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# Indifference and incompleteness distinguished by rational trade ☆

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ABSTRACT

We use an agent's strict preferences to define indifference and incompleteness relations that identify the sequences of trades that are rational to undertake. If an agent makes sequences of trades of options labeled indifferent, the agent will never be led to an inferior outcome, but trades of options where no preference judgments obtain can lead to diminished welfare. For one-shot choices, in contrast, there can be no behavioral distinction between indifference and incompleteness. Applications include: an isomorphism for incomplete preferences that indicates when weak and strict preferences contain interchangeable information, a characterization of the (possibly incomplete) preference relations consistent with a one-shot choice function, and an equivalent definition of incompleteness that relies on the philosophical theory of incommensurability.



## 1. Introduction

It is often said that indifference and incompleteness are behaviorally indistinguishable. Both an agent who is indifferent between two options and an agent who cannot rank the options can rationally choose either one. And if indifference and incompleteness lead to the same choice behavior, the entire concept of incompleteness would seem to be superfluous. The trouble with these views is that they look at each of an agent's choices in isolation. When decisions are strung together, the terrain looks different. Indifference and incompleteness relations can then demarcate which sequences of trades are rational: the options that we will label 'indifferent' can always be traded in either direction without leading an agent to a dispreferred final outcome, while trades of the options we label 'unranked' can make an agent worse off. (We use 'unranked' as the adjective to describe incompleteness: $x$ and $y$ are unranked if $x$ is not weakly preferred to $y$ and $y$ is not weakly preferred to $x$.)

We call these indifference and incompleteness relations 'behavioral' because they are determined solely by properties of an agent's strict preferences $\succ$, which can be inferred from choice behavior, and because they make testable predictions. Let the notation $x \sim^* y$ mean not $x \succ y$ and not $y \succ x$. It can be inappropriate to view $\sim^*$ as an indifference relation; even when $\succ$ is transitive, the relations $\succ$ and $\sim^*$ taken together need not form a transitive weak preference relation. Instead, we define the behavioral indifference relation $\sim_B$ by $x \sim_B y$ if and only if, for every option $z$, $x \sim^* z \Leftrightarrow y \sim^* z$. The relation $\sim_B$ is a well-established equivalence relation for strict partial orders (see, e.g., Fishburn, 1970) but has not been tied to a theory of choice behavior. Our behavioral incompleteness relation, $\perp_B$, holds for the remaining pairs in $\sim^*$: $x \perp_B y$ if and only if $x \sim^* y$ and not $x \sim_B y$.

☆ I am grateful to an advisory editor and two referees for several valuable suggestions. I wrote this paper following heated arguments with Marco Mariotti about the relative merits of taking weak and strict preferences as primitive. In light of Section 5 and Observation 2, I no longer see any merit in either of our positions but my conversations with Mariotti were indispensable. We have agreed not to argue any further about the subject.

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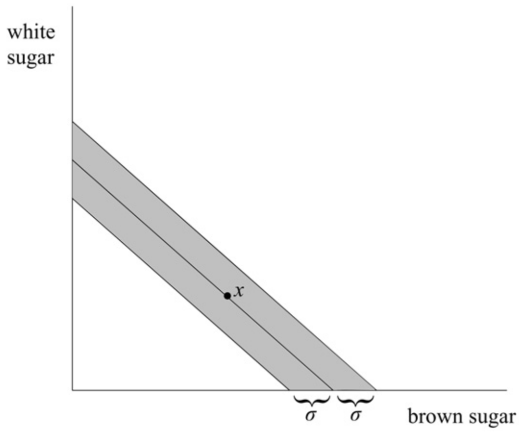


**Fig. 1.** Bundles that are $\sim_B$ related (the line through $x$) and bundles that are $\perp_B$ related to $x$ (shaded) in a semiorder.

The following examples illustrate the implications of $\sim_B$ and $\perp_B$ for rational trade.

**Example 1** *(A semiorder).* In the canonical semiorder (Luce, 1956), an agent chooses over the (positive, real) number of grains of sugar in a cup of coffee, and prefers more sugar to less so long as the difference in the number of grains is greater than some number $\sigma > 0$, the agent's capacity to detect sweetness. When two cups of coffee contain sugar levels that differ by $\sigma$ or less, the cups have the same taste. So, for nonnegative real numbers $x$ and $y$, the agent's strict preferences are given by $x \succ y$ if and only if $x > y + \sigma$. Since $\sim_B$ is uninteresting in this case, we add the following twist: let there be two colors of sugar (white and brown) that are equally sweet. So, for any two bundles $x = (x_w, x_b)$ and $y = (y_w, y_b)$ in $\mathbb{R}^2_+$, the agent's strict preferences are given by

$$(x_w, x_b) \succ (y_w, y_b) \quad \Leftrightarrow \quad x_w + x_b > y_w + y_b + \sigma.$$

Suppose the agent makes a one-shot binary choice that determines consumption. For a choice between $x$ and $y$ with $x_w + x_b > y_w + y_b + \sigma$, the agent's only rational decision is to select $x$. But if $|(x_w + x_b) - (y_w + y_b)| \leqslant \sigma$, in which case $x \sim^* y$, then choosing either $x$ or $y$ will leave the agent with a bundle that is not $\succ$-worse than the other option: either decision is rational.

The relation $\sim_B$ nevertheless singles out some pairs with $x \sim^* y$ for special treatment: $x \sim_B y \Leftrightarrow (x_w + x_b = y_w + y_b)$ (see Fig. 1).[1] From the vantage point of the agent's sensory experience, this may seem like an arbitrary classification. But if the agent faces a sequence of decisions prior to consumption rather than a one-shot choice, $\sim_B$ gives the agent the right advice about which bundles to treat as interchangeable. If at any stage the agent agrees to trade $y$ for $x$ if and only if either $x \succ y$ or $x \sim_B y$, then the agent's final outcome cannot be $\succ$-worse than any bundle the agent had a chance to consume: if the agent makes a chain of trades $x(1) \to x(2) \to \cdots \to x(n)$, the total $x_b(i) + x_w(i)$ will be weakly increasing in $i$. But if the agent agrees to trade in either direction any pair with $x \perp_B y$ then there will be a chain of trades that moves the agent to an inferior outcome. Since the agent could agree to a chain that lowers $x_w(i) + x_b(i)$ by up to $\sigma$ at each step, just two trades of $\perp_B$-related options can lead the agent to a $\succ$-worse outcome. The label 'indifferent' for $\sim_B$-related bundles therefore fits: any trade in either direction of a pair with $x \sim_B y$ cannot possibly lead to harm, hence a rational agent can agree to such trades, while a trade in one direction or the other of a pair with $x \perp_B y$ can do harm, hence a rational agent should not treat such bundles as fully interchangeable.

**Example 2** *(Status quo bias).* An agent with status quo bias demands a premium to trade away from the status quo. Suppose there are two goods and that the bundle $x$ is the status quo. An agent with the strict preferences $\succ$ then displays status quo bias if $B(x) = \{y \in \mathbb{R}^2_+ \colon\ y \succ x\}$ has a kink at $x$: the minimum quantity of good 2 the agent will accept in compensation for a small loss of good 1 is strictly larger than the amount of 2 the agent will pay for a small gain of 1. If one of the two goods is money, this premium is called a *willingness-to-pay/willingness-to-accept* disparity. Suppose the status quo bias is systematic: for any $x$, let $B(x)$ have a kink at $x$. While $B$ sets will then overlap (see Fig. 2), this fact is consistent with the transitivity of $\succ$ since any weak preference relation that lies behind $\succ$ can be incomplete (Mandler, 2004).[2] It is therefore

[1] That $x_w + x_b = y_w + y_b$ implies $x \sim_B y$ follows immediately from the definitions of $\succ$ and $\sim_B$. To show that $x_w + x_b \neq y_w + y_b$ implies not $x \sim_B y$, observe that if $x_w + x_b > y_w + y_b + \sigma$ then $x \succ y$. The other possibility (up to an interchange of $x$ and $y$) is for $x_w + x_b > y_w + y_b$ and $x_w + x_b \leqslant y_w + y_b + \sigma$ to hold. Then, for $z$ such that $z_w + z_b$ lies in the open interval $(y_w + y_b + \sigma, x_w + x_b + \sigma)$, we have $z_w + z_b > y_w + y_b + \sigma$, $z_w + z_b < x_w + x_b + \sigma$, and $x_w + x_b < z_w + z_b + \sigma$. So $z \succ y$ and $z \sim^* x$, hence not $x \sim_B y$.

[2] The connection between incomplete preferences and status quo bias dates at least to Bewley (1986). See also Mandler (2005) and Masatlioglu and Ok (2005).

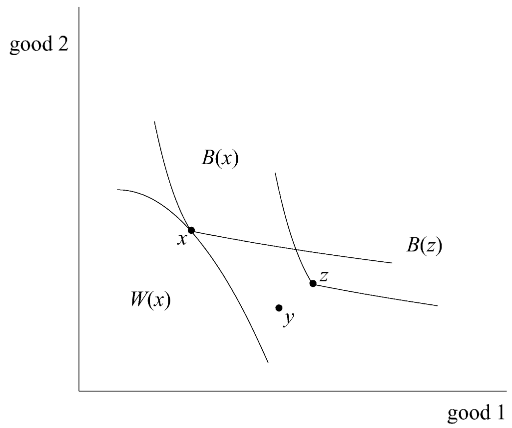


**Fig. 2.** An irrational chain of trades, $z \to x \to y$, with status quo bias.

permissible to assume that $\succ$ is transitive, and we assume that $\succ$ is convex and monotone as well. One can then show that in some neighborhood of $x$, the closures of the sets $B(x)$ and $W(x) = \{y\colon\ x \succ y\}$ will not intersect except possibly at $x$. Hence we can find $y$ and $z$ that are not in $B(x)$ or $W(x)$ and such that $z \succ y$. For any such $y$ and $z, x \perp_B y$ and $x \perp_B z$.[3] Since the chain of trades $z \to x \to y$ moves the agent from a better to a worse option, $\perp_B$ identifies pairs that should not be treated as interchangeable, just as in Example 1. In a neighborhood of $x$, there will in fact be no bundles $\sim_B$-related to $x$ except $x$ itself (Mandler, 2008). As we will see (Theorem 3), it follows that for *any* $y \notin B(x) \cup W(x)$ near $x$ there is a third option that with $x$ and $y$ forms a chain of two $\perp_B$ trades that leaves the agent worse off.

In our abstract account of the implications of $\sim_B$ and $\perp_B$ for rational trade, an agent will face a linked sequence of choice sets. Rationality demands that the agent take advantage of every opportunity to trade to strictly preferred options. If the agent in addition makes any trades among $\sim_B$-related options then the agent cannot be led through time to a dispreferred outcome (Theorem 1). On other hand, if there are $\perp_B$-related options, then there are always $\perp_B$ trades that leave the agent worse off (Theorem 2). For rational agents, $\sim_B$ and $\perp_B$ therefore lead to different predictions of behavior. It turns out that if the agent is willing to trade some pair of $\perp_B$-related options, then the agent is always just one further trade away from completing a chain of trades that leaves the agent worse off (Theorem 3). Examples 1 and 2 illustrate this feature: in both cases, it takes only two $\perp_B$ trades for the agent to end up with a strictly inferior outcome.

The relation $\sim_B$ is defined from an agent's strict preference relation $\succ$ but other indifference relations $\sim$ besides $\sim_B$ are consistent with $\succ$ (i.e., $\succ \cup \sim$ will form a bona fide weak preference relation). An agent may therefore have an underlying set of indifference judgments $\sim$ that does not coincide with $\sim_B$; specifically, $\sim_B$ can declare more options equivalent than $\sim$ does. So, while if $x \sim_B y$ then an exchange of $x$ and $y$ can do no harm, an agent could nevertheless insist that $x$ and $y$ are not psychologically interchangeable. But it still makes sense from the ordinal or behavioral perspective to identify $\sim_B$ as an agent's indifference relation. Even if agents mentally distinguish between $\sim_B$-related options, there is never a preference cost to treating these options as interchangeable. Since the mental distinction need not show up in behavior, an agent can treat $\sim_B$-related options 'as if' he or she were indifferent.

It nevertheless proves valuable to identify the entire set of indifference relations consistent with an agent's strict preference judgments (Theorem 4). We can then characterize those weak preferences whose indifference relations do coincide with $\sim_B$ (Theorem 5), and thereby generalize to incomplete preferences the well-known isomorphism between complete and transitive weak preferences on one hand and negatively transitive and asymmetric strict preferences on the other (Observation 2). The generalized isomorphism reports when it is immaterial if one takes weak or strict preference as primitive and how to translate between theorems on strict preferences and theorems on incomplete weak preferences.

If only one-shot rather than sequential choices are observable, one can make the case that each of the indifference relations $\sim$ consistent with $\succ$ is as legitimate as any of the others. Adopting this point of view, we address the classical question of revealed preference analysis: when can a one-shot choice function be rationalized by a (possibly incomplete) preference relation? As we will see, one-shot revealed preference analysis cannot go beyond recovery of an agent's strict preferences $\succ$. Any of the $\sim$ relations consistent with $\succ$ are therefore reasonable candidates for indifference: nothing in traditional revealed preference analysis of one-shot choices can distinguish among them. To single out $\sim_B$ as the right definition of indifference, we must appeal to the fact that there is no welfare cost to *sequentially* trading $\sim_B$-related options.

Eliaz and Ok (2006) develop an alternative account of how to distinguish indifference from incompleteness; they take a one-shot choice function as primitive and ask when a choice function can be rationalized by an incomplete preference relation. Their behavioral definitions differ from ours: they infer indifference or incompleteness between two options only if

[3] These conclusions follow from $x \sim^* y$, $x \sim^* z$, and $z \succ y$ since then it is not the case that $x \sim^* z \;\Rightarrow\; y \sim^* z$ and not the case that $x \sim^* y \;\Rightarrow\; z \sim^* y$.

whenever both options are available and one is selected then so is the other. Since for one-shot choices this inference rule cannot distinguish indifference from incompleteness, we might as well let agents make arbitrary one-shot choices among alternatives that are not strictly ranked. Still, the accounts connect: if we translate Eliaz and Ok into our terminology they contend that $\sim_B$ is the appropriate indifference relation to use in rationalizations. Since a lot of other indifference relations will do just as well for one-shot choices, an argument in favor of $\sim_B$ is needed; our answer is that $\sim_B$ identifies the options that are never irrational to trade sequentially.

A final motivation for $\sim_B$ stems from philosophical analyses of incommensurability, most prominently Raz (1986). Raz argued that $x$ and $y$ are unranked if $x \sim^* y$ and if in addition a slight improvement to $x$ or to $y$ does not break $\sim^*$ and lead strict preference to hold. In the typical philosophical example, $x$ and $y$ represent two starkly different modes of life, say life as artist vs. life as a lawyer, but Raz's relation can be applied to any setting. As we will see, Raz's definition of incompleteness and his implicit definition of indifference coincide with $\perp_B$ and $\sim_B$.

## 2. Notation

Fix a set of options $X$. $\mathcal{F}_X$ will denote the set of nonempty finite subsets of $X$.

A *binary relation* $R$ on $X$ is a subset of $X \times X$; $xRy$ means $(x, y) \in R$ and we then say $x$ and $y$ are *R-related*. The binary relation $R$ is *transitive on* $Y \subset X$ if and only if, for all $x, y, z \in Y$, $xRy$ and $yRz$ imply $xRz$; *transitive* if and only if $R$ is transitive on $X$; *negatively transitive* if and only if the relation $T$ defined by $xTy \Leftrightarrow$ (not $yRx$) is transitive; *reflexive* if and only if, for all $x \in X$, $xRx$; *symmetric* if and only if for all $x, y \in X$, $xRy$ implies $yRx$; *asymmetric* if and only if, for all $x, y \in X$, $xRy$ implies not $yRx$; an *equivalence relation* if and only if $R$ is reflexive, symmetric, and transitive. Given a binary relation $R$, its *symmetric part* is the relation $T$ defined by $xTy \Leftrightarrow$ ($xRy$ and $yRx$) and its *asymmetric part* is the relation $W$ defined by $xWy \Leftrightarrow$ ($xRy$ and not $yRx$).

A binary relation $\succ$ is a *strict preference* if and only if $\succ$ is asymmetric and transitive. We define the relation $\not\succ$ by $x \not\succ y \Leftrightarrow$ not $x \succ y$ and the relation $\sim^*$ by $x \sim^* y \Leftrightarrow (x \not\succ y$ and $y \not\succ x)$.

A binary relation $\succcurlyeq$ is a *weak preference* if and only if $\succcurlyeq$ is reflexive and transitive. Given a weak preference $\succcurlyeq$, we let $\sim$ denote the symmetric part of $\succcurlyeq$. Less conventionally, we define the relation $\perp$ by $x \perp y \Leftrightarrow$ (not $x \succcurlyeq y$ and not $y \succcurlyeq x$). We will say that $x$ and $y$ are *unranked* when $x \perp y$ and *indifferent* when $x \sim y$. If the strict preference $\succ$ is the asymmetric part of $\succcurlyeq$ then, with this language, $x \sim^* y$ if and only if $x$ and $y$ are either unranked or indifferent.

## 3. Indifference and incompleteness distinguished by sequential choice

We take as given some agent's strict preference $\succ$ on a set of options $X$. In the next section we discuss when an agent's choice behavior will reveal $\succ$. There is no harm in imagining that behind $\succ$ lies a set of weak preference judgments $\succcurlyeq$ where the asymmetric part of $\succcurlyeq$ is $\succ$.

We define behavioral indifference and incompleteness relations (with $B$ subscripts that stand for 'behavioral') and then examine the implications of these relations for choice behavior. Recall that $\sim^*$ is defined by $x \sim^* y \Leftrightarrow (x \not\succ y$ and $y \not\succ x)$.

**Definition 1.** The binary relation $\sim_B$ is defined by $x \sim_B y$ if and only if, for every $z \in X$, $x \sim^* z \Leftrightarrow y \sim^* z$.

Given that $\sim^*$ is reflexive, $x \sim_B y$ implies $x \sim^* y$. One may also readily show that $\sim_B$ is an equivalence relation. For a concrete illustration of $\sim_B$, see the semiorder in Example 1.

The relation $\sim_B$ is a well-known equivalence relation in decision theory, where sometimes it is designated $\approx$, but it has lacked motivation as a definition of indifference. We offer a rationale for $\sim_B$ by using it to distinguish between rational and irrational trades, and give several characterizations of $\sim_B$ that can serve as richer definitions.

**Definition 2.** The binary relation $\perp_B$ is defined by $x \perp_B y$ if and only if $x \sim^* y$ and not $x \sim_B y$.

Thus, for any pair $(x, y)$ one of four mutually exclusive possibilities obtains: $x \succ y$, $y \succ x$, $x \sim_B y$, or $x \perp_B y$.

**Example 3.** Suppose that the strict preference $\succ$ is negatively transitive. That is, $\preccurlyeq \equiv \not\succ$ is transitive and, by the asymmetry of $\succ$, complete. Letting $\sim$ be the symmetric part of $\preccurlyeq$, we have $\sim = \sim^*$. To see that $\sim_B = \sim^*$, suppose first that $x \sim^* y$. The transitivity of $\sim$ and hence $\sim^*$ then gives $x \sim^* z \Leftrightarrow y \sim^* z$ for all $z$. Conversely, if $x \sim^* z \Leftrightarrow y \sim^* z$ for all $z$ then the reflexivity of $\sim^*$ gives $x \sim^* y$. Hence $\sim_B = \sim^*$, and thus $\sim_B$ identifies the traditional concept of indifference.

We turn to the implications of $\sim_B$ and $\perp_B$ for sequential decision-making. It is not enough to consider the chains of trades $x(1) \to x(2) \to \cdots \to x(n)$ that an agent will agree to, as we did in the examples in the introduction; to vindicate the rationality of a trading strategy we have to consider the trades that an agent refuses. Suppose an agent faces a sequence of choice sets $(S_1, S_2, \ldots)$ with each $S_i \subset X$. In each round $i$, the agent makes a set of selections from $S_i$. One of the alternatives in $S_i$ is a 'holding' $h_i$ that is either drawn from the choices the agent made in the previous round or, in the case of $S_1$, an initial endowment. The agent thus has the opportunity in round $i$ to trade away a previous selection and acquire a new option. The agent can also condition his selections from $S_i$ on his current holding $h_i$; indeed we will see that it is imperative that the agent does so.

The agent does not know in advance when the sequence of trading opportunities will come to an end, but when the sequence does end, say at $n$, the agent will consume one of the options chosen from $S_n$. Since the agent can guarantee himself any $x \in \bigcup_{i \leqslant n} S_i$ simply by choosing $x$ when it is first available and then refusing to trade to any alternative, a rational agent should not end up with an option that is $\succ$-worse than any option available at or before $n$.

Since the agent may decide to condition his selections on his current holding, we cannot represent the agent's behavior by a classical one-shot choice function that specifies one set of selections for every choice set $S$. Letting $S$ and $h$ denote generic $S_i$ and $h_i$, we instead denote the agent's selections from $S$ when the agent's holding is $h$ by $c(S, h)$ and we call the map $c$ a *sequential choice function*.

Formally, let $\mathcal{F}_X$ be the finite subsets of $X$. A sequential choice function is a map $c$ with the domain $\Sigma \equiv \{(S, h)\colon\ S \in \mathcal{F}_X$ and $h \in S$ or $h = \emptyset\}$ that assigns to any $(S, h) \in \Sigma$ a nonempty $c(S, h) \subset S$. We allow $h = \emptyset$ to cover agents who have no initial endowment.

As with a classical choice function, a sequential choice function selects a set of options from each $S$. Only one of the these options becomes the holding in the next round, just as in a classical choice function only one selected item becomes the agent's actual consumption. Alternatively we could suppose that the agent chooses just one option but that this selection can vary across the instances at which the agent faces $(S, h)$; $c(S, h)$ would then be the set of the agent's possible selections. As long as each option in $c(S, h)$ is a possible selection whenever the agent faces $S$, our analysis would remain effectively unchanged.

An agent *trades* $x$ for $y$ when $x \in S_i$ and $y \in c(S_i,\ h_i)$: the agent has the right to choose $x$ by itself, but by letting $y \in c(S_i, h_i)$ the agent can end up with $y$ or in the next round face a $S_{i+1}$ that contains $y$ but not $x$. If $\{x, y\} \subset c(S_i, h_i)$ then the agent trades $x$ for $y$ and $y$ for $x$; we then say that the agent *freely exchanges* $x$ and $y$. If an agent trades $x_1$ for $x_2$, then trades $x_2$ for $x_3, \ldots,$ then trades $x_{n-1}$ for $x_n$, but $x_1 \succ x_n$, the agent makes an irrational chain of trades:

**Definition 3.** A sequential choice function $c$ *makes an irrational chain of trades* if and only if there exist finite choice sets $(S_1, \ldots, S_n)$ and $(h_1, \ldots, h_n, x, y)$ such that $h_i \in c(S_{i-1}, h_{i-1})$ and $h_i \in S_i$ for $i = 2, \ldots, n$, $x \in S_1$, $y \in c(S_n, h_n)$, and $x \succ y$. A sequential choice function is *sequentially rational* if and only if it does not make an irrational chain of trades.[4,5]

One requirement of sequential rationality is that a $c$ must select only undominated options: if $x \in c(S, h)$ and $y \in S$ then it must not be the case that $y \succ x$. The main tradition of one-shot revealed preference theory goes further and assumes that a $c$ will select *all* undominated options: if $x \in S$ and $y \nsucc x$ for all $y \in S$ then $x \in c(S, h)$. An agent with such a $c$ freely exchanges any pair of undominated options. But selecting all undominated options can be a poor way to make a sequence of decisions, as the following example illustrates.

**Example 4.** First, let $X = \{x, y, z\}$ and suppose the agent has one pair ordered by strict preference, say $x \succ z$. Let $c$ select all undominated options. Then for $S_1 = \{x, y\}$ we have $c(S_1, h_1) = \{x, y\}$: the agent freely exchanges $x$ and $y$. And if $S_2 = \{y, z\}$ and $h_2 = y$, then $c(S_2, h_2) = \{y, z\}$: the agent freely exchanges $y$ and $z$. We conclude that $c$ fails to be sequentially rational since $x \in S_1$, $h_2 \in c(S_1, h_1)$, $h_2 \in S_2$, $z \in c(S_2, h_2)$, and $x \succ z$. If instead the agent selects either $c(S_1, h_1) = \{x\}$ or $c(S_2, h_2) = \{y\}$, then there cannot be an irrational chain in the first two rounds of choice. Since $x \perp_B y$ and $z \perp_B y$, we see that freely exchanging $\perp_B$-related options can be welfare-diminishing.

Not surprisingly, if the strict preference $\succ$ is negatively transitive, or equivalently $\preccurlyeq \equiv \nsucc$ is complete and transitive, then the $c$ that selects all undominated options is sequentially rational (for any $X$). Moreover, since $\sim_B = \sim^*$ for any negatively transitive $\succ$ (Example 3), trades of $\sim_B$-related options do no harm. The proof of sequential rationality is simple. Suppose we are given $(S_1, \ldots, S_n)$ and $(h_1, \ldots, h_n, x, y)$ such that $h_i \in c(S_{i-1}, h_{i-1})$ and $h_i \in S_i$ for $i = 2, \ldots, n$, $x \in S_1$, and $y \in c(S_n, h_n)$. If $n = 1$ then the fact that $c$ selects all undominated options implies $x \preccurlyeq y$. When $n > 1$, the same fact implies that $x \preccurlyeq h_2$, $h_{i-1} \preccurlyeq h_i$ for $i = 3, \ldots, n$, and $h_n \preccurlyeq y$. Since $\preccurlyeq$ is transitive, $x \preccurlyeq y$.

Example 4 repeats the pattern found in Examples 1 and 2 where the free exchange of $\perp_B$-related options generates irrational chains of trades but free exchanges of $\sim_B$-related options do not. To develop this argument, we consider the sequential choice functions that lie at the opposite end of the spectrum from selecting all undominated options. First, suppose that $c$ cautiously requires that some selection dominates the agent's holding $h$ (when the agent has a holding):

[4] For the modifier 'rational' to fit, we must assume that if an agent changes $c$ to $c'$ then the new alternatives offered at any $S_i$ do not change. That is, if an agent uses $c$ and faces a sequence of choice sets $\langle S_i \rangle$ and a sequence of holdings $\langle h_i \rangle$, where $h_i \in c(S_{i-1}, h_{i-1})$ and $h_i \in S_i$ for $i = 2, \ldots, n$, then a change to $c'$ must cause the agent to face sequences $\langle \hat{S}_i \rangle$ and $\langle \hat{h}_i \rangle$ such that $\hat{S}_i \setminus \{\hat{h}_i\} = S_i \setminus \{h_i\}$, $\hat{h}_i \in c(\hat{S}_{i-1}, \hat{h}_{i-1})$, and $\hat{h}_i \in \hat{S}_i$ for $i = 2, \ldots, n$, and $\hat{S}_1 = S_1$. An agent will then have no incentive to choose counterpreferentially in order to face different sets of alternatives. Since this requirement matters only for the rationality interpretation (no result hinges on it), we let it remain implicit.

[5] We need consider only those irrational chains that run from period 1 to some $n$: a chain that begins at some $k \neq 1$ can be relabeled to run from 1 to $n$. For similar reasons, we do not need to consider $(S_1, \ldots, S_n; h_1, \ldots, h_n)$ such that $h_i \notin c(S_{i-1}, h_{i-1})$ for some $i \in \{2, \ldots, n\}$.

**Definition 4.** The sequential choice function $c$ is *cautious* if and only if, for all $(S,h) \in \Sigma$,

$$h \notin c(S,h) \text{ implies either } x \succ h \text{ for some } x \in c(S,h) \text{ or } h = \emptyset.$$

Second, suppose that $c$ always selects a single undominated option. We call a $c$ with these two features *trade-avoiding*: a trade-avoiding $c$ begins by selecting an undominated option and then trades away a holding $h$ only when some alternative that is $\succ$-superior to $h$ becomes available. Such a $c$ keeps to a holding unless offered an unambiguously superior alternative and is a minimal trading strategy in that it chooses a single option and then makes the fewest possible changes in $h$. As long as there is some pair with $x \sim^* y$, there will be multiple trade-avoiding $c$'s. It is easy to confirm that any trade-avoiding $c$ is sequentially rational.[6]

So we have two polar opposite ways to make sequential decisions: the maximal strategy of freely exchanging all undominated options, which can lead to a failure of sequential rationality, and the minimal strategy of selecting singletons and only switching one's holding when offered a superior option, which will ensure sequential rationality. We now pin down where exactly the dividing line between sequential rationality and irrationality falls.

Sequential choice functions can depart from the minimal end of the spectrum in two ways: by selecting more than one option and by dropping the caution requirement that some selection dominates the agent's holding. We begin with the first departure. Let $V$, which will denote a binary relation on $X$, be the pairs of options that the agent will freely exchange (subject to the constraint of not selecting any dominated option). $V$ thus contains 'candidates' for indifference and we examine the sequential rationality of freely exchanging these candidates.

**Definition 5.** The sequential choice function $c$ *makes $V$ exchanges* if and only if, for all $(S,h) \in \Sigma$, the set $C = c(S,h)$ satisfies

(1) $C$ contains only undominated options: $(x \in C \text{ and } y \in S) \Rightarrow y \not\succ x$;
(2) $C$ contains only $V$-related options: $(x, y \in C \text{ and } x \neq y) \Rightarrow (xVy \text{ or } yVx)$[7]

and there is no larger set $C \supsetneq c(S,h)$ with $C \subset S$ that satisfies (1) and (2).

So if $c$ makes $V$ exchanges and (i) $x \in S$, (ii) $x$ and $y$ are $V$-related for all $y \in c(S,h)$, and (iii) $x$ is not dominated by some other option in $S$, then $c$ must also select $x$. Notice that $V$ may contain a $(x,y)$ with $x \succ y$, but condition (1) makes this possibility irrelevant: if $x \succ y$ then any $c$ that makes $V$ exchanges also makes $V \setminus \{(x,y)\}$ exchanges.

We can use Definition 5 to define the polar opposite sequential choice functions discussed earlier: the $c$ that selects all undominated options is the $c$ that makes $\sim^*$ exchanges and a trade-avoiding $c$ is a $c$ that is cautious and makes $\emptyset$ exchanges. Though there is only one $c$ that selects all undominated options, for some $V$'s there can be multiple sequential choice functions that make $V$ exchanges.

The rationality of freely exchanging options $x$ and $y$ hinges on whether $x \sim_B y$ or $x \perp_B y$ obtains:

**Theorem 1.** *If $V$ contains only $\sim_B$-related pairs then any cautious $c$ that makes $V$ exchanges is sequentially rational.*

**Theorem 2.** *If $\perp_B$ is nonempty then there is a $V$ that contains only $\perp_B$-related pairs such that any $c$ that makes $V$ exchanges is not sequentially rational.*

Proofs are in Appendix A.

A pair of options that is $\sim^*$-related will be $\sim_B$-related or $\perp_B$-related but not both. Theorems 1 and 2 come close to but do not quite characterize this partition of $\sim^*$ in terms of rational trades. While Theorem 1 vindicates freely exchanging $\sim_B$-related options, Theorem 2 says only that *some* exchanges of $\perp_B$-related options are irrational. There are in fact sequentially rational $c$'s that freely exchange some $\perp_B$-related pairs.

But we can characterize the danger of $\perp_B$-related options as follows: an agent who freely exchanges a pair of $\perp_B$-related options is always 'one trade away' from making an irrational chain of trades. Curiously, to ensure that agents who do *not* freely exchange $\perp_B$-related options do not have to be one trade away from an irrational chain, we must drop the caution requirement; the added flexibility ensures that there is always some $c$ that freely exchanges a single $\perp_B$-related pair (and arbitrarily many $\sim_B$-related pairs) that is sequentially rational.

**Theorem 3.** *The relation $V$ contains one or more $\perp_B$-related pairs if and only if there exists a pair $(x,y)$ such that no $c$ that makes $\{(x,y)\} \cup V$ exchanges is sequentially rational.*

So, if an agent freely exchanges just one pair $(a,b)$, Theorem 3 implies that $a \perp_B b$ holds if and only if the exchange of one further $\sim^*$-related pair will necessarily expose the agent to an irrational chain of trades. In fact, when an agent freely

[6] Given $(S_1,\ldots,S_n)$ and $(h_1,\ldots,h_n,x,y)$ such that $h_i \in c(S_{i-1},h_{i-1})$ and $h_i \in S_i$ for $i = 2,\ldots,n$, $x \in S_1$, and $y \in c(S_n,h_n)$, we have $y \rhd h_n \rhd \cdots \rhd h_2$, where $a \rhd b$ means either $a \succ b$ or $a = b$. If $x \succ y$ then the transitivity of $\succ$ implies $x \succ h_2$, violating the assumption that $c$ selects undominated options.

[7] We could use weaker definitions of what it means for chosen options to be $V$-related without changing any of our results. For example, we could replace (2) with: if $|C| \geqslant 2$ and $x \in C$ then there exists $z \in C$ such that $xVz$ or $zVx$.

exchanges some $(a, b)$ with $a \perp_B b$ and also exchanges the additional pair provided by Theorem 3 then there will be an irrational chain of trades that consists of just two links.

Theorem 3 thus characterizes when a single pair of options is $\perp_B$-related. Alternatively, we can use Theorem 3 to give a new definition of $\sim_B$:

**Corollary of Theorem 3.** $a \sim_B b$ if and only if for every pair $(x, y)$ there is a $c$ that makes $\{(a, b), (x, y)\}$ exchanges that is sequentially rational.

So options are $\sim_B$-related if and only if freely exchanging these options need not put the agent within one link of an irrational chain of trades.

The caution requirement is absent from Theorem 3 for the sake of the 'if' half of the theorem: there can be a $V$ with no $\perp_B$-related pairs and a $(x, y)$ such that *every* cautious $c$ that makes $V \cup \{(x, y)\}$ exchanges fails to be sequentially rational.[8] The added flexibility that comes with not having to select some option that dominates the holding $h$ ensures that some $c$ can achieve sequential rationality.

To sum up, an agent can make as many $\sim_B$ exchanges as he or she likes, and no further exchange of a pair of $\sim^*$-related options need do any harm, but if the agent treats any $\perp_B$-related pair as interchangeable then there will be just one further exchange of $\sim^*$-related options that will lead the agent to a worse option.

The surprising twist is that if a sequentially rational $c$ makes some set of $\perp_B$ exchanges, it may well be that among the additional exchanges that complete an irrational chain are exchanges of $\sim_B$-related options: making $\perp_B$ exchanges is *so* dangerous that they can lead even $\sim_B$ exchanges to be harmful. The following example illustrates.

**Example 5.** Let $X = \{x, z, a, b\}$ and suppose $\succ$ ranks only one pair, $\succ = \{(z, x)\}$. Then $\sim_B = \{(a, b), (b, a)\} \cup I$, where $I$ is the identity relation $\{(x, x)\colon\ x \in X\}$, and

$$\perp_B = \bigl\{(r, s)\colon\ \bigl(r \in \{x, z\} \text{ and } s \in \{a, b\}\bigr) \text{ or } \bigl(r \in \{a, b\} \text{ and } s \in \{x, z\}\bigr)\bigr\}.$$

Setting $V = \{(z, a), (x, b)\}$, one may readily confirm that a cautious $c$ that makes $V$ exchanges is sequentially rational. But let $c'$ in addition make $\{(a, b)\}$ exchanges: for $V' = V \cup \{(a, b)\}$, suppose $c'$ makes $V'$ exchanges. Set $S_1 = \{z, a\}$, $S_2 = \{a, b\}$, $S_3 = \{x, b\}$, $h_2 = a$, and $h_3 = b$. Then, since $h_2 \in c'(S_1, h_1) = \{z, a\}$, $h_2 \in S_2$, $h_3 \in c'(S_2, h_2) = \{a, b\}$, $h_3 \in S_3$, $z \in S_1$, $x \in c'(S_3, h_3) = \{x, b\}$, and $z \succ x$, $c'$ fails to be sequentially rational.

## 4. Inferring strict preferences from sequential decisions

So far, we have taken an agent's strict preference relation $\succ$ as given. When will $\succ$ be revealed by an agent's choices?

In traditional revealed preference theory, $x$ is revealed to be strictly preferred to $y$ if $y$ is never chosen when $x$ is available. This definition readily translates to our sequential setting: given a sequential choice function $c$, the strict revealed preference relation $\succ_c$ is defined by

$$x \succ_c y \quad \text{if and only if, for all } (S, h) \in \Sigma,\ (x, y \in S) \quad \Rightarrow \quad y \notin c(S, h).$$

To infer an agent's strict preferences from sequential choice data, the agent's strict preference relation $\succ$ must coincide with $\succ_c$.

Sequential rationality by itself does not imply $\succ = \succ_c$. Although sequential rationality ensures that $x \succ y \Rightarrow x \succ_c y$, the reverse implication need not hold; an agent with $x \sim^* y$ could use a sequentially rational $c$ that never chooses $y$ when $x$ is available, thereby introducing into $\succ_c$ additional strict revealed preferences not in $\succ$.

The norm in revealed preference theory is to assume that agents 'always select all undominated options.'[9] Under this assumption, it is easy to confirm that $\succ = \succ_c$. While the inference problem can be formally disposed of in this way, 'always selecting all undominated option' is not a principle of rational self-interest: rationality requires only that agents avoid choosing dominated options. But matters are worse when choice is sequential: then always selecting all undominated options can lead to an irrational chain of trades, as we saw in Example 4.

Luckily there are alternatives. Let us say that a sequential choice function $c$ satisfies *weak revelation* if and only if

$$x \sim^* y \quad \Rightarrow \quad \text{there exists } (S, h) \in \Sigma \text{ such that } y \in c(S, h) \text{ and } x \in S.$$

A $c$ that always selects all undominated options requires, when $x \sim^* y$, that $c(\{x, y\}, h) = \{x, y\}$, while weak revelation imposes only the weaker requirement that there is some $S$ at which $c$ selects $y$ when $x$ is available. It could well be with weak revelation that there is no $S$ at which both $x$ and $y$ are chosen. Observation 1 below is immediate.

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[8] For example, if $z \succ x$, $x \perp_B y$, $z \perp_B y$ and $V = \emptyset$, then any cautious $c$ that makes $\{(x, y)\} \cup V$ exchanges will fail to be sequentially rational: if $S_1 = \{y, z\}$, $h_1 = y$, $S_2 = \{x, y\}$, and $h_2 = y$, then $c(S_1, h_1) = \{y\}$ and $c(S_2, h_2) = \{x, y\}$. To be sequentially rational, a $c$ that makes $\{(x, y)\}$ exchanges must not choose $y$ when $z$ is available.

[9] That is, when revealed preference theory allows a choice function to be rationalized by an incomplete preference relation, $R$ is defined to rationalize a choice function only when the function always selects all $R$-undominated options. See, e.g., Moulin (1985).

**Observation 1.** If $c$ is sequentially rational and satisfies weak revelation, then $\succ = \succ_c$.

The antecedent of Observation 1 is not vacuous: we know from the previous section that any trade-avoiding $c$ is sequentially rational, and any trade-avoiding $c$ satisfies weak revelation since if $x \sim^* y$ then $c(\{x, y\}, x) = \{x\}$. It is not hard to show that any cautious $c$ that freely exchanges any set of $\sim_B$-related pairs also satisfies weak revelation (and by Theorem 1 is sequentially rational).

Of course, a rational agent could reject weak revelation: when $x \sim^* y$ the agent could insist on never choosing $y$ when $x$ is available. It is tempting in such a case nevertheless to label $x$ as strictly preferred to $y$. Since the agent across all possible choice sets gives no behavioral evidence that $x \sim^* y$ (the agents always acts as if $x$ is better), for all positive or predictive purposes we can declare strict preference to obtain. In any event, since weak revelation is weaker than the assumption that $c$ always selects all undominated options, the latter is needlessly strong for the purpose of deducing $\succ$ from $c$.

Nothing about the difficulty of inferring strict preference is particularly related to sequential choice. Suppose an agent makes a single decision, as summarized by a traditional one-shot choice function $\mathcal{C}$, which is a correspondence from $\mathcal{F}_X$ to $\mathcal{F}_X$ such that $\mathcal{C}(S) \subset S$ for all $S \in \mathcal{F}_X$. Either weak revelation or the assumption that $\mathcal{C}$ always selects all undominated options will imply that the strict preferences revealed by $\mathcal{C}$ will coincide with $\succ$.[10] Just as with sequential choice functions, a rational agent could violate either condition without having to select a dominated option. Moreover, in the setting of one-shot choices, where many partitions of $\sim^*$ into indifference and incompleteness relations would seem to be equally valid, the stronger assumption that $\mathcal{C}$ always selects all undominated options is of no help in discriminating indifferent from unranked options: the assumption therefore does not serve any of the goals of this paper. We will see in the next section just how many classifications of indifference and incompleteness there are for one-shot choices.

## 5. The family of rationalizations

We have used an agent's strict preferences $\succ$ to define an indifference relation $\sim_B$ that characterizes which sequences of trades are rational. But how many indifference relations, whether equal to $\sim_B$ or not, are consistent with $\succ$? That is, how many indifference relations, when joined to $\succ$, define a weak preference? The answer will shed light on two points. First, instead of tying indifference to the sequential trades that a rational agent can undertake, one could instead adopt a psychological definition of indifference that does not purport to predict choice behavior. It is valuable to know how precise a psychological theory of indifference can be. While under mild conditions we can unambiguously infer an agent's strict preferences $\succ$ from choice behavior, there will usually be many psychological indifference relations that an agent with a given $\succ$ might have. In static settings where agents know that they will not be able to trade repeatedly, a purely psychological definition of indifference may be the way to go: if agents are sure they are making once-and-for-all decisions, the behavioral predictions of the $\sim_B$ theory of indifference have no bite. Second, specifying the weak preferences consistent with a strict preference will generate an isomorphism between weak and strict preferences that indicates when they are equivalent starting points.

We again take as given an arbitrary strict preference $\succ$ (an asymmetric and transitive relation) and $\sim_B$ and $\perp_B$ are derived from $\succ$ using Definitions 1 and 2. For a weak preference $\succcurlyeq$ (a reflexive and transitive relation), recall that we define $\sim$ to be the symmetric part of $\succcurlyeq$ (the traditional definition of indifference) and that $\perp$, where $x \perp y \Leftrightarrow$ (not $x \succcurlyeq y$ and not $y \succcurlyeq x$), denotes the pairs unranked by $\succcurlyeq$.

**Definition 6.** The relation $\succcurlyeq$ *rationalizes* $\succ$ if and only if $\succcurlyeq$ is a weak preference with an asymmetric part equal to $\succ$.

Each rationalizing $\succcurlyeq$ defines an indifference relation by its symmetric part. It will come as no surprise that this $\sim$ need not coincide with $\sim_B$: while $\sim_B$-related options can be freely exchanged without leading an agent to harm, an agent's weak preferences could nevertheless judge such options to be unranked rather than indifferent. For example, $\succcurlyeq = \succ \cup I$, where $I$ is the identity relation $\{(x, x)\colon\ x \in X\}$, will always rationalize $\succ$ but $I$ will often be a strict subset of $\sim_B$. We can build less trivial rationalizations too. Observe that $x \sim_B y$ if and only if, whenever some $z$ is strictly ranked relative to either $x$ or $y$, the same strict ranking must obtain between $z$ and the other option in $\{x, y\}$. For example, if $x \sim_B y$ and $z \succ x$ then Definition 1 implies that either $z \succ y$ or $y \succ z$. But by transitivity $y \succ z$ yields a contradiction. Hence $z \succ y$. We can conclude that $x \sim_B y$ implies

$$\succ \cup \big\{(x, y), (y, x)\big\} \cup I$$

is transitive on $\{x, y, z\}$. The above relation is in fact transitive on all of $X$ – we therefore have a rationalization – and $\{(x, y), (y, x)\} \cup I$ is an equivalence relation. The following theorem extends this observation and gives a converse.

**Theorem 4.** ($\succcurlyeq$ *rationalizes* $\succ$) *if and only if* ($\succcurlyeq = \succ \cup E$ *for some equivalence relation* $E \subset \sim_B$).

[10] For a one-shot choice function $\mathcal{C}$, weak revelation is the assumption that $x \sim^* y \Rightarrow$ there exists $S \in \mathcal{F}_X$ such that $y \in \mathcal{C}(S)$ and $x \in S$, and 'always select all undominated options' is the assumption that $\mathcal{C}(S) = \{a \in S\colon\ b \in S \Rightarrow b \not\succ a\}$ for all $S \in \mathcal{F}_X$.

So we can generate any of the rationalizations of $\succ$ by selecting an equivalence relation in $\sim_B$ and joining it to $\succ$.

One way to view Theorem 4 is as a delineation of just how many weak preferences will rationalize the observable behavior of an agent who makes one-shot choices, where perhaps the theory of rational sequential trade is irrelevant. Suppose an agent with a strict preference $\succ$ generates a one-shot choice function $\mathcal{C}$ that never selects dominated options,

$$x \in S \quad \text{and} \quad y \in \mathcal{C}(S) \quad \text{imply} \quad x \not\succ y,$$

or even a $\mathcal{C}$ that satisfies the stronger assumption of always selecting all undominated options. Then Theorem 4 identifies the set of weak preferences $\succcurlyeq$ that the agent with $\mathcal{C}$ might possess: these $\succcurlyeq$ all have $\succ$ as their asymmetric part. So typically $\sim^*$ can be partitioned into $\sim$ and $\perp$ in many different ways. Notice that the assumption that $\mathcal{C}$ always selects all undominated options is, as in Section 4, pointlessly strong: it is not needed to identify $\succ$ and does not pin down indifference.

Among the rationalizations given in Theorem 4, two stand out: the unique minimal rationalization, $\succ \cup I$, and the unique maximal rationalization, $\succ \cup \sim_B$. So, if $\succcurlyeq$ rationalizes $\succ$ then $I \subset \sim \subset \sim_B$.

We can also use Theorem 4 to give yet another definition of $\sim_B$.

**Corollary of Theorem 4.** $\sim_B$ is the largest equivalence relation $E$ such that $\succ \cup E$ is transitive.[11]

On what grounds can we single out $\succ \cup \sim_B$ from the family of rationalizing weak preference relations? To remain faithful to the ordinalist principle that preferences should not reflect purely psychological distinctions with no implications for choice behavior, we should classify $\sim^*$-related options as unranked rather than indifferent only when trades of those options can harm an agent and hence will not be executed. In a setting of one-shot choices, this rule has no traction since any trade of $\sim^*$-related options is harmless; any $\sim$ identified by Theorem 4 is then suitable. But if it is possible that there will be further offers to trade, then the distinction between $\sim_B$ and $\perp_B$ is the relevant guide to decision-making. An agent may still *feel* that some pairs of $\sim_B$-related options are unranked, but this feeling has no impact on rational behavior. In sequential settings, we can therefore state the ordinalist principle that separates indifference and incompleteness as follows (see the closing discussion in Section 8 as well):

**Definition 7.** If the weak preference $\succcurlyeq$ has $\succ$ as its asymmetric part, then $\succcurlyeq$ satisfies *incompleteness recommends against trade* (IRAT) if and only if $x \perp y \Rightarrow x \perp_B y$.[12]

The role of $\succ$ in Definition 7 is that $\perp_B$ is derived from $\succ$ via Definition 2.

**Theorem 5.** *If the weak preference $\succcurlyeq$ has $\succ$ as its asymmetric part, then*

$$(\sim = \sim_B) \quad \Leftrightarrow \quad \succcurlyeq \textit{ satisfies IRAT}.$$

So the ordinalist principle IRAT is equivalent the conclusion that $\sim_B$ is the proper definition of indifference. Notice that IRAT is vacuously satisfied when $\succcurlyeq$ is complete. Hence, as we saw in Example 3, any complete and transitive $\succcurlyeq$ has $\sim = \sim_B$.

Theorem 5 also answers the question: when does it make a difference if weak or strict preference is primitive? If $\succcurlyeq$ is complete and transitive, it is well known that $\succcurlyeq$ and $\succ$ (defined as the asymmetric part of $\succcurlyeq$) contain interchangeable information: since $\succcurlyeq = \succ \cup \sim^*$, we can deduce $\succcurlyeq$ from $\succ$. Thus, an isomorphism obtains between the set of complete and transitive binary relations and the set of asymmetric and negatively transitive binary relations (where each set contains only relations on the same $X$): call it the 'standard isomorphism.' Since $\sim^* = \sim_B$ for a complete and transitive $\succcurlyeq$, we can equivalently express the standard isomorphism by the equality $\succcurlyeq = \succ \cup \sim_B$.

Theorem 5 generalizes the interchangeability of weak and strict preferences to IRAT preferences: if a weak preference $\succcurlyeq$ satisfies IRAT, then the equality $\succcurlyeq = \succ \cup \sim_B$ still obtains. So, in a world that admits only IRAT weak preferences, it makes no difference whether we begin with weak or strict preference. Put formally,

**Observation 2.** For any $X$, an isomorphism obtains between the set of IRAT weak preferences on $X$ and the set of strict preferences on $X$.

[11] The relation $\succ \cup \sim_B$ usually will not be the maximal transitive *extension* of $\succ$. If $\succ \cup \sim_B$ leaves some pair of options $(x, y)$ unranked, i.e., $x \perp_B y$, then

$$\succ \cup \sim_B \cup \{(x, y), (y, x)\}$$

will be acyclic and the asymmetric part of its transitive closure will extend $\succ$ (see the proof of Theorem 3). Indeed one may apply Szpilrajn's theorem to the equivalence classes defined by $\sim_B$ to construct a complete and transitive relation on $X$ that contains $\succ \cup \sim_B$ and whose asymmetric part extends $\succ$. But these relations incorporate additional strict rankings not in $\succ$. If we label any more options indifferent beyond those designated by $\sim_B$ then to preserve transitivity we will also have to declare additional pairs to be ordered by strict preference.

[12] IRAT is similar to *nontrivial incompleteness* in Mandler (2005), which identifies when incompleteness allows for intransitive but rational choices, and to *regularity* in Eliaz and Ok (2006).

The bijection $f$ that establishes Observation 2 takes each IRAT $\succcurlyeq$ to its asymmetric part $\succ$, and $f^{-1}$ takes $\succ$ to $\succ \cup \sim_B$. So, if we begin with a IRAT $\succcurlyeq$, and then take its asymmetric part $\succ$, applying $f^{-1}$ to $\succ$ will return us to $\succcurlyeq$. Since a complete weak preference satisfies IRAT, Observation 2 extends the standard isomorphism. The present isomorphism gives a recipe for translating theorems on transitive (but not necessarily negatively transitive) strict preferences into theorems on possibly incomplete weak preferences, and vice versa. In the recent literature on incomplete preferences, see Ok (2002) and Manzini and Mariotti (2008) for samples of taking weak and strict preference as starting places.

## 6. Application 1: the philosophical theory of incommensurability

A rich philosophical theory of incomplete preferences explains incompleteness as the outcome when agents lack clear-cut decision criteria or are in the midst of deliberation. See in particular Raz (1986) and Chang (2002) for variants of the Raz position and for philosophical context. The terminology in philosophy is different; when an agent has no weak preference judgment between a pair of options, the options are 'incommensurate' or 'incomparable' but we will keep to our expression 'unranked.'

Raz (1986) provides a behavioral definition of incompleteness that at first glance appears to differ from $\perp_B$. Raz in effect labels $x$ and $y$ unranked if $x \sim^* y$ and there is a third option that is strictly preferred (resp. dispreferred) to one of the options but not strictly preferred (resp. dispreferred) to the other.[13] We then write $x \perp_R y$, see Definition 8. For example, $x \perp_R y$ if $x \sim^* y$ and there is a $z$ such that $z \succ x$ but $z \not\succ y$. We would not want to label $x$ and $y$ indifferent in such a case since, if we did, the agent could not have transitive weak preferences: we would have $z \succ x$, $x$ and $y$ indifferent, and yet $z \not\succ y$. Raz called the presence of $z$ the 'mark of incommensurability' of $x$ and $y$. Backers of this definition argue that $z$ is sufficient empirical evidence of incompleteness and also that when $x$ and $y$ are unranked it is likely that such a $z$ will exist; it will typically be found among small improvements to $x$ or $y$.

Once $\perp_R$ is defined, we have a new behavioral indifference relation $\sim_R$ given by the complement of $\perp_R$ relative to $\sim^*$.

**Definition 8.** The relation $\perp_R$ is defined by: for all $x, y \in X$,

$$(x \perp_R y \text{ and } y \perp_R x) \quad \Leftrightarrow \quad \big[x \sim^* y \text{ and } \exists z \in X \text{ such that either } (z \succ x \text{ and } z \not\succ y) \text{ or } (x \succ z \text{ and } y \not\succ z)\big].$$

The relation $\sim_R$ is defined by $x \sim_R y \Leftrightarrow (x \sim^* y \text{ and not } x \perp_R y)$.

The theorem below reports that $\sim_R$ in fact coincides with $\sim_B$ (and hence $\perp_R$ coincides with $\perp_B$). This equivalence has gone unnoticed, possibly because $\sim_B$ and $\sim_R$ arise in different literatures, or because of the veil that the decision theory literature defines an equivalence relation while the philosophical literature defines the complementary incompleteness relation.

**Theorem 6.** $\sim_R = \sim_B$.

Raz takes the view that although $x \perp_R y$ is sufficient evidence that $x$ and $y$ are preferentially unranked, $x \perp_R y$ is not entailed by $x$ and $y$ being unranked. To assess this position, suppose that the agent possesses a weak preference $\succcurlyeq$ that lies behind $\succ$ – that is, the asymmetric part of $\succcurlyeq$ is $\succ$ – with the interpretation that $\succcurlyeq$ indicates the agent's 'true' preference judgments. The strict preference $\succ$ still defines $\sim_R$ and $\perp_R$, while $\sim$, the symmetric part of $\succcurlyeq$, indicates the agent's judgments of indifference and $\perp = \sim^* \setminus \sim$ indicates the options the agent cannot rank. The Raz position is that if $x \perp_R y$ we can be sure the agent's underlying weak preferences judge $x$ and $y$ to be unranked, $x \perp y$, but if $x \sim_R y$ then the agent's preferences might judge $x$ and $y$ to be unranked. We can confirm this position exactly. Theorems 4 and 6 report that $\sim \subset \sim_R$. Hence $\perp \supset \perp_R$ (since $\perp = \sim^* \setminus \sim$ and $\perp_R = \sim^* \setminus \sim_R$). In fact, $\perp = \perp_R$ if and only if $\succcurlyeq$ satisfies IRAT. So $x \perp_R y$ implies $x \perp y$ but not vice versa, which is Raz's view: $x \perp_R y$ is sufficient but not necessary for $x \perp y$.

There is one minor difference between Definition 8 and the approach taken by Raz and other philosophers of incomparability. The philosophers usually reason that $x$ and $y$ are unranked or incomparable if there is some $z$ that is a 'small' improvement (resp. worsening) relative to $x$ such that $z$ is not superior (resp. inferior) to $y$. Since money can be doled out in small quantities, it often serves as the vehicle of improvement. Smallness in contrast plays no role here. The difference is that we aim only to define incomparability in terms of the existence of a $z$, whereas the philosophers aim to show that it is plausible that such a $z$ will exist in certain types of decision problems and where to find $z$. When $x$ and $y$ represent alternatives that an agent cannot judge between, say the careers mentioned in the introduction, then the most fruitful place to hunt for a $z$ that satisfies Definition 8 is indeed among the slight improvements or worsenings of $x$ or $y$: small changes are unlikely to clear up an agent's inability to rank.

## 7. Application 2: the family of rationalizations in one-shot revealed preference theory

We turn to the revealed preference program that takes an agent's one-shot choices as given rather than the agent's preferences. If the agent's choices are summarized by the one-shot choice function $\mathcal{C}$ – one set of selections for every $S$ –

[13] See also De Sousa (1974) who may be the originator of the definition.

the classical question of revealed preference theory is: 'which weak preference relations rationalize $\mathcal{C}$?'[14] As in Section 5, there will be a multiplicity of rationalizations, reinforcing our conclusion that the choice environment must be sequential in order for rational behavior to single out a unique indifference relation. We consider two main definitions of rationalization, the traditional one where a preference relation rationalizes $\mathcal{C}$ if and only if $\mathcal{C}$ always selects all undominated options and a more permissive definition. With either definition, only an agent's strict preferences are unambiguously revealed. Theorem 4 consequently identifies a multiplicity of rationalizing weak preferences, just as in Section 5. Since the assumption that agents choose all undominated options therefore does not distinguish indifference from incompleteness, we might as well let agents choose arbitrarily among indifferent or unranked options.

Recall that a one-shot choice function is a $\mathcal{C}: \mathcal{F}_X \to \mathcal{F}_X$ such that $\mathcal{C}(S) \subset S$ for all $S \in \mathcal{F}_X$. Strict revealed preference is defined as follows.

**Definition 9.** Given a one-shot choice function $\mathcal{C}$, the binary relation $\succ_{\mathcal{C}}$ is defined by

$$x \succ_{\mathcal{C}} y \quad \text{if and only if, for all } S \in \mathcal{F}_X, \quad (x, y \in S) \quad \Rightarrow \quad y \notin \mathcal{C}(S).$$

The relation $\succ_{\mathcal{C}}$ must be acyclic and hence asymmetric: if not, then $x_1 \succ_{\mathcal{C}} x_2 \succ_{\mathcal{C}} \cdots \succ_{\mathcal{C}} x_T \succ_{\mathcal{C}} x_1$ for some $Y \equiv \{x_1, \ldots, x_T\} \subset X$, but since there must be some $x_i \in \mathcal{C}(Y)$, it then could not be the case that $x_{i-1} \succ_{\mathcal{C}} x_i$ (where if necessary $x_0$ denotes $x_T$).

Now consider two definitions of rationalization. The permissive approach defines the weak preference $\succcurlyeq$ to *strict preference* (*SP*) *rationalize* $\mathcal{C}$ if and only if the asymmetric part of $\succcurlyeq$ equals $\succ_{\mathcal{C}}$. Any $\mathcal{C}$ can be SP-rationalized by a $\succcurlyeq$ whose asymmetric part is acyclic since $\succ_{\mathcal{C}}$ itself SP-rationalizes $\mathcal{C}$. SP rationalization does not impose any restrictions on how agents choose among options that are not strictly ranked and thus follows the Afriat (1967) branch of revealed preference theory which requires that each observation of a consumer's purchases be *one* of the solutions to the consumer's maximization problem. In the choice function literature, the traditional way to proceed is to say that $\succcurlyeq$ rationalizes $\mathcal{C}$ if $\mathcal{C}$ selects all options that $\succcurlyeq$ classifies as undominated. Formally, the weak preference $\succcurlyeq$ *nondomination* (*ND*) *rationalizes* $\mathcal{C}$ if and only if, for every $S \in \mathcal{F}_X$,

$$\mathcal{C}(S) = \{x \in S\colon\ y \in S \ \Rightarrow\ \text{not } y \succ x\},$$

where, as throughout this section, $\succ$ denotes the asymmetric part of $\succcurlyeq$. If $\succcurlyeq$ ND-rationalizes $\mathcal{C}$ then $\succcurlyeq$ SP-rationalizes $\mathcal{C}$, but not necessarily vice versa.

The distinction between SP and ND rationalization runs parallel to the distinction in Section 4 between weak revelation and the assumption that a choice function always selects all undominated options. It is again hard to see why a rational agent should feel compelled to use a $\mathcal{C}$ that selects all undominated options.

**Definition 10.** The weak preference $\succcurlyeq$ *transitively SP* (resp. *ND*) *rationalizes* $\mathcal{C}$ if and only if $\succcurlyeq$ is reflexive and transitive and $\succcurlyeq$ *SP* (resp. *ND*) rationalizes $\mathcal{C}$. $\mathcal{C}$ is *transitively SP* (resp. *ND*) *rationalizable* if and only if there is a weak preference $\succcurlyeq$ that transitively SP (resp. ND) rationalizes $\mathcal{C}$.

The question of when a $\mathcal{C}$ is transitively ND-rationalizable is answered by Plott (1973), Blair et al. (1976), Schwartz (1976), Suzumura (1983, Ch. 2), Bandyopadhay and Sengupta (1993), and Eliaz and Ok (2006).[15] Most of these papers characterize transitive ND-rationalizability via expansion and contraction axioms; the 'strict partial order axiom' of Bandyopadhay and Sengupta (1993) and WARNI of Eliaz and Ok (2006) are exceptions.

**Observation 3.** A choice function $\mathcal{C}$ is transitively SP-rationalizable if and only if $\succ_{\mathcal{C}}$ is transitive.

Let $\sim_B(\mathcal{C})$ denote the behavioral indifference relation derived from $\succ_{\mathcal{C}}$, that is, the binary relation given by Definition 1 when $\sim^*$ is replaced by $\sim^*(\mathcal{C}) \equiv \{(x, y)\colon \text{not } x \succ_{\mathcal{C}} y \text{ and not } y \succ_{\mathcal{C}} x\}$.

**Example 6.** To see that the transitivity of $\succ_{\mathcal{C}}$ is weaker than any of the conditions that characterize transitive ND rationalization, consider the following two cases where a $\mathcal{C}$ is transitively SP rationalizable but not transitively ND rationalizable. In both cases, $X = \{x, y, z\}$. First, let $\mathcal{C}$ be defined by $\mathcal{C}(\{x, z\}) = \{z\}$, $\mathcal{C}(\{y, z\}) = \{y\}$, $\mathcal{C}(\{x, y\}) = \{x, y\}$, $\mathcal{C}(\{x, y, z\}) = \{z\}$. Then the only strict revealed preference we can infer is $z \succ_{\mathcal{C}} x$ and $\succ_{\mathcal{C}} \cup I$ transitively SP rationalizes $\mathcal{C}$ (where $I$ is the identity relation). Yet $\mathcal{C}$ cannot be ND rationalized: if $\mathcal{C}$ were ND rationalized by $\succcurlyeq$ then from $\mathcal{C}(\{y, z\}) = \{y\}$ we infer $y \succ z$, but from $\mathcal{C}(\{x, y, z\}) = \{z\}$ we infer $z \succ y$. That $\succ_{\mathcal{C}}$ consists of just a single ordered pair is not important: for a second example, let $\mathcal{C}'$ be defined by $\mathcal{C}'(\{z, x\}) = \{z\}$, $\mathcal{C}'(\{y, z\}) = \{y\}$, $\mathcal{C}'(\{x, y\}) = \{y\}$, $\mathcal{C}'(\{x, y, z\}) = \{z\}$. Then $\succ_{\mathcal{C}'}$ is given by $z \succ_{\mathcal{C}'} x$

[14] We briefly considered one-shot choice functions in Sections 4 and 5 but there we endowed the agent with exogenously given strict preferences.

[15] With the exception of Eliaz and Ok, the papers ask 'when can $\mathcal{C}$ be ND-rationalized by quasitransitive $\succcurlyeq$?' (A binary relation is quasitransitive if its asymmetric part is transitive.) The two questions are the same since $\mathcal{C}$ can be transitively ND-rationalized if and only if $\mathcal{C}$ can be ND-rationalized by a quasitransitive $\succcurlyeq$ (the options labeled unranked in transitive ND-rationalization are labeled indifferent in quasitransitive ND-rationalization). We follow Eliaz and Ok's way of putting the matter to underscore the possibility that options can be unranked.

and $y \succ_{\mathcal{C}'} x$, and hence $z \sim_B(\mathcal{C}')y$. Like the first case, $\mathcal{C}'$ can be transitively SP rationalized but not ND rationalized: if $\mathcal{C}'$ were ND rationalized by $\succcurlyeq$ then $\mathcal{C}'(\{y,z\}) = \{y\}$ gives $y \succ z$ but $\mathcal{C}'(\{x,y,z\}) = \{z\}$ gives $z \succ y$. In contrast to the first case, $\succ_{\mathcal{C}'} \cup \sim_B(\mathcal{C}')$ is complete and transitive.

The relation $\succcurlyeq$ SP-rationalizes $\mathcal{C}$ if and only if the asymmetric part of $\succcurlyeq$ equals $\succ_{\mathcal{C}}$. Hence if $\succcurlyeq$ SP-rationalizes $\mathcal{C}$ then so does any $\succcurlyeq'$ that has the same asymmetric part as $\succcurlyeq$. Similarly, since $\succcurlyeq$ ND-rationalizes $\mathcal{C}$ if and only if, for all $S \in \mathcal{F}_X$,

$$x \in \mathcal{C}(S) \quad \Leftrightarrow \quad \big(x \in S \text{ and } (y \in S \;\Rightarrow\; \text{not } y \succ x)\big),$$

where $\succ$ is the asymmetric part of $\succcurlyeq$, any $\succcurlyeq'$ with the same asymmetric part as $\succcurlyeq$ will also ND-rationalize $\mathcal{C}$. Given Theorem 4 we have:

**Observation 4.** If $\mathcal{C}$ is transitively SP (resp. ND) rationalizable, then, for any equivalence relation $E \subset \sim_B(\mathcal{C})$, $\succ_{\mathcal{C}} \cup E$ transitively SP (resp. ND) rationalizes $\mathcal{C}$.

Notice that since the only pertinent feature of a ND-rationalizing $\succcurlyeq$ is that its asymmetric part matches $\succ_{\mathcal{C}}$, ND-rationalization cannot distinguish between indifference and incompleteness.

The multiplicity of transitive SP or ND rationalizations persists when $\succ_{\mathcal{C}}$ is negatively transitive and hence $\succ_{\mathcal{C}} \cup \sim^*(\mathcal{C})$ is complete (and in which case $\sim^*(\mathcal{C}) = \sim_B(\mathcal{C})$). Even WARP (Arrow, 1959), which ensures that $\mathcal{C}$ is transitively ND-rationalizable, permits multiple rationalizations: there is one complete ND-rationalization, namely $\succ_{\mathcal{C}} \cup \sim_B(\mathcal{C})$, but there will be incomplete rationalizations as well (assuming that $\sim_B(\mathcal{C})$ is not just the identity relation).

For the general problem, where $\succ_{\mathcal{C}}$ need not be negatively transitive, Eliaz and Ok (2006) favor the ND-rationalization $\succ_{\mathcal{C}} \cup \sim_B(\mathcal{C})$. We concur that $\succ_{\mathcal{C}} \cup \sim_B(\mathcal{C})$ deserves special mention but only because of the sequential rationality arguments we gave in Section 3. To put those arguments in the present terminology, an agent will never end up with a $\succ_{\mathcal{C}}$-worse option if along a sequence of trades he or she freely accepts any $\sim_B(\mathcal{C})$-related option and refuses to exchange any options that are $\perp_B(\mathcal{C})$-related. But this argument does not fit well in the present context: it applies to sequences of choices, not the static choices that classical revealed preference theory addresses.

Classical revealed preference analysis *can* allow a finer discrimination of options into indifferent and unranked subsets if we assume agents treat indifferent and unranked options differently. For example we could define $\succcurlyeq$ to *weak preference* (*WP*) *rationalize* $\mathcal{C}$ if and only if $\succcurlyeq$ SP-rationalizes $\mathcal{C}$ and, for all $S \in \mathcal{F}_X$ and all $x, y \in X$,

$$\big(y \in \mathcal{C}(S) \text{ and } x \in S\big) \quad \Rightarrow \quad \big(x \succcurlyeq y \;\Leftrightarrow\; x \in \mathcal{C}(S)\big).$$

So we infer $y \succ x$ if and only if $x$ is never chosen when $y$ is available and infer $x \sim y$ if and only if $x$ and $y$ are always both chosen when one is chosen and the other is available. If $\mathcal{C}$ is WP-rationalizable then only one $\succcurlyeq$ may serve as the rationalization, namely the $\succcurlyeq$ defined by $x \succcurlyeq y$ if and only if $x \in \mathcal{C}(\{x,y\})$. As with ND rationalization, it is hard to see why a self-interested agent must use a $\mathcal{C}$ that can be WP rationalized. The task of inferring indifference or incompleteness would certainly be easier if agents conveniently discriminated between indifferent and unranked options, but the rationality of one-shot decision-making does not demand that they do so. We again come to the conclusion a behavioral distinction between indifference and incompleteness requires choice to be sequential.

## 8. Conclusion: ordinal versus psychological indifference

When an agent fails to systematically choose $x$ over $y$ or $y$ over $x$, one could simply ask the agent, 'are you indifferent?' This question unfortunately does not get us very far since it has many interpretations, including 'would you trade one option for the other?', 'do you view the options as interchangeable?', and 'can you compare the options?'. Various social scientists and philosophers implicitly adopt one of these interpretations. Following the ordinalist tradition, we have pursued a definition of indifference, $\sim_B$, that draws on the trade interpretation.

The classification that $\sim_B$ provides will not always be consistent with the other interpretations of indifference. In particular $\sim_B$ can deviate from the utilitarian understanding of preference as raw sensory experience. Ordinal preference theory officially does not say much about how an agent psychologically responds to options $x$ and $y$ unranked by $\succ$, where $x \sim^* y$ obtains. Most ordinalists stipulate only that the agent does not regard either $x$ or $y$ as superior to the other. But it is plausible that an agent facing options where $x \sim^* y$ would have one of two psychological responses: a 'psychological indifference' reaction when the agent feels the options are completely interchangeable and an 'psychological incompleteness' reaction when the agent feels the options cannot be compared. While an agent's immediate psychological responses may divide in this way, the distinction need not match the partition of $\sim^*$ into $\sim_B$ and $\perp_B$ subsets. This partition gives the implications of $\succ$ for the trades that a rational agent will make. But it may take some thinking on the agent's part, not just an immediate psychological response, to deduce this trading advice. The semiorder example (Example 1) illustrates the difference. As we saw, $x \perp_B y$ if and only if total number of grains of sugar that $x$ and $y$ represent are different but within $\sigma$ of each other. Although an exchange of $x$ and $y$ can be unwise (it can put the agent within one trade of a welfare loss), the agent undergoes the same psychological experience when consuming $x$ and consuming $y$. The agent's immediate sensory response and the $\sim_B$ classification do not coincide.

## Appendix A

**Lemma 1.** *$(r \succ s(\succ \cup \sim_B)t \Rightarrow r \succ t)$ and $(r(\succ \cup \sim_B)s \succ t \Rightarrow r \succ t)$.*

**Proof.** Assume $r \succ s(\succ \cup \sim_B)t$. If in addition $s \succ t$ then the transitivity of $\succ$ gives $r \succ t$. So assume $s \sim_B t$. Then since $r \succ s$ we must have either $r \succ t$ or $t \succ r$. If $t \succ r$ then $t \succ s$, contradicting $s \sim_B t$. Hence $r \succ t$. The proof of $r(\succ \cup \sim_B)s \succ t \Rightarrow r \succ t$ is similar. □

**Lemma 2.** *$\succ \cup \sim_B$ is transitive.*

**Proof.** Assume $r(\succ \cup \sim_B)s(\succ \cup \sim_B)t$. If $r \succ s \succ t$ then the transitivity of $\succ$ gives $r \succ t$. If $r \sim_B s \sim_B t$, then $r \sim_B s$ gives $(r \sim^* d \Leftrightarrow s \sim^* d)$ and $s \sim_B t$ gives $(s \sim^* d \Leftrightarrow t \sim^* d)$: hence $r \sim^* d \Leftrightarrow t \sim^* d$ and therefore $r \sim_B t$. The remaining two possibilities are covered by Lemma 1. □

**Proof of Theorem 1.** Let $c$ be cautious and make $V$ exchanges, where $V \subset \sim_B$. Letting $V^{-1}$ denote the binary relation defined by $xV^{-1}y \Leftrightarrow yVx$ and letting $I$ denote the identity relation $\{(x,x)\colon x \in X\}$, define the reflexive and symmetric relation $E = V \cup V^{-1} \cup I$. Since $\sim_B$ is reflexive and symmetric and $V \subset \sim_B$, $E \subset \sim_B$.

Suppose $(S_1, \dots, S_n)$ and $(h_1, \dots, h_n, x, y)$ satisfy $h_i \in c(S_{i-1}, h_{i-1})$ and $h_i \in S_i$ for $i = 2, \dots, n$, $x \in S_1$, and $y \in c(S_n, h_n)$. For $i = 2, \dots, n$, Definition 4 and the second half of Lemma 1 imply that if $z \in c(S_i, h_i)$ then $z(\succ \cup E)h_{i-1}$. Hence

$$h_i(\succ \cup E)h_{i-1} \quad \text{for } i = 3, \dots, n-1, \quad \text{and} \quad y(\succ \cup E)h_{n-1}. \tag{A.1}$$

Definition 5 implies that

$$\text{if } z \in c(S_1, h_1) \quad \text{then not } x \succ z. \tag{A.2}$$

Suppose $n > 1$. Since $h_2 \in c(S_1, h_1)$, (A.2) gives not $x \succ h_2$. Since $(\succ \cup E) \subset (\succ \cup \sim_B)$, Lemma 2 applied to (A.1) implies $y(\succ \cup \sim_B)h_2$. So $x \succ y$ would by Lemma 2 imply $x \succ h_2$, a contradiction. If $n = 1$, the theorem follows directly from (A.2). □

**Proof of Theorem 2.** Since $\perp_B \neq \emptyset$ there exist $x$, $y \in X$ such that $x \perp_B y$. Hence there exists a $z$ such that one of the following four possibilities holds: (1) $x \sim^* z$ and $y \succ z$, (2) $x \sim^* z$ and $z \succ y$, (3) $y \sim^* z$ and $x \succ z$, (4) $y \sim^* z$ and $z \succ x$. In case (1), we must have $x \perp_B z$ since if $x \sim_B z$ then (since $\sim_B$ is symmetric) Lemma 1 implies $y \succ x$, which contradicts $x \perp_B y$. We set $V = \{(x, y), (x, z)\}$ and suppose $c$ makes $V$ exchanges. Then for $S_1 = \{x, y\}$, $S_2 = \{x, z\}$, $h_2 = x$,

$$c(S_1, h_1) = \{x, y\}, \qquad c(S_2, h_2) = \{x, z\}, \quad h_2 \in c(S_1, h_1) \text{ and } h_2 \in S_2.$$

Since $y \succ z$, $y \in S_1$, and $z \in c(S_2, h_2)$, we have a violation of sequential rationality. Case (2) is similar. We have $x \perp_B z$, set $V = \{(x, y), (x, z)\}$, and suppose $c$ makes $V$ exchanges. Then for $S_1 = \{x, z\}$, $S_2 = \{x, y\}$, $h_2 = x$,

$$c(S_1, h_1) = \{x, z\}, \qquad c(S_2, h_2) = \{x, y\}, \quad h_2 \in c(S_1, h_1) \text{ and } h_2 \in S_2.$$

Since $z \succ y$, $z \in S_1$, and $y \in c(S_2, h_2)$, we again have a violation of sequential rationality. Since $\perp_B$ is symmetric, cases (3) and (4) repeat (1) and (2). □

**Proof of Theorem 3.** Assume $V \cap \perp_B \neq \emptyset$ and let $(x, y) \in V \cap \perp_B$. Showing that there exists a pair with $x \sim^* z$ such that any $c$ that makes $\{(x, z)\} \cup V$ exchanges fails to be sequentially rational requires only minor adjustments to the proof of Theorem 2. One of the four cases given in that proof must obtain, and only two of these cases are distinct given that $\perp_B$ is symmetric. In the first case there is a $z$ such that $x \sim^* z$ and $y \succ z$, and hence $x \perp_B z$. If $c$ makes $\{(x, z)\} \cup V$ exchanges, then for $S_1 = \{x, y\}$, $S_2 = \{x, z\}$, $h_2 = x$, we have $c(S_1, h_1) = \{x, y\}$, $c(S_2, h_2) = \{x, z\}$, $h_2 \in c(S_1, h_1)$ and $h_2 \in S_2$. Since $y \succ z$, $y \in S_1$, and $z \in c(S_2, h_2)$, we have a violation of sequential rationality. In the second case there is a $z$ such that $x \sim^* z$ and $z \succ y$, and hence $x \perp_B z$. If $c$ makes $\{(x, z)\} \cup V$ exchanges, then for $S_1 = \{x, z\}$, $S_2 = \{x, y\}$, $h_2 = x$, we have $c(S_1, h_1) = \{x, z\}$, $c(S_2, h_2) = \{x, y\}$, $h_2 \in c(S_1, h_1)$ and $h_2 \in S_2$. Since $z \succ y$, $z \in S_1$, and $y \in c(S_2, x)$, we again have a violation of sequential rationality.

Now assume $V \cap \perp_B = \emptyset$ and suppose $x \sim^* y$. Define$\hat{V} = V \cap \sim_B$ and $\succcurlyeq\, = \,\succ \cup \hat{V} \cup \hat{V}^{-1} \cup I$, and let $\succcurlyeq^\alpha$ denote the transitive closure of

$$\succcurlyeq^\beta \equiv \succcurlyeq \cup \{(x, y), (y, x)\}.$$

We will use $\succcurlyeq^\alpha$ and its extension to a complete and transitive relation to show that there is a $\hat{c}$ that makes $\{(x, y)\} \cup V$ exchanges that is sequentially rational.

To begin, we establish that the asymmetric part of $\succcurlyeq^\alpha$, say $\succ^\alpha$, extends $\succ$, that is, $a \succ b \Rightarrow a \succ^\alpha b$. If, to the contrary, there exist $a, b \in X$ such that $a \succ b$ but not $a \succ^\alpha b$, then there are $s_1, \dots, s_r$ such that

$$b \succcurlyeq^\beta s_1 \succcurlyeq^\beta \cdots \succcurlyeq^\beta s_r \succcurlyeq^\beta a. \tag{A.3}$$

Since $\succ \cup \sim_B$ is transitive (Lemma 2) and $\succcurlyeq \subset (\succ \cup \sim_B)$, if

$$b \succcurlyeq s_1 \succcurlyeq \cdots \succcurlyeq s_r \succcurlyeq a, \tag{A.4}$$

then $b\ (\succ \cup \sim_B)\ a$. But since $a \succ b \Rightarrow$ (not $b \sim_B a$ and not $b \succ a$), $b\ (\succ \cup \sim_B)\ a$ is impossible. Hence it must be that at least one pair of adjacent options in (A.3) is $\{x, y\}$. In fact, we may assume that only one pair of adjacent options in (A.3) is $\{x, y\}$ since otherwise we could eliminate all but one of the adjacent $\{x, y\}$ and arrive at a $s_1, \ldots, s_k$ such that $b \succcurlyeq^\beta s_1 \succcurlyeq^\beta \cdots \succcurlyeq^\beta s_k \succcurlyeq^\beta a$ with a single adjacent $\{x, y\}$ (if the leftmost appearance of an adjacent $\{x, y\}$ in (A.3) has $x \succcurlyeq^\beta y$ and the rightmost has $y \succcurlyeq^\beta x$, or vice versa, then we could assemble a set of options satisfying (A.4)). We label the left option of the sole adjacent pair $x$ and the right option $y$. Thus,

$$(b \succcurlyeq \cdots \succcurlyeq x) \quad \text{and} \quad (y \succcurlyeq \cdots \succcurlyeq a).$$

These conditions permit $b = x$ and/or $y = a$. Since $\succ \cup \sim_B$ is transitive and $\succcurlyeq \subset (\succ \cup \sim_B)$,

$$\big(b\ (\succ \cup \sim_B)\ x\big) \quad \text{and} \quad \big(y\ (\succ \cup \sim_B)\ a\big).$$

Since $b\ (\succ \cup \sim_B)\ x$, $a \succ b$ implies $a \succ x$ (see Lemma 1). And since $y\ (\succ \cup \sim_B)\ a$, we have $y \succ x$ (again by Lemma 1), which contradicts $x \sim^* y$. Hence $\succ^\alpha$ extends $\succ$.

By applying Szpilrajn's theorem (see, e.g., Fishburn, 1970) to the indifference classes of $\succcurlyeq^\alpha$, we can extend $\succcurlyeq^\alpha$ to a complete and transitive relation $\succcurlyeq^\gamma$ such that $a \succ^\alpha b \Rightarrow a \succ^\gamma b$ and $a \succcurlyeq^\alpha b \Rightarrow a \succcurlyeq^\gamma b$. So $\succ^\gamma$ extends $\succ$.

To conclude, we define a sequentially rational $\hat{c}$ that makes $\{(x, y)\} \cup V$ exchanges. For each $(S, h) \in \Sigma$, set $\hat{c}(S, h)$ to equal a set $C$ such that

(1) $C \subset \{a \in S\colon\ a \succcurlyeq^\gamma b \text{ for all } b \in S\}$,
(2) $(a \neq b \text{ and } a, b \in C) \Rightarrow a\ (\{(x, y)\} \cup V)\ b \text{ or } b\ (\{(x, y)\} \cup V)\ a$

and there is no larger set $C \supsetneq \hat{c}(S, h)$ that satisfies (1) and (2). The set $\hat{c}(S, h)$ is well-defined since $S$ is finite for $(S, h) \in \Sigma$ and therefore $\{a \in S\colon\ a \succcurlyeq^\gamma b \text{ for all } b \in S\}$ is nonempty. It is easy to confirm that $\hat{c}$ makes $\{(x, y)\} \cup V$ exchanges.

To establish sequential rationality, suppose $(S_1, \ldots, S_n)$ and $(h_1, \ldots, h_n)$ satisfy $h_i \in c(S_{i-1}, h_{i-1})$ and $h_i \in S_i$ for $i = 2, \ldots, n$, and that $w \in S_1$ and $v \in \hat{c}(S_n, h_n)$. Since $\hat{c}(S, h) \subset \{a \in S\colon\ a \succcurlyeq^\gamma b \text{ for all } b \in S\}$, $h_i \succcurlyeq^\gamma h_{i-1}$ for $i = 3, \ldots, n-1$, $h_2 \succcurlyeq^\gamma w$, and $v \succcurlyeq^\gamma h_{n-1}$. Since $\succcurlyeq^\gamma$ is transitive, $v \succcurlyeq^\gamma w$. Since $\succ^\gamma$ extends $\succ$, we conclude that not $w \succ v$. $\square$

**Proof of Theorem 4.** Suppose that $E$ is an equivalence relation, and that $E \subset \sim_B$, that is, $aEb \Rightarrow a \sim_B b$. Since $E$ is reflexive, so is $\succ \cup E$. Given the transitivity of both $E$ and $\succ$, to conclude that $\succ \cup E$ is transitive it is sufficient to show ($x \succ y$ and $yEz) \Rightarrow x \succ z$, and ($xEy$ and $y \succ z) \Rightarrow x \succ z$. Since $E \subset \sim_B$, both implications follow from Lemma 1. So $\succ \cup E$ is a weak preference. Since $E$ is symmetric, $\succ \cup E$ has the same asymmetric part as $\succ$.

Conversely, suppose $\succcurlyeq$ rationalizes $\succ$. We set $F = \succcurlyeq \setminus \succ$. Since $\succcurlyeq$ is reflexive and has an asymmetric part equal to $\succ$, $F$ must be reflexive and symmetric. Given that $\succcurlyeq$ is transitive and $F$ is symmetric part of $\succcurlyeq$, $F$ must be transitive. To see that $F \subset \sim_B$, suppose to the contrary that $xFy$ does not imply $x \sim_B y$. Then there exist $x$, $y$, and $z$ such that $xFy$, $x \sim^* z$, and not $y \sim^* z$. Not $y \sim^* z$ implies either $y \succ z$ or $z \succ y$. Since $F \subset \succcurlyeq$ and the asymmetric part of $\succcurlyeq$ is $\succ$, the transitivity of $\succcurlyeq$ implies either $x \succ z$ or $z \succ x$, contradicting $x \sim^* z$. $\square$

**Proof of Theorem 5.** Since $\perp = \sim^* \setminus \sim$ and $\perp_B = \sim^* \setminus \sim_B$, $\sim = \sim_B$ implies $\perp = \perp_B$, and hence $\succcurlyeq$ satisfies IRAT. Conversely, suppose $\succcurlyeq$ satisfies IRAT. Given Theorem 4, we only need to show that $a \sim_B b \Rightarrow a \sim b$. As we observed following Definition 1, $a \sim_B b \Rightarrow a \sim^* b$. But if $a \perp b$ then IRAT gives $a \perp_B b$, a contradiction. Hence $a \sim b$. $\square$

**Proof of Theorem 6.** It is simpler to show $\perp_R = \perp_B$. If $x \perp_R y$ then there exists a $z$ such that one of the following conditions is satisfied: (1) ($z \succ x$ and $z \not\succ y$), (2) ($x \succ z$ and $y \not\succ z$), (3) ($z \succ y$ and $z \not\succ x$), (4) ($y \succ z$ and $x \not\succ z$). Given Lemma 1, each of these conditions is inconsistent with $x \sim_B y$. Since $x \perp_R y \Rightarrow x \sim^* y$, we have $x \perp_B y$. Conversely, if $x \perp_B y$ then there exists a $z$ such that one of the following four conditions is satisfied: (i) $z \sim^* y$ and $z \succ x$, (ii) $y \sim^* z$ and $x \succ z$, (iii) $z \sim^* x$ and $z \succ y$, (iv) $x \sim^* z$ and $y \succ z$. Replacing $\sim^*$ with $\not\succ$, we have cases (1)–(4). Since $x \perp_B y \Rightarrow x \sim^* y$, we have $x \perp_R y$.

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