 1$ and hence $\binom{r}{2} t > \log_2 r^t + 1$.

Let $C = (C_1, \dots, C_T)$ be a more-than-binary checklist that leads to some \succ with n indifference classes and let $\bar{e} = \max\{e(C_1), \dots, e(C_T)\}$. So $\bar{e} > 2$. Letting t_j be the number of criteria in C with exactly j equivalence classes, the cost of C is given by $\sum_{j=2}^{\bar{e}} \binom{j}{2} t_j$. Since $\binom{j}{2} t_j > \log_2 j^{t_j} + 1$ for $j > 2$ (by the previous paragraph) and since $\binom{2}{2} t_2 = \log_2 2^{t_2}$,

$$\sum_{j=2}^{\bar{e}} \binom{j}{2} t_j > \sum_{j=2}^{\bar{e}} \log_2 j^{t_j} + 1 \geq \log_2 \left(\prod_{j=2}^{\bar{e}} j^{t_j} \right) + 1,$$

and given that Theorem 1 shows that $\prod_{j=2}^{\bar{e}} j^{t_j} \geq n$,

$$\log_2 \left(\prod_{j=2}^{\bar{e}} j^{t_j} \right) + 1 \geq \log_2 n + 1 > \lceil \log_2 n \rceil.$$

■

So binary criteria have the potential to generate preference discriminations more efficiently than any other type of criteria – though keep in mind that to achieve this potential the binary criteria have to form a quick binary checklist. It is therefore only the agents who end up being rational who have preferences that can be built with minimum effort.

There is practical advice in Theorem 5. Returning to the restaurant meals that opened

the paper, suppose you want a method for ordering meals that discriminates sufficiently into preference classes but that requires the fewest decisions. Theorem 5 instructs you to use binary criteria with a specific structure: as you proceed through your criteria, each binary distinction must divide the not-yet-discriminated meals into halves. It is not hard to identify categories that will accomplish this goal. As mentioned in section 5, if the attributes of alternatives can be put on a one-dimensional scale – for example, in the case of meals, their calorie count, their cost, their spiciness, etc., then a line can be drawn to divide each set of not-yet-discriminated alternatives into equal halves with more and less respectively of the attribute. Some attributes that are not one-dimensional – meat versus fish – may also happen to divide sets of meals into approximately equal halves. The upshot of Theorem 5 is that a binary structure such as this is the most efficient way to partition meals into n indifference classes. Notice that if one can find the required binary distinctions, the efficiency advantage will remain even if on each day you go out to eat your ranking of some of the binary categories changes – one day you prefer fish and the next you prefer meat.

The binary checklists we have described order the set of all possible meals while a particular restaurant offers a smaller choice set, namely its menu, but a binary checklist can evidently also serve as a way to search a menu. Of course, a checklist might not generate enough indifference classes to pick out a single meal from a menu; at this point the meals that have survived all of the binary comparisons would be declared indifferent and the agent would pick an arbitrary survivor. This binary method may not be a bad description of how people sometimes decide. Our analysis pushes Rubinstein (1996) one step further: not only do linear binary *relations* stand out in their usefulness but those that stem from binary *categories* stand out within the class of linear binary relations as the easiest to construct.

8 Minimum expected run time

While rational preferences agents can always be the outcome of checklists with the minimum number of criteria, the possibility remains that compared to their irrational coun-

terparts rational preferences may be the outcome of checklists that in expectation proceed through more criteria before coming to a decision. Subject to one proviso this turns out not to be the case; rational preferences are not only the outcome of the shortest checklists but also of checklists that on average execute the most quickly.

Let each preference in this section be defined on a common domain X that has n elements. Suppose the preference \succ is the outcome of the checklist $C = (C_1, \dots, C_T)$. For any pair $\{x, y\} \subset X$, let $t_C(x, y) = t_C(y, x)$ be the index of the criterion that determines the agent's decision between x and y : $t_C(x, y)$ is the largest integer $i \in \{1, \dots, T\}$ such that (not $x C_j y$ and not $y C_j x$ for $j \in \{1, \dots, i-1\}$). So if x and y are not ranked by \succ then $t_C(x, y) = T$. We define the *expected run time* of C to be

$$Et_C = \sum_{\{x,y\} \subset X: x \neq y} t_C(x,y) \frac{1}{2^{n(n-1)/2}}.$$

Since $2^{n(n-1)/2} = \binom{n}{2}$ is the number of unordered pairs in X , Et_C is the expectation of the number of criteria the agent must examine, assuming each pair is equally likely. We could apply any strictly increasing transformation to t_C (to measure psychic costs that might not be a linear function of time) without affecting the conclusion of the theorem below.

A preference \succ has *singleton indifference classes* if each \succ -indifference class consists of a single element of X .

A preference \succ with singleton indifference classes has *minimum expected run time* if, for any discriminatory capacity d , \succ is the outcome of a d -checklist C and $Et_C \leq Et_{C'}$ for any \succ' with singleton indifference classes and any d -checklist C' that leads to \succ' . Also, \succ has *strictly smaller expected run time than* \succ' if, for any d , \succ is the outcome of a d -checklist C that satisfies $Et_C \leq Et_{C'}$ for all d -checklists C' that lead to \succ' , and, for some admissible \hat{d} such that \succ' is the outcome of a \hat{d} -checklist, strict inequality always holds, that is, \succ is the outcome of a \hat{d} -checklist C where $Et_C < Et_{C'}$ for all \hat{d} -checklists C' that lead to \succ' .

Theorem 6 *Any rational preference \succ with singleton indifference classes has minimum expected run time. If a \succ' with singleton indifference classes is not rational and $n \geq 5$*

then \succ has strictly smaller expected run time than \succ' .

A \succ with minimum expected run time in fact terminates *very* quickly in expectation: one may easily show that if \succ has minimum expected run time and C achieves this minimum for some d then $Et_C < 2$.

Why the restriction to singleton indifference classes? If C is a checklist that leads to \succ and \succ has multiple-element indifference classes then any pair of indifferent elements will be decided only by the final criterion in C . So preferences with more pairs of indifferent options will run more slowly, even when they are rational. And even looking at preferences with the same number of indifferent pairs, it could be that the quickest checklist that leads to a preference decides between singleton indifference classes with its initial criteria and between multiple-element indifference classes with its later criteria, which again will slow average run time.

9 Conclusion

Rationality may not be as difficult as Herbert Simon supposed. If we identify a preference relation with a decision procedure and not with an agent who must uncover from his mind a preexisting utility function, then there need be no intrinsic challenge to choosing rationally. For agents who use checklists, rational preferences will emerge if the constituent building blocks of a checklist – the criteria – are themselves rational; moreover criteria are necessarily rational when agents use simple binary criteria. Irrationality is still possible, of course, but there is no computational argument in its favor in the present setting. Indeed, it is the rational agents who hold the advantage. The agents who use rational criteria can potentially make choice discriminations with strictly greater efficiency than the agents who do not.

While irrational agents are all slow compared to the potential speed of rational agents, the techniques in this paper can be used to determine the comparative slowness of different irrational preferences. Given a preference \succ , let $A \subset X$ and $B \subset X$ contain disjoint sets of \succ -indifference classes. If the \succ -indifference classes in A and B are \succ -order isomorphic then there is a quick checklist on the subdomain $A \cup B$: the restriction of \succ to

$A \cup B$ will be the outcome of a $(2, d_2)$ -checklist where d_2 is the number of \succ -indifference classes in A (or B). Subdomains can be interpreted as restricted sets of conceivable consumption alternatives – suppose that every so often, say on the first of the month, the agent discovers that decisions in the coming month will involve only alternatives drawn from the subdomain. Of course, analogous checklists arise for all k -tuples (rather than pairs) of disjoint sets of \succ -order isomorphic indifference classes. A preference's endowment of order-isomorphic subsets thus measures its decision-making speed across domains and discriminatory capacities. As we have seen, by this measure too the rational agents always defeat the irrational agents.

A Appendix: Remaining results and proofs

Singleton Indifference Class Lemma. If a preference with n indifference classes is the outcome of a length T checklist then there is a preference with n singleton indifference classes that is also the outcome of a length T checklist.

Proof. For each \succ -indifference class I , let e_I be an arbitrary element of I and for any $x \in X$, let $[x]$ denote the \succ -indifference class that contains x . Given a d -checklist (C_1, \dots, C_T) that leads to \succ , define the checklist $C' = (C'_1, \dots, C'_T)$ by

$$x C'_i y \Leftrightarrow e_{[x]} C_i e_{[y]}$$

for $i = 1, \dots, T$ and all $x, y \in X$. It is easy to confirm that each C'_i is asymmetric and has no more than d_i equivalence classes. Letting \succ' be the outcome of C' ,

$$\begin{aligned} x \succ y &\Leftrightarrow e_{[x]} \succ e_{[y]} \Leftrightarrow \text{the first } C_i \text{ that ranks } e_{[x]}, e_{[y]} \text{ has } e_{[x]} C_i e_{[y]} \\ &\Leftrightarrow \text{the first } C'_j \text{ that ranks } x, y \text{ has } x C'_j y \Leftrightarrow x \succ' y, \end{aligned}$$

where in fact the indices i and j coincide. Since any $a, b \in X$ with $a \sim b$ are always in the same C'_i -equivalence class for $i = 1, \dots, T$, the binary relation \triangleright on $\{I \subset X : I = [x] \text{ for some } x \in X\}$ defined by $[a] \triangleright [b]$ if and only if $a \succ b$ must also be the outcome of a d -checklist of length T . ■

Remainder of Proof of Theorem 2.

It remains to show that if a \succ with $n \geq 5$ indifference classes is the outcome of a quick d -checklist for each admissible d then \succ is rational. In step 1 we establish a simple consequence of the assumption that some $(2, \lceil \frac{n}{2} \rceil)$ -checklist leads to \succ . We use this consequence in step 2 to show that any $(3, \lceil \frac{n}{3} \rceil)$ -checklist that leads to \succ must have a linear C_1 . We in turn use this fact in step 3 to show that if \succ is also the outcome of a quick $(3, 2, \dots, 2)$ -checklist then \succ is rational. Given the proof of the Singleton Indifference Class Lemma, we can assume for each checklist C that each \succ -indifference class is contained in exactly one C_i -equivalence class for each C_i in C . Also, for two \succ -indifference classes x

and y , we use $x \perp y$ to mean $\neg(x \succ y)$ and $\neg(y \succ x)$.

Step 1. Let L and U be the C_1 -equivalence classes for some (C_1, C_2) that is a $(2, \lceil \frac{n}{2} \rceil)$ -checklist that leads to \succ where UC_1L and therefore $U \succ L$. Given the proof of Theorem 1, any pair of \succ -indifference classes must either be in distinct C_1 or C_2 equivalence classes. Hence if n is even the C_1 -equivalence classes contain $\frac{n}{2}$ \succ -indifference classes each and if n is odd the C_1 -equivalence classes contain $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ \succ -indifference classes.

Step 2. Let I, J, K be the C_1 -equivalence classes for some (C_1, C_2) that is a $(3, \lceil \frac{n}{3} \rceil)$ -checklist that leads to \succ . Using the fact that the intersection of each C_1 -equivalence class and each C_2 -equivalence class can contain at most one \succ -indifference class and that C_2 has $\lceil \frac{n}{3} \rceil$ -equivalence classes, it is easy to see that: if $0 \equiv n \pmod{3}$ then each C_1 -equivalence class contains $\frac{n}{3}$ \succ -indifference classes, if $2 \equiv n \pmod{3}$ then the C_1 -equivalence classes contain $\lfloor \frac{n}{3} \rfloor$, $\lceil \frac{n}{3} \rceil$, and $\lceil \frac{n}{3} \rceil$ \succ -indifference classes, and if $1 \equiv n \pmod{3}$ then the C_1 -equivalence classes contain either $(\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor, \text{ and } \lceil \frac{n}{3} \rceil)$ or $(\lfloor \frac{n}{3} \rfloor - 1, \lceil \frac{n}{3} \rceil, \text{ and } \lceil \frac{n}{3} \rceil)$ \succ -indifference classes. It follows that the union of the two C_1 -equivalence classes with the fewest \succ -indifference classes has at least $2 \lfloor \frac{n}{3} \rfloor$ \succ -indifference classes. If $n \geq 6$ then $2 \lfloor \frac{n}{3} \rfloor \geq \lceil \frac{n}{2} \rceil$ and if $n = 5$ then $\lfloor \frac{n}{3} \rfloor + \lceil \frac{n}{3} \rceil = \lceil \frac{n}{2} \rceil$. Thus the same union always has at least $\lceil \frac{n}{2} \rceil$ \succ -indifference classes, a fact we use below.

We show that $\{I, J, K\}$ is linearly ordered by C_1 and hence by \succ . Since C_1 must have exactly 3 equivalence classes – if C_1 had $k < 3$ equivalence classes then $k \lceil \frac{n}{3} \rceil < n$ violating Theorem 1 – the only other possibilities (up to order isomorphism) are (i) $IC_1JC_1KC_1I$, (ii) IC_1JC_1K , and (iii) IC_1J , considered as (i)-(iii) below.

Given a $(3, \lceil \frac{n}{3} \rceil)$ -checklist (C_1, C_2) that leads to \succ , call a C_2 -equivalence *full* if it consists of exactly 3 \succ -indifference classes.

Lemma. At most two C_2 -equivalence classes are not full.

Proof. If 3 C_2 -equivalence classes were not full then those equivalence classes could contain at most 6 \succ -equivalence classes. The remaining $\lceil \frac{n}{3} \rceil - 3$ C_2 -equivalence classes could together contain at most $3(\lceil \frac{n}{3} \rceil - 3)$ \succ -indifference classes. Thus the entire set of $\lceil \frac{n}{3} \rceil$ C_2 -equivalence classes could contain at most $3 \lceil \frac{n}{3} \rceil - 3 < n$ \succ -indifference classes, a contradiction. \square

Case (i). We then have $I \succ J \succ K \succ I$. Choose the labels of the C_1 -equivalence classes so that $I \cap U \neq \emptyset$. Then, since $K \succ I$ and $U \succ L$, $K \cap L = \emptyset$. Hence $K \cap U \neq \emptyset$ and, applying the same argument, $J \cap L = \emptyset$. Hence $J \cap U \neq \emptyset$ and, again, $I \cap L = \emptyset$. Thus $L = \emptyset$, a contradiction.

Case (ii). We have $I \succ J \succ K$. For $n \geq 7$ the Lemma implies there must be a C_2 -equivalence class E that is full. For $5 \leq n \leq 6$, C_2 has 2 equivalence classes, hence again a full C_2 -equivalence class E . First suppose that $J \cap L \neq \emptyset$ and $J \cap U \neq \emptyset$. Then $I \subset U$ and $K \subset L$. Since the three \succ -indifference classes in E must be in separate C_1 -equivalence classes, $x \equiv I \cap E \neq \emptyset$ and $z \equiv K \cap E \neq \emptyset$. Given that I and K are not ranked by C_1 , we have $x \perp z$, which contradicts the fact that $I \subset U$ and $K \subset L$. Next suppose that either $J \subset L$ or $J \subset U$. If say $J \subset U$ then $I \subset U$ and, since $I \cup J$ must contain at least $\lceil \frac{n}{2} \rceil$ \succ -indifference classes, $K = L$. But then the presence of a full C_2 -equivalence class E again implies that there are two \succ -indifference classes x and z in I and K respectively with $x \perp z$, again contradicting the fact that $I \subset U$ and $K \subset L$. The subcase where $J \subset L$ is similar.

Case (iii). We have $I \succ J$. Observe first that each full C_2 -equivalence class E must

either be contained in L or U : letting the three \succ -indifference classes in $I \cap E$, $J \cap E$, and $K \cap E$ be denoted, respectively by x , y , and z , we have $x \perp z$ and $y \perp z$ and so $x \cup z$ and $y \cup z$ must both be contained in either L or U . In the following subcases there must be a set \mathcal{E} of full C_2 -equivalence classes such that $|\mathcal{E}| > \lceil \frac{n}{2} \rceil$: $n \geq 10$ (since, by the Lemma, at most 4 \succ -indifference classes are in C_2 -equivalence classes that are not full), $n = 6, 9$ (since $0 \equiv n \pmod{3}$ and hence every C_2 -equivalence class is full), $n = 8$ (since $\lceil \frac{n}{3} \rceil = 3$ and hence 2 C_2 -equivalence classes must be full), and $n = 7$ and there are 2 full C_2 -equivalence classes. Since $|\mathcal{E}| > \lceil \frac{n}{2} \rceil$, there must two full C_2 -equivalence classes, E and F , such that $E \subset L$ and $F \subset U$. But for the indifference classes $x = E \cap I$ and $y = E \cap J$ we have $y \succ x$, a contradiction.

Two tedious subcases remain. The first is where $n = 7$ and there is one full C_2 -equivalence class E . Call the two nonfull C_2 -equivalence classes F and G . Assume for concreteness that $E \subset U$ (the $E \subset L$ possibility is similar). Then $I \subset U$ since $IC_1(J \cap E)$. Hence $(J \cup K) \cap (F \cup G)$ must contain at least three \succ -indifference classes that belong to L and consequently both $J \cup K$ and $F \cup G$ must each contain at least one \succ -indifference class that belongs to L . Since $E \subset U$ and E intersects each C_1 -equivalence class, we must have $EC_2(F \cup G)$. Now suppose $I \cap (F \cup G) \neq \emptyset$. Let us then choose the labels F and G so that $I \cap F \neq \emptyset$. Then the two \succ -indifference classes in G are both in L : one must be in J and the other in K . Consequently since $(I \cap F) \subset U$ we must have FC_2G . Given that EC_2G , that FC_2G , and that J and K are not ranked by C_1 , we conclude that the items in G cannot form two \succ -indifference classes (each item in G is \succ -worse than the other 5 \succ -indifference classes). On the other hand suppose $I \cap (F \cup G) = \emptyset$. Then, if F and G are unranked by C_2 , $(F \cup G) \cap J$ cannot contain two \succ -indifference classes (each item in $(F \cup G) \cap J$ is \succ -worse than the \succ -indifference classes in E and \succ -unranked relative to the \succ -indifference classes in $K \setminus E$). While if F and G are ranked by C_2 – say FC_2G – then G cannot contain two \succ -indifference classes (each item in G is \succ -worse than the other 5 \succ -indifference classes). The second subcase is where $n = 5$. Since $\lceil \frac{n}{3} \rceil = 2$, one of the C_2 -equivalence classes, say E , must be full and C_2 must rank E and the other C_2 -equivalence class, F . Since E is full and $3 = \lceil \frac{5}{2} \rceil$, either $E = L$ or $E = U$. Suppose for concreteness that $E = U$ (the $E = L$ possibility is similar). If $F \cap I \neq \emptyset$ then $F \cap I \subset U$ and hence U would contain at least 4 \succ -indifference classes. So $F \cap J \neq \emptyset$ and $F \cap K \neq \emptyset$ but, since $E \succ F$ and C_2 does not rank J and K , $F \cap (J \cup K)$ cannot form two distinct \succ -indifference classes.

Thus the set of C_1 -equivalence classes above, which we now denote \mathcal{Z} , is linearly ordered by \succ .

Step 3. We conclude by showing that if \succ is the outcome of a $(3, 2, \dots, 2)$ -checklist \widehat{C} with length $1 + \lceil \log_2 \frac{n}{3} \rceil$ in addition to the steps 1 and 2 checklists, then \succ must be linear. The length is chosen so that $\lceil \log_2 \frac{n}{3} \rceil$, the number of binary criteria in \widehat{C} , equals the minimum integer t such that $3 \times 2^t \geq n$. Each \widehat{C}_1 -equivalence class consists of no more than $2^{\lceil \log_2 \frac{n}{3} \rceil}$ \succ -indifference classes and since each \widehat{C}_i , $i \geq 2$, is linear these \succ -indifference classes are linearly ordered by \succ . So we may instead consider a $(3, 2^{\lceil \log_2 \frac{n}{3} \rceil})$ -checklist (C_1, C_2) that leads to \succ where C_2 is linear and show that \succ is linear. As in step 2, C_1 must have exactly 3 equivalence classes, which we again label I , J , and K . Since \succ is linear if C_1 is, it is sufficient to consider the nonlinear possibilities for C_1 , which (up to

order isomorphism) are (i) $IC_1JC_1KC_1I$, (ii) IC_1JC_1K , and (iii) IC_1J . Case (i) repeats case (i) in step 2.

Case (iii). We have $I \succ J$. For any $z \in X$ or \succ -indifference class z , we use $E(z)$ to denote the C_2 -equivalence class that contains z .

For any nonlinear \succ , there exists a \succ -indifference class $k \subset K$ such that (I) there is a \succ -indifference class $i \subset I$ with $\neg iC_2k$ and (II) there is a \succ -indifference class $j \in J$ with $\neg kC_2j$. To see this, suppose to the contrary that for every \succ -indifference class $k \subset K$ either (I) or (II) is violated. For example, let (I) be violated, i.e., every \succ -indifference class $i \subset I$ satisfies iC_2k . Then any \succ -indifference class x with $x \succ E(k) \cap J$ satisfies xC_2k and any \succ -indifference class x with $E(k) \cap J \succ x$ satisfies kC_2x – so in fact $E(k)$ consists of only one \succ -indifference class. Consequently the checklist C' where k is ‘transferred’ into J but is otherwise identical to C is a checklist that leads to \succ ; formally $C'_2 = C_2$ and $C'_1 = C_1 \cup \{(\alpha, \beta) : \alpha \in I \text{ and } \beta \in k\}$. Notice that either C' is not of type (iii) or, if it is (with C'_1 -equivalence classes I', J' , and K' that satisfy $I'C_1J'$), then $k \subset J'$. A similar construction applies when (II) is violated. Proceeding sequentially through the \succ -indifference classes in K , we eliminate all of the \succ -indifference classes in K and conclude that there is a checklist $(\tilde{C}_1, \tilde{C}_2)$ that leads to \succ where each \tilde{C}_i is linear. Hence \succ is linear.

Given that any \succ -indifference class $k \subset K$ satisfies (I) and (II), ($k \succ i$ or $k \perp i$), ($j \succ k$ or $k \perp j$), and $i \succ j$. Step 2 then implies that $i \cup j \cup k \in Z$ for some $Z \in \mathcal{Z}$. Suppose $Y \in \mathcal{Z}$ and $Y \succ Z$. Any \succ -indifference class $x \subset J$ satisfies $i \succ x$ and so x cannot be a subset of Y . For a \succ -indifference classes $x \subset (Y \cap I)$ then we must have xC_2k (since otherwise $k \succ x$ or $k \perp x$). To pin down the features of \succ -indifference classes in $Y \cap K$, we use $\gamma(z)$, for any \succ -indifference class z , to denote the set in \mathcal{Z} that contains z . Let x be a \succ -indifference classes contained by $Y \cap K$. To show that xC_2w for all $w \subset J$, suppose to the contrary that $\neg xC_2w$ for some $w \subset J$. Then $\gamma(x) \neq \gamma(w)$. Since furthermore $i \succ w$, we have $\gamma(w) \neq Z$ and, since \neq is transitive on \mathcal{Z} , we get the contradiction that x cannot be a subset of Y . For $x \subset (Y \cap K)$ it must in addition be the case that xC_2k . Since xC_2k and xC_2w for all \succ -indifference classes $w \subset J$, $E(x)$ must contain just one \succ -indifference class: since $i \succ J$ the only other possible \succ -indifference class in $E(x)$ must be a subset of I and any \succ -indifference class u with $u \succ E(x) \cap I$ satisfies uC_2x and any \succ -indifference class u with $E(x) \cap I \succ u$ satisfies xC_2u . We conclude that if v is a \succ -indifference class in Y then v is a subset of $\{a \in I : aC_2k\}$ or $\{a \in K : aC_2k \text{ and } E(a) \text{ contains one } \succ\text{-indifference class}\}$. Hence each \succ -indifference class in Y is in a distinct C_2 -equivalence class F such that FC_2k . A similar argument shows that if $Y \in \mathcal{Z}$ and $Z \succ Y$ then any \succ -indifference class $v \subset Y$ is a subset of $\{a \in J : kC_2a\}$ or $\{a \in K : kC_2a \text{ and } E(a) \text{ contains one } \succ\text{-indifference class}\}$, and so each \succ -indifference class in Y is in a distinct C_2 -equivalence class G such that kC_2G . Since the union of the two sets in \mathcal{Z} with the smallest number of \succ -indifference classes contains at most $2 \lfloor \frac{n}{3} \rfloor$ \succ -indifference classes (see step 2), there must be at least $2 \lfloor \frac{n}{3} \rfloor + 1$ C_2 -equivalence classes. But one may readily show that $2 \lfloor \frac{n}{3} \rfloor + 1 > 2^{\lceil \log_2 \frac{n}{3} \rceil}$.¹⁰

Case (ii). This case is similar to but simpler than (iii). We have $I \succ J \succ K$. If \succ is not linear then there must be \succ -indifference classes $i \subset I$ and $k \subset K$ such that $\neg iC_2k$;

¹⁰For any $x \geq 1$, including $x = \frac{n}{3}$, $1 + \log_2 \lfloor x \rfloor \geq \lceil \log_2 x \rceil$ and so $2 \lfloor x \rfloor \geq 2^{\lceil \log_2 x \rceil}$ and $2 \lfloor x \rfloor + 1 > 2^{\lceil \log_2 x \rceil}$.

otherwise IC_2K and hence $I \succ K$ and so we could add IC_1K to the original C_1 thus creating a checklist of two linear orders. Similarly to case (iii), i, k, J , any \succ -indifference class $x \subset I$ with iC_2x , and any \succ -indifference class $y \subset K$ with yC_2k must all be subsets of some $Z \in \mathcal{Z}$. So if $Y \in \mathcal{Z}$ and $Y \succ Z$ and v is a \succ -indifference class, then $v \subset I$ and vC_2i and therefore each \succ -indifference class in Y is in a distinct C_2 -equivalence class F such that FC_2i . Similarly, if $Y \in \mathcal{Z}$ and $Z \succ Y$ and u is a \succ -indifference class, then $u \subset K$ and kC_2u and therefore each \succ -indifference class in Y is in a distinct C_2 -equivalence class G such that kC_2G . Again the union of the two sets in \mathcal{Z} with the smallest number of \succ -indifference classes contains at most $2 \lfloor \frac{n}{3} \rfloor$ \succ -indifference classes and so there must be at least $2 \lfloor \frac{n}{3} \rfloor + 1$ C_2 -equivalence classes, a contradiction. ■

Proof of Theorem 3.

Since there are $\frac{n(n-1)}{2}$ unordered pairs of elements in X_n and since for any $x, y \in X_n$ an asymmetric relation \succ must specify one of three possibilities ($x \succ y$, $y \succ x$, or (not $x \succ y$ and not $y \succ x$)), $|\succ(n)| = 3^{\frac{n(n-1)}{2}}$.

The number of ways that X_n can be partitioned into \bar{d} cells is given by the Stirling number of the second kind $S(n, \bar{d})$ (see Goldberg et al. (1972) and Knuth (1997a) and for the formula for $S(n, \bar{d})$ we use below). The number of asymmetric binary relations on the domain $\{1, \dots, \bar{d}\}$ is $3^{\frac{\bar{d}(\bar{d}-1)}{2}}$, a total that includes all of the asymmetric binary relations with m equivalence classes for any $m \leq \bar{d}$. Hence $S(n, k)3^{\frac{\bar{d}(\bar{d}-1)}{2}}$ provides an upper bound on the number of asymmetric binary relations C_j on X_n with \bar{d} or fewer C_j -equivalence classes. Finally, since a preference on X_n has at most n indifference classes, a quick d -checklist has no more than $\lceil \log_{\bar{d}} n \rceil$ criteria (Theorem 1). Hence an upper bound $U(n)$ on $|q(n)|$ is given by

$$U(n) = \left(S(n, k) 3^{\frac{\bar{d}(\bar{d}-1)}{2}} \right)^{\lceil \log_{\bar{d}} n \rceil}.$$

Since (Goldberg et al. (1972)) we may write $S(n, \bar{d})$ as

$$S(n, \bar{d}) = \frac{1}{\bar{d}!} \sum_{i=0}^{\bar{d}} (-1)^{\bar{d}-i} \binom{\bar{d}}{i} i^n,$$

we have

$$S(n, \bar{d}) \leq \frac{1}{\bar{d}!} \sum_{i=0}^{\bar{d}} \binom{\bar{d}}{i} \bar{d}^n.$$

Hence there is a $r > 0$ such that $S(n, k)3^{\frac{\bar{d}(\bar{d}-1)}{2}}$ is bounded above by r^n . Hence

$$U(n) \leq (r^n)^{1+\log_{\bar{d}} n} = e^{(1+\log_{\bar{d}} n)n \ln r}.$$

So, for any fixed $k > 0$, $e^{kn} |q(n)|$ is bounded above by $e^{(1+\log_{\bar{d}} n)n \ln r + kn}$. Since $|\succ(n)| = 3^{\frac{n(n-1)}{2}} = e^{\frac{n(n-1)}{2} \ln 3}$ and since

$$\lim_{n \rightarrow \infty} \left(\frac{n(n-1)}{2} \ln 3 - ((1 + \log_{\bar{d}} n)n \ln r + kn) \right) = \infty,$$

$e^{kn} \frac{|q(n)|}{|\succ(n)|} \rightarrow 0$ as $n \rightarrow \infty$. ■

Proof of Theorem 4.

Only one adjustment is needed to the previous proof. For each asymmetric binary relation \succ_{n-1} on X_{n-1} we may select a binary relation R on X_n that only ranks n relative to the elements of X_{n-1} (each ordered pair in R has n as one of its coordinates) such that $\succ_{n-1} \cup R$ is in $\succ_{\text{ir}}(n)$. Hence $3^{\frac{(n-1)(n-2)}{2}}$ can serve as a lower bound for $|\succ_{\text{ir}}(n)|$. Since $3^{\frac{(n-1)(n-2)}{2}} = e^{\frac{(n-1)(n-2)}{2} \ln 3}$ and since

$$\lim_{n \rightarrow \infty} \left(\frac{(n-1)(n-2)}{2} \ln 3 - ((1 + \log_{\bar{d}} n)n \ln r + kn) \right) = \infty,$$

we conclude that $e^{kn} \frac{|q_{\text{ir}}(n)|}{|\succ_{\text{ir}}(n)|} \rightarrow 0$ as $n \rightarrow \infty$.

Proof of Theorem 6.

Let \succ on X be a rational preference with n singleton indifference classes and fix some discriminatory capacity d . We recursively define a d -checklist $\widehat{C} = (\widehat{C}_1, \dots, \widehat{C}_{\lceil \log_d n \rceil})$ that leads to \succ as follows.

Set $\widehat{C}_0 = \emptyset$. Given the binary relation \widehat{C}_i on X , let $\widehat{\mathcal{I}}_i$ denote the partition of X whose cells are the \widehat{C}_i -equivalence classes. So in particular $\widehat{\mathcal{I}}_0 = \{X\}$. Given the partitions $\widehat{\mathcal{I}}_0, \dots, \widehat{\mathcal{I}}_r$ of X , we let $\widehat{\mathcal{J}}_r = \bigvee_{j=0, \dots, r} \widehat{\mathcal{I}}_j$ denote the coarsest common refinement of $\widehat{\mathcal{I}}_0, \dots, \widehat{\mathcal{I}}_r$. For the recursion, take $\widehat{C}_0, \dots, \widehat{C}_{i-1}$ as given and define for each $J \in \widehat{\mathcal{J}}_{i-1}$ a partition $\{J(1), \dots, J(q)\}$ of J , where $q = \min[d_i, |J|]$, such that (1) each cell contains either $\lfloor \frac{|J|}{q} \rfloor$ or $\lceil \frac{|J|}{q} \rceil$ elements and (2) $J(r) \succ J(s)$ if $r < s$. If $|J| < d_i$, set $J(m) = \emptyset$ for $|J| < m \leq d_i$. For any $k = 1, \dots, d_i$ such that $\bigcup_{J \in \widehat{\mathcal{J}}_{i-1}} J(k) \neq \emptyset$, define a \widehat{C}_i -equivalence class by $\widehat{I}_i(k) = \bigcup_{J \in \widehat{\mathcal{J}}_{i-1}} J(k)$ and then set \widehat{C}_i by $I_i(r) \widehat{C}_i I_i(s)$ if and only if $r < s$ and both $\widehat{I}_i(r)$ and $\widehat{I}_i(s)$ are nonempty. It is easy to check that each $J \in \widehat{\mathcal{J}}_{\lceil \log_d n \rceil}$ is a singleton and that $\widehat{C} = (\widehat{C}_1, \dots, \widehat{C}_{\lceil \log_d n \rceil})$ is a d -checklist that leads to \succ .

The nested division property of ceilings and floors, $\lfloor \frac{\lfloor n/d_i \rfloor}{d_{i+1}} \rfloor = \lfloor \frac{n}{d_i d_{i+1}} \rfloor$ and $\lceil \frac{\lceil n/d_i \rceil}{d_{i+1}} \rceil = \lceil \frac{n}{d_i d_{i+1}} \rceil$, implies that for any $i = 1, \dots, \lceil \log_d n \rceil$ and any $J, K \in \widehat{\mathcal{J}}_i$, $||J| - |K|| \leq 1$. It will be useful for the final paragraph to note that if a partition \mathcal{L} of X contains only cells with cardinalities that differ by at most 1 then \mathcal{L} contains only sets with either $\lfloor \frac{n}{|\mathcal{L}|} \rfloor$ or $\lceil \frac{n}{|\mathcal{L}|} \rceil$ elements.

Let \succ' be an arbitrary preference on X with singleton indifference classes and let $C' = (C'_1, \dots, C'_T)$ be a d -checklist that leads to \succ' . It is sufficient to show that for any $\tau \leq T$ the first τ criteria in C' make no more discriminations than do the first $\min[\tau, \lceil \log_d n \rceil]$ criteria in \widehat{C} . Let τ_{\min} henceforth denote $\min[\tau, \lceil \log_d n \rceil]$. That is, if we define, for an arbitrary checklist C ,

$$D_C(\tau) = \{\{x, y\} \subset X : \exists j \text{ with } 1 \leq j \leq \tau \text{ such that either } x C_j y \text{ or } y C_j x\},$$

it is sufficient to show that $|D_{\widehat{C}}(\tau_{\min})| \geq |D_{C'}(\tau)|$ for any $\tau \leq T$.

Before proceeding, define for any partition \mathcal{L} of X the set

$$D_{\mathcal{L}} = \{\{x, y\} \subset X : \exists I, J \in \mathcal{L} \text{ such that } x \in I, y \in J, \text{ and } I \neq J\}.$$

Given criteria (C_1, \dots, C_t) and the partition \mathcal{I}_j of X that consists of the equivalence classes of C_j , we have $\mathcal{J}_t = \bigvee_{j=1, \dots, t} \mathcal{I}_j$ as usual. So $D_{\mathcal{J}_t}$ is well-defined. Since x and y are in different cells of \mathcal{J}_t only if $x C_j y$ for some $j \in \{1, \dots, t\}$, $|D_{\mathcal{J}_t}| \geq |D_{C'}(t)|$. Since inequality can occur only if some C_j is incomplete, for any initial segment $\widehat{C}_1, \dots, \widehat{C}_i$ of the criteria in the \widehat{C} we defined that leads to \succ , we have $|D_{\widehat{\mathcal{J}}_i}| = |D_{\widehat{C}}(i)|$. Notice also that if each C_j in (C_1, \dots, C_t) has at most d_j equivalence classes then \mathcal{J}_t can have at most $\min[d_1 \cdots d_t, n]$ cells but $\widehat{\mathcal{J}}_i$ has exactly $\min[d_1 \cdots d_i, n]$ cells.

To get our result, fix some $\tau \leq T$. For $i = 1, \dots, \tau$, let \mathcal{I}'_i denote the partition formed by the equivalence classes of C'_i . Beginning with $\mathcal{J}'_{\tau} = \bigvee_{j=1, \dots, \tau} \mathcal{I}'_j$, we define a new partition by taking any pair $I, J \in \mathcal{J}'_{\tau}$ of maximal and minimal cardinality and replacing them with two sets that partition $I \cup J$ into sets with cardinalities $\left\lfloor \frac{|I \cup J|}{2} \right\rfloor$ and $\left\lceil \frac{|I \cup J|}{2} \right\rceil$. By iterating this operation, we arrive at a partition \mathcal{K} such that if $K, J \in \mathcal{K}$ then $||K| - |J|| \leq 1$ and where $|\mathcal{K}| = |\mathcal{J}'_{\tau}|$. It is easy to confirm that when this operation is applied to \mathcal{L}_1 to generate \mathcal{L}_2 , we have $|D_{\mathcal{L}_2}| \geq |D_{\mathcal{L}_1}|$. So $|D_{\mathcal{K}}| \geq |D_{\mathcal{J}'_{\tau}}|$. As noted earlier, each $K \in \mathcal{K}$ has $\left\lfloor \frac{n}{|\mathcal{K}|} \right\rfloor$ or $\left\lceil \frac{n}{|\mathcal{K}|} \right\rceil$ elements and each $J \in \widehat{\mathcal{J}}_{\tau_{\min}}$ has $\left\lfloor \frac{n}{|\widehat{\mathcal{J}}_{\tau_{\min}}|} \right\rfloor$ or $\left\lceil \frac{n}{|\widehat{\mathcal{J}}_{\tau_{\min}}|} \right\rceil$ elements. Hence, since $|\mathcal{K}| \leq \min[d_1 \cdots d_{\tau}, n]$ and $|\widehat{\mathcal{J}}_{\tau_{\min}}| = \min[d_1 \cdots d_{\tau}, n]$, we have $|K| \geq |J|$ for any $K \in \mathcal{K}$ and $J \in \widehat{\mathcal{J}}_{\tau_{\min}}$. It follows that $|D_{\widehat{C}}(\tau)| = |D_{\widehat{\mathcal{J}}_{\tau_{\min}}}| \geq |D_{\mathcal{K}}|$. Since $|D_{\mathcal{J}'_{\tau}}| \geq |D_{C'}(\tau)|$, we have $|D_{\widehat{C}}(\tau_{\min})| \geq |D_{C'}(\tau)|$ as desired.

As for the second sentence of the theorem, Theorem 2 implies that for any \succ' with $n \geq 5$ that is not rational there is an admissible d such that \succ' is not the outcome of a quick d -checklist. By replacing enough coordinates d_i where $i > \lceil \log_d n \rceil$ with 3's, we arrive at a \widetilde{d} such that \succ' must be the outcome of a \widetilde{d} -checklist. So there is a \widetilde{d} -checklist $C' = (C'_1, \dots, C'_T)$ that leads to \succ' where $t_{C'}(x, y) = T$ for some $\{x, y\} \subset X$ and $T > \lceil \log_{\widetilde{d}} n \rceil$. Since $t_{\widehat{C}}(x, y) < T$ for all $\{x, y\} \subset X$ and $|D_{\widehat{C}}(\tau_{\min})| \geq |D_{C'}(\tau)|$ for any $\tau < T$, \succ must have strictly smaller expected run time than \succ' . ■

References

- [1] Abramowitz, M. and I. Stegun (eds.), 1972, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover: New York.
- [2] Apestequia J. and M. Ballester, 2009a, 'Choice by sequential procedures' Working Paper, Department of Economics, Universitat Pompeu Fabra.
- [3] Apestequia J. and M. Ballester, 2009b, 'The computational complexity of rationalizing behavior' *Journal of Mathematical Economics*, forthcoming.
- [4] Chipman, J., 1960, 'The foundations of utility' *Econometrica* 28: 193-224.

- [5] Chipman, J., 1971, ‘On the lexicographic representation of preference orderings’ in J. Chipman, L. Hurwicz, M. Richter, and H. Sonnenschein (eds.) *Preferences, Utility, and Demand*. Harcourt: New York.
- [6] Cuesta Dutari, N., 1943, ‘Teoria decimal de los tipos de orden’ *Revista Matematica Hispano-Americana* 3: 186-205, 242-268.
- [7] Cuesta Dutari, N., 1947, ‘Notas sobre unos trabajos de Sierpinski’ *Revista Matematica Hispano-Americana* 7: 128-131.
- [8] Fishburn, P., 1970, *Utility Theory for Decision Making*. Wiley: New York.
- [9] Fishburn, P., 1974, ‘Lexicographic orders, utilities and decision rules: a survey’ *Management Science* 20: 1442-1471.
- [10] Goldberg, K., M. Newman, and E. Haynsworth, 1972, ‘Combinatorial analysis’ in M. Abramowitz and I. Stegun (eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover: New York.
- [11] Harary, F., 1969, *Graph Theory*. Addison-Wesley, Reading, MA.
- [12] Kalai, G., A. Rubinstein, and R. Spiegler, 2002, ‘Rationalizing choice functions by multiple rationales’ *Econometrica* 70: 2481-88.
- [13] Knuth, D., 1997a, *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*. Addison-Wesley: Reading MA.
- [14] Knuth, D., 1997b, *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms*. Addison-Wesley: Reading MA.
- [15] Mandler, M., P. Manzini, and M. Mariotti, 2008, ‘A million answers to twenty questions: choosing by checklist’ available at <http://personal.rhul.ac.uk/uhte/035/million.answers.to.twenty.questions.pdf>
- [16] Mandler, M., 2009, ‘Rationality and the speed of decision-making (extended abstract)’ in A. Heifetz (ed.) *Proceedings of the 12th Conference on Theoretical Aspects of Rationality and Knowledge (TARK XII)*. ACM Digital Library. Available at <http://personal.rhul.ac.uk/uhte/035/Mandler.TARK.2009.pdf>
- [17] Manzini, P. and M. Mariotti, 2007, ‘Sequentially rationalizable choice’ *American Economic Review* 97: 1824-1839.
- [18] Rubinstein, A., 1996, ‘Why are certain properties of binary relations relatively more common in natural language?’ *Econometrica* 64: 343-355.
- [19] Salant, Y., 2003, ‘Limited computational resources favor rationality’ mimeo.
- [20] Simon, H., 1990, ‘Bounded rationality’ in J. Eatwell, M. Milgate and P. Newman (eds.) *The New Palgrave: Utility and Probability*. Macmillan, London.
- [21] Tversky, T., and I. Simonson, ‘Context-dependent preferences’ *Management Science* 39: 1179-1189.