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Sraffian Indeterminacy in General Equilibrium

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The indeterminacy claim for competitive price systems made by Sraffa (1960) is examined by placing Sraffa's work in an intertemporal general equilibrium model. We show that indeterminacy occurs at a natural type of equilibrium. Moreover, the presence of linear activities instead of a differentiable technology is crucial and the indeterminacy is constructed, as in Sraffa, by fixing some or all of the economy's aggregate quantities. On the other hand, an extra condition, that some factors have inelastic excess demand is necessary, and, unlike Sraffa's model, relative prices must be allowed to vary through time. Sraffian indeterminacy and the generic finiteness of the number of equilibria are reconciled by showing that indeterminacy occurs at a measure-zero set of endowments. We use an overlapping-generations model to show that these endowments nevertheless arise systematically and that indeterminacy does not occur when relative prices are constant through time.

1. INTRODUCTION

Near the beginning of his *Production of Commodities by Means of Commodities* (1960), Sraffa specifies a system of equations that prices must obey if the capital invested in each sector of an economy is to earn the same rate of profit. The system contains one more unknown than equation, leading Sraffa to conclude that prices, and wage and interest rates in particular, are indeterminate. Sraffa's book has since been read as arguing that some outside, non-economic consideration must therefore set prices and the distribution of income. Part of the controversy the book has provoked is due to Sraffa's apparent agreement with the neoclassical emphasis on optimization and competitive markets. The standard Sraffian system of equations is nothing other than the first-order conditions that hold for price-taking, profit-maximizing firms using linear activities. But if Sraffa's assumptions are in accord with neoclassical theory, how can his assertions of indeterminacy be correct?

I examine Sraffa's indeterminacy argument and his critique of neoclassical economics by embedding his equations in a general equilibrium model. As we will see, the indeterminacy claim, properly interpreted, correctly describes the equilibria with strictly positive prices. Moreover, the indeterminacy demonstrated here closely duplicates the indeterminacy Sraffa claimed. First, it is prices alone that are indeterminate. I follow Sraffa's striking procedure of fixing the quantities of goods produced: an infinity of (normalized) price vectors support a single vector of aggregate quantities. Second, Sraffa seemed to claim, and his advocates (*e.g.* Robinson (1961)) have certainly claimed that his book is a critique of marginal productivity theory. Correspondingly, indeterminacy in the present model hinges on marginal products not being everywhere well-defined when there are linear activities; with smooth production sets, our indeterminacy arguments do not work. Third, the dimension of the indeterminacy I demonstrate matches Sraffa's accounting of the dimension of indeterminacy.

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On the other hand, some parts of Sraffa's framework must be altered. Most prominently, I abandon the requirement that relative prices are constant through time.[1] Indeed, allowing relative prices to vary is indispensable for the indeterminacy argument. In a dynamic extension of the paper's basic two-period model, I show that the steady-state equilibria (where relative prices are unchanging) are generically determinate. The literature on Sraffa has therefore concentrated on an equilibrium concept inhospitable to indeterminacy while criticizing the equilibrium concept where indeterminacy arises robustly. Second, placing Sraffa in a general equilibrium model requires a specification of demand functions. By itself, this might be a minor departure. But for indeterminacy to occur, at least some factors of production must be inelastically supplied. No mention of any analogous point occurs in Sraffa or the subsequent literature. Third, I require markets to clear. Sraffa does not discuss whether demand must equally supply; so this requirement may well be compatible with his economic philosophy.

A common neoclassical response to Sraffa has been simply to point out that standard Arrow–Debreu general equilibrium models are logically consistent in the sense that equilibria exist under a broad range of circumstances. See, *e.g.* Hahn (1982), and Harcourt (1974) for a qualified acceptance of this view from the opposing camp. More generally, the consistency of the general equilibrium model has often been invoked as a response to criticisms alleging that neoclassical economics rests on faulty marginal productivity foundations (*e.g.* Bliss (1975)). This consensus is puzzling since the existence of equilibrium has no bearing on the determinacy question. Moreover, only following Debreu (1970), some ten years after Sraffa's book, did a systematic study of determinacy in the general equilibrium model begin to emerge. Prior to this development, arguments for determinacy simply relied on counting equations and unknowns. In an n-good exchange economy, for example, the equilibrium values of the $n-1$ relative prices can be seen as the solution of $n-1$ supply-equals-demand equations, with Walras' law guaranteeing that the n-th market-clearing condition is also satisfied. The gap in the traditional argument is that an equality of equations and unknowns is not sufficient for determinacy. The equilibrium conditions must also be independent; more precisely, the gradients defined by the equilibrium equations must be linearly independent. The theory of regular economies that developed in the wake of Debreu's paper addressed this problem by demonstrating that when the set of models is parameterized in a plausible way this independence condition will be satisfied by almost every possible model. See Mas-Colell (1975) or Kehoe (1982) for papers covering the linear activities model and Mas-Colell (1985) for an overview. It follows from the theory of regular economies that general equilibrium models displaying Sraffian indeterminacy must be exceptional or non-generic. Indeed, if the endowments of resources are chosen by standard random processes, then with probability 1 all equilibria will be determinate.

Sraffa would likely have remained unperturbed by this fact. From the point of view of a multi-period model of capital accumulation the endowments of produced resources are endogenous variables, not randomly chosen parameters. Moreover, past production can generate the seemingly unlikely endowment levels at which indeterminacy occurs. In our basic model, indeterminacy occurs only if no resource is in excess supply; given the technology of this model, such endowments indeed form a negligible subset of all possible endowment levels. But it will be the norm for factors not to be in excess supply when resource levels are determined by past production decisions. A model of ongoing economic

1. Many of Sraffa's followers, *e.g.* Garegnani (1976), Eatwell (1982), have argued that modern general equilibrium theory's use of varying relative prices is a prime analytical error. See also Sraffa (1932), where economies without constant relative prices are described as not being "in equilibrium".

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activity—the production of commodities by means of commodities—can therefore systematically induce indeterminacy. The appearance of indeterminacy helps explain Sraffa's criticism of theories that treat the economy as a "one-way avenue that leads from 'factors of production' to 'consumption goods'". The difficulty with such theories does not lie in assuming that current production is constrained by available resources, but in taking endowments to be arbitrary rather than determined by past economic activity.

The significance of Sraffian indeterminacy lies in three related points. The first is simply to clarify what is necessary for Sraffa's analysis to constitute a valid internal criticism of mainstream theory. Second, the appearance of indeterminacy corrects a common misconception about general equilibrium theory. Even in a conventional finite-horizon setting, inelastically demanded factors and linear activities can readily lead to indeterminacy. The Sraffian focus on dynamic models is ideally suited to bring out this phenomenon. Third, Sraffa's criticisms of neoclassical economics are linked to longstanding disputes about the plausibility of marginal productivity theory. Outside of general equilibrium theory, it is common to think that the marginal-productivity explanation of factor demand, which assumes that factors can be combined productively in any proportion, is central to neoclassical economics. Contemporary general equilibrium theory in contrast is agnostic about how substitutable factors are and has left the impression that such details about technology are irrelevant to pure theory. The present paper casts Sraffa as a traditional critic of marginal productivity theory: his analysis points to difficulties of models with limited factor substitutability and implicitly challenges the general equilibrium view of the importance of factor substitution assumptions.

The paper begins (Section 2) with a presentation of Sraffa's indeterminacy argument and a general equilibrium model that incorporates Sraffa's price equations. I then indicate the standard, informal argument for why such general equilibrium models have locally unique equilibria. In Section 3, I show that Sraffa's claim that the equilibria with positive prices are indeterminate is nevertheless correct as long as a subset of equilibrium conditions satisfy an independence or rank condition. I also show how this claim and the literature on generic determinacy are consistent with each other.

The remainder of the paper undertakes various adjustments of the basic model in order to pin down what drives the indeterminacy conclusion. Section 4 surveys these variations and the underlying logic of when indeterminacy occurs. The basic model of Sections 2 and 3, like Sraffa's, allows only a single activity for each produced good. Section 5 allows multiple activities. In the extreme case where factors can be combined in any proportion—the "neoclassical" description of technology—I show that the earlier indeterminacy arguments can never succeed. But a general model with an arbitrary finite number of linear activities and joint production (multiple outputs) allows no universal conclusion to be drawn: determinacy depends on how many activities are in use. The general case also elucidates Sraffa's accounting of multiple dimensions of indeterminacy. Sraffa's arguments, although incomplete, led him to the correct conclusion. I explain how Sraffa, although looking at equilibria with constant relative prices, reasoned correctly about the indeterminacy of non-steady-state equilibria.

Section 6 uses an overlapping generations model to place the two-period model in a dynamic setting. I show that the steady-state equilibria are determinate generically (echoing Kehoe and Levine's (1984) study of the pure exchange case), thus demonstrating that constant relative prices preclude indeterminacy. Also, we will see that the endowment points that yield indeterminacy in the two-period model (where equilibria can be non-steady state) arise robustly.

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Section 7 concludes with comments on the size (rather than the dimension) of Sraffian indeterminacy and Sraffa's famous remark that his analysis makes no assumption of constant returns.

Before proceeding, let me mention another general-equilibrium interpretation of Sraffa, Geanakoplos (1980), which both explored Sraffa's indeterminacy claim and provided a pioneering study of indeterminacy in the overlapping generations model. Geanakoplos connects the two species of indeterminacy by linking Sraffa's description of a production process "without beginning or end" to the fact that OLG indeterminacy can arise when time extends "indefinitely into the past and into the future". On the other hand, as later research has revealed, OLG indeterminacy can arise when there is only a one-sided infinity stretching into the future. Furthermore, unlike indeterminacy in the present paper, OLG indeterminacy is unrelated to the nature of technology (or indeed to whether models even have a production sector). But analogously to our results in Section 6, Geanakoplos argues in a one-sector example that steady-state OLG equilibria are determinate.

2. SRAFFA'S MODEL AND THE GENERAL EQUILIBRIUM RESPONSE

Sraffa's model uses n material inputs and labour to produce the same n physical commodities at a later point in time. In the simplest version, each commodity is produced by one linear activity ("technique of production"). For each $j = 1, \ldots, n$, let the material inputs and labour necessary to produce one unit of j be represented by the n-vector $a_j = (a_{1j}, \ldots, a_{nj}) \geqq 0$ and the scalar $\ell_j \geqq 0$. We assume that each produced good requires a positive amount of some input.

Assumption A2.1. For each j, if $a_{ij} = 0$, $i = 1, \ldots, n$, then $\ell_j > 0$.

Sraffa requires that capital invested in each sector of the economy earns the same rate of profit. Assuming that labour is purchased when output is sold, we have

$$p_j = (1+r)(p_1 a_{1j} + \cdots + p_n a_{nj}) + w\ell_j, \qquad j = 1, \ldots, n,$$

where p_j is the price of good j, r is the interest rate, and w is the wage. Defining the column vectors $p = (p_1, \ldots, p_n)$ and $\ell = (\ell_1, \ldots, \ell_n)$, the matrix $A = [a_1 \ldots a_n]$, and letting A' denote the transpose of A, rewrite the above equations as

$$p = (1+r)A'p + w\ell. \tag{2.1}$$

Sraffa then notes that after normalization, say setting $p_1 = 1$, (2.1) constitutes n equations in the $n+1$ unknowns $(p_2, \ldots, p_n,\ w,\ 1+r)$. Since there is one degree of freedom, one variable—Sraffa chooses r—can be chosen exogenously while still satisfying (2.1). Sraffa concludes that competition leaves relative prices indeterminate.

There are at least three prominent neoclassical objections to the set-up of Sraffa's model. First, the equations in (2.1) make no mention of the demand for commodities, endowments, or the equilibrium of demand and supply. Consequently a $(p_2, \ldots, p_n,\ w,\ 1+r)$ satisfying (2.1) may not be consistent with market-clearing. Second, the same prices p appear on both the left- and right-hand sides of (2.1). Inputs and outputs, which appear at different times, should be distinguished by date and relative prices allowed to vary through time. Only in exceptional circumstances will endowments and preferences be consistent with the existence of an equilibrium where prices in different time periods are

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proportional. Third, producers are permitted no choice of technique; there is only one activity per output.

The third criticism is the least fundamental since the production set of the model, $Y=\{(y, -x, -l)\in R^n_+\times -R^n_+\times -R_+ \colon Ay\leqq x,\ \ell\cdot y\leqq l\}$, obeys standard general-equilibrium assumptions (convexity, free disposal, *etc.*). In Section 5, I introduce substitution possibilities and argue that their presence can limit the scope for indeterminacy. But even if empirically unreasonable, the lack of choice of technique is acceptable according to traditional general-equilibrium modeling practices.

To deal with the first two objections, we distinguish between the period 0 input prices $p^0=(p^0_1,\ldots,p^0_n)$ and the period 1 output prices $p^1=(p^1_1,\ldots,p^1_n)$. It is simplest to think of the outputs as consumption goods and the inputs as unconsumed factors whose endowment levels are determined by production decisions made prior to time 0. Since labour appears only at time 1, we retain the unsuperscripted w as the wage. When convenient, we use π to denote $(p^0, p^1, w, 1+r)$.

To derive the equilibrium conditions analogous to (2.1), consider the profit-maximization problem facing a typical firm with access to the n activities and able to borrow at interest rate r. Since firms pay for labour at the time output is sold, activity j run at the unit level yields revenue $\rho_j=p^1_j-w\ell_j$ at cost $c_j=p^0\cdot a_j$. Operating activity j at level y_j therefore requires borrowing c_jy_j at time 0 and paying $(1+r)c_jy_j$ back at time 1. Hence profits at time 1 are $\Pi(y)=\sum_{j=1}^n(\rho_j-(1+r)c_j)y_j$. If a maximum to $\Pi(y)$ exists, we must have $\rho_j\leqq(1+r)c_j$, and $y_j>0$ only if $\rho_j=(1+r)c_j$. Hence, $p^1_j\leqq(1+r)(p^0\cdot a_j)+w\ell_j$, for all j. If all activities are operating,

$$p^1=(1+r)A'p^0+w\ell.$$

Compared to (2.1), the difference between the number of unknowns and the number of equations has become more acute. Even allowing for the fact that we are now allowed two normalizations (I explain this presently), say $p^0_1=1$ and $p^1_1=1$, $2n$ unknowns remain: $2n-2$ goods prices, w, and r. According to the general equilibrium criticism, however, this surplus of endogenous variables is an artifact of having neglected equations setting supply equal to demand.

We therefore add a demand side to the model. The aggregate endowments of the factors of production are given by $(\omega_1,\ldots,\omega_n)=\omega\gg 0$ for the material inputs and $\omega_l>0$ for labour. We assume that factors are inelastically supplied; excess demand for material inputs and labour therefore equal $-\omega$ and $-\omega_l$, respectively, at all π. I relax this assumption in Section 6.

Aggregate demand for the n goods is given by $z(\pi)$. We assume that $z(\pi)\gg 0$ at all $\pi\gg 0$. To explain the further restrictions that z should satisfy, suppose that all resources are owned by utility-maximizing agents. A typical agent k has the endowment (ω^k, ω^k_l) and ω^k can be loaned to firms in period 0; the interest (or profits) earned are used to finance consumption purchases in period 1. Letting q^k denote the dollar amount lent in period 0 and $z^k\geqq 0$ the consumption level of the n outputs, agent k faces the budget constraints: $q^k\leqq p^0\cdot\omega^k$ and $p^1\cdot z^k\leqq(1+r)q^k+w\omega^k_l$. It follows that z^k must satisfy

$$p^1\cdot z^k\leqq(1+r)p^0\cdot\omega^k+w\omega^k_l. \tag{2.2}$$

Multiplying (p^0, p^1, w) or $(p^1, w, 1+r)$ by any positive scalar leaves the set of z^k satisfying (2.2) unchanged. We therefore require that aggregate demand, $\sum_k z^k$, is unchanged when prices are scaled in these ways and that $\sum_k z^k$ is consistent with each z^k satisfying (2.2).

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Assumption A2.2. The function z is continuously differentiable and satisfies

(a) $z(p^1, p^0, w, 1+r) = z(\lambda p^1, \lambda p^0, \lambda w, 1+r) = z(\lambda p^1, p^0, \lambda w, \lambda(1+r))$ for all $\lambda > 0$, and

(b) $p^1 \cdot z(\pi) = (1+r)p^0 \cdot \omega + w\omega_l$ (Walras' law).

An *economy* is a technology matrix A satisfying A2.1, a vector of labour input requirements, ℓ, and a function z and endowments ω and ω_l satisfying A2.2. We use the standard general equilibrium definition of equilibrium: demand should be no greater than supply and prices must be consistent with profit maximization. We do not require that p^0 be proportional to p^1.

Definition D2.1. An equilibrium is a $(y, \pi) \geqq 0$ such that (a) $z(\pi) \leqq y$, $Ay \leqq \omega$, $\ell \cdot y \leqq \omega_l$, (b) $p^1 \leqq (1+r)A'p^0 + w\ell$.

The equilibrium occurs through two rounds of trading. At $t=0$, agents sell $\sum_k \omega^k = \omega$ to firms, receiving $\sum_k q^k$ in return; at $t=1$, agents use $\sum_k (1+r)q^k$ and $\sum_k w\omega_l^k = w\omega_l$ to purchase $\sum_k z^k = z(\pi)$. Note that D2.1 and A2.2(b) imply that any good in excess supply has a zero price.

To facilitate comparison with Sraffa, we have deviated slightly from the general equilibrium system for representing prices. Standard accounting uses one set of prices for all goods at all dates and does not include an interest rate. We use an interest rate and distinct price vectors in the two periods. This choice amounts to an accounting convention; later-period prices are preceded by the coefficient $1/(1+r)$, or, equivalently, earlier-period prices by $1+r$. Given an economy and an equilibrium (y, π), $(y, \hat{p} = ((1+r)p^0, p^1, w))$ is an equilibrium in the standard sense for the general equilibrium model with excess demands $\hat{z}(\hat{p}) = z(p^0, p^1, w, 1+r)$, $-\omega$, and $-\omega_l$ and production set Y.

We concentrate on the equilibria of natural interest to Sraffa, where prices are strictly positive. Let a *Sraffa equilibrium* be a (y, π) such that $\pi \gg 0$. To study determinacy, we need a system of equations that locally characterize these equilibria. First, observe that A2.2(a) and the homogeneity of D2.1(b) imply an uninteresting nominal indeterminacy of equilibrium: multiplying (p^0, p^1, w) or $(p^1, w, 1+r)$ by a scalar will produce another vector of equilibrium prices consistent with the same values of z and y. Since two such scalings are possible, we can make two normalizations. When all prices are positive, the first allows us to set $p_1^0 = 1$, the second $p_1^1 = 1$. In terms of notation, let $\bar{p}^0 = (1, p_2^0, \ldots, p_n^0)$, $\bar{p}^1 = (1, p_2^1, \ldots, p_n^1)$, and let $\bar{\pi} = (\bar{p}^0, \bar{p}^1, w, 1+r)$. Also, it is easy to show that if $\pi \gg 0$, then all of the inequalities in D2.1 hold with equality. Finally, in a Sraffa equilibrium, Walras' law implies that one of the inequalities in D2.1(a) is redundant. For example, to see that $z_1(\pi) = y_1$ is redundant, suppose that $Ay = \omega$, $\ell \cdot y = \omega_l$, and that $z_i(\pi) = y_i$, $i = 2, \ldots, n$, are satisfied. Then A2.2(b) implies that $p_1^1 z_1(\pi) + p_2^1 y_2 + \cdots + p_n^1 y_n = (1+r)A'p^0 \cdot y + w\ell \cdot y$. Given that D2.1(b) holds with equality, $z_1(\pi) = y_1$.

The following system of equations therefore locally determines the Sraffa equilibria. We use the notation $\bar{y} = (y_2, \ldots, y_n)$ and $\bar{z}(\pi) = (z_2(\pi), \ldots, z_n(\pi))$ and indicate the number of equations in parentheses.

$$\bar{z}(\bar{p}^1, \bar{p}^0, w, 1+r) = \bar{y} \qquad (n-1) \tag{2.3}$$

$$Ay = \omega \qquad (n) \tag{2.4}$$

$$\ell \cdot y = \omega_l \qquad (1) \tag{2.5}$$

$$\bar{p}^1 = (1+r)A'\bar{p}^0 + w\ell. \qquad (n) \tag{2.6}$$

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This system of $3n$ equations has $3n$ endogenous variables, $\bar{p}^1$, $\bar{p}^0$, w, r, and y. This equality of equations and unknowns underlies the standard presumption that this model is determinate: if the equations were independent, solutions of the above system of equations would be locally unique.

Note also that the original Sraffian requirement that $\bar{p}^0 = \bar{p}^1$, allowing (2.1) to be satisfied, would overdetermine the system; $n-1$ new equations would be added without any corresponding new variables. Consequently, the parameters of the model—for example, the endowments ω and ω_l—may well be inconsistent with the existence of such an equilibrium. Of course, at steady states, where the endowments of produced resources are endogenously determined, $\bar{p}^0 = \bar{p}^1$ will be satisfied.

3. SRAFFA'S THEOREM

Despite initial appearances, the system of equations (2.3)–(2.6) typically has a continuum of solutions: if we fix the vector y, as in Sraffa's book, then independent of the value of $\bar{\pi}$, (2.4) and (2.5) remain satisfied.[2] Having fixed y, the remaining equations (2.3) and (2.6) consist of $2n-1$ equations in the $2n$ price variables $\bar{\pi}$. As long as a rank condition is met, assuring us that the gradients of (2.3) and (2.6) are linearly independent and that the implicit function theorem can be applied, this excess of variables over equations is enough to prove indeterminacy.

Definition D3.1. An equilibrium $(y, \bar{\pi})$ is regular if the $2n-1$ by $2n$ matrix

$$\begin{pmatrix} D_{\bar{p}^1}\bar{z} & D_{\bar{p}^0}\bar{z} & D_w\bar{z} & D_r\bar{z} \\ \bar{I}_n & -(1+r)\bar{A}' & -\ell & -A'\bar{p}^0 \end{pmatrix}$$

has rank $2n-1$.[3]

In the Appendix, I show that for typical demand functions z and input requirement coefficients, all equilibria are regular.

An equilibrium $(y, \bar{\pi})$ is *indeterminate* if, for any $\varepsilon > 0$, there is an equilibrium price vector $\bar{\pi}^* \neq \bar{\pi}$ such that the Euclidean distance between $\bar{\pi}^*$ and $\bar{\pi}$ is less than ε.

Theorem 3.1. *Any regular Sraffa equilibrium is indeterminate.*

Proof. Let $(\bar{p}^1, \bar{p}^0, w, 1+r)^*$ be the equilibrium price vector corresponding to a regular Sraffa equilibrium. It is sufficient to show that a solution $(\bar{p}^1, \bar{p}^0, w, 1+r)$ to the equation system

$$\bar{z}(\bar{p}^1, \bar{p}^0, w, 1+r) - \bar{y} = 0,$$

$$\bar{p}^1 - (1+r)A'\bar{p}^0 - w\ell = 0,$$

can be found that is arbitrarily close to but distinct from $(\bar{p}^1, \bar{p}^0, w, 1+r)^*$. Given regularity, there exists at least one column corresponding to, say, the variable t, that we may eliminate from the matrix in D3.1, and produce a $2n-1$ by $2n-1$ matrix with rank $2n-1$. The implicit function theorem then guarantees that there exists an open set $T \subset R$ with

2. Since y consists of n variables and (2.4) and (2.5) constitute $n+1$ equations, the gradients of the system (2.3)–(2.6) must be linearly dependent. The standard proof of local uniqueness therefore fails.

3. Given an n by m matrix C, let $\bar{C}$ indicate the matrix consisting of columns 2 through m of C and let $\bar{C}'$ indicate columns 2 through n of C'. I_n is the n by n identity matrix.

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$t^* \in T$ and a continuously differentiable function $F: T \to R^{2n}$ such that $F(t^*) = (\bar{p}^1, \bar{p}^0, w, 1+r)^*$ and such that for all $t \in T$, $F(t)$ solves the above equation system. The continuity of F implies indeterminacy. ||

A couple of remarks are in order. First, for typical functions z, the one-dimensional set of (normalized) equilibrium prices constructed in the proof can be parameterized by any of the $2n$ variables $(\bar{p}^1, \bar{p}^0, w, 1+r)$. We can exogenously vary r, as Sraffa does, or any of the other prices. Second, the utilities of the agents underlying the aggregate demand function will typically change as prices vary. As an example, suppose $n = 1$; equilibrium prices are then defined by the single equation $1 = (1+r)a + w\ell$. An agent k's second-period wealth and level of consumption equals $w\omega_l^k + (1+r)\omega^k$. A simple calculation shows that changes in r will alter k's consumption as long as k's endowments do not satisfy the knife-edge condition

$$\frac{a}{l} = \frac{\omega^k}{\omega_l^k}. \tag{3.1}$$

How is the indeterminacy of (2.3)–(2.6) consistent with the generic determinacy of equilibrium?[4] To answer this question, note that (2.4) and (2.5) constitute $n+1$ equations. It follows that for almost every value of ω and ω_l, there will be no n-vector y satisfying (2.4) and (2.5). In the $n = 1$ case, for example, (ω, ω_l) must obey the condition $a/\ell = \omega/\omega_l$ if (2.4) and (2.5) both hold. Hence, the (ω, ω_l) satisfying (2.4) and (2.5) are only a one-dimensional subset of the two-dimensional set of possible endowments.[5] When the parameters ω and ω_l satisfy (2.4) and (2.5), we will say they are *Sraffa endowments*. Since the Sraffa endowments have measure zero (as a subset of R^{n+1}), Theorem 3.1 does not contradict generic determinacy. Outside of this measure-zero set, at least one of the $n+1$ factors will be in excess supply and its price will equal zero. Our focus on Sraffa equilibria, which have $(\bar{p}^0, w) \gg 0$ and thus no factors in excess supply, has guaranteed that (ω, ω_l) are Sraffa endowments and hence that equilibrium is indeterminate.

As argued in the Introduction, the fact that the Sraffa endowments have measure zero is not, by itself, of great significance. If the material inputs are themselves produced, then ω is determined by earlier savings decisions; measure-zero sets of endowments are thus not *prima facie* unlikely. Suppose that the vector of required material inputs, Ay, is strictly positive in equilibrium. If ω is determined by past production, it is implausible that any capital good k would have been produced in such quantities that it will later be in excess supply (*i.e.* that $A_k y < \omega_k$, where A_k is the k-th row of A), since resources would then have been wasted in the production of ω_k. Furthermore if agents have perfect foresight, they would have anticipated that $p_k^0 = 0$ and would not have invested in capital good k. So it would seem, at least informally, that capital goods are produced in quantities that yield Sraffa endowments. See Mandler (1995) for a demonstration that the zero-measure endowments that cause indeterminacy appear routinely in the later periods of finite-horizon general equilibrium models. Also, as we will see in Section 6, the endowments that yield indeterminacy in the 2-period model occur robustly at steady-state equilibria.

4. For a generic determinacy theorem that applies to the current context, see Mas-Colell (1985), Theorem 8.7.3, where technology is described by linear activities and where a subset of commodity excess demands can be (locally) inelastic with respect to price.

5. More generally, the range space of the linear transformation $\binom{A}{\ell'}$ has dimension less than or equal to n. Hence, almost all values of the $n+1$-vector (ω, ω_l) will not be elements of this space and (2.4) and (2.5) will have no solution. Note that aggregate endowments can permit indeterminacy even though variations in factor prices change individual consumption. For example, if $n = 1$, aggregate endowments can satisfy $a/\ell = \omega/\omega_l$ while each (ω^k, ω_l^k) violates (3.1).

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4. WHAT CAUSES INDETERMINACY?

The remainder of the paper adjusts various ingredients of the basic model and examines the consequences for indeterminacy. In this section, I give an overview of the intuition underlying these variations. Fundamentally, indeterminacy in the Sraffa model is caused by the shortage of zero-profit conditions relative to the number of inelastically supplied factors: factor prices can therefore change without violating any of the zero-profit conditions, and consequently neither the demand for nor the supply of factors varies.

Changes to the model that redress the imbalance between zero-profit conditions and inelastically supplied factors consequently restore determinacy. As we will see in Section 5, a differentiable (neoclassical) description of technology guarantees that there are always as many marginal productivity conditions—conditions setting production function derivatives equal to factor prices—as factors. Determinacy therefore typically obtains. We then consider a general model of linear activities. With an arbitrary number of activities (rather than n as in the basic model) and if $n+1$ activities are used in equilibrium, the added zero-profit condition is enough to ensure that equilibria are typically determinate. But the linear activities model also allows joint production, which can reduce the number of activities in use and thus the number of binding zero-profit conditions. Unlike the differentiable case and the Sraffa equilibria of Section 2, therefore, no general conclusion is possible.

A second way to ensure determinacy is to require that relative prices are constant through time: $\bar{p}^1 = \bar{p}^0$. We establish this principle formally in the model of steady states in Section 6. But the underlying point can be seen in the basic model. The requirement that $\bar{p}^1 = \bar{p}^0$ adds $n-1$ new restrictions on factor prices. Hence, since there is only one dimension of indeterminacy in the basic model, these equations usually eliminate indeterminacy when $n \geqq 2$.[6] Steady-state equilibria prohibit variations in the period 0 prices of produced factors independently of the period 1 prices of the same physical goods currently in production. Sraffa's demonstration of indeterminacy when relative prices are constant therefore cannot be validated.

The model of steady states also allows us to relax one final feature of the basic model, the highly restrictive assumption that *all* period 0 factors are supplied inelastically. Although the exact indeterminacy argument of Section 3 no longer applies when only some of the factors are supplied inelastically, a slight variant of the reasoning can still be invoked. If some subset of the $n+1$ factors is supplied inelastically and is used in production by fewer than $n+1$ activities, then once again there are too few zero-profit conditions to pin down factor prices. Hence it is the combination of non constant relative prices, linear activities, and the presence of *some* inelastically supplied resources that explains the occurrence of indeterminacy.

5. CHOICE OF TECHNIQUE

The proof of Theorem 3.1 parallels Sraffa's reasoning in that profit-maximizing firms will agree to produce the same vector of aggregate quantities at multiple price vectors. Differentiable technologies preclude this form of argument.

To see this, let the production set be $Y = \{(y, -x, -l) \in R_+^n \times -R_+^n \times -R_+ : g(y, x, l) \leqq 0\}$, where y, x, and l represent output produced, material inputs consumed, and labour consumed, respectively, and where g is convex, differentiable, and homogeneous of degree one. Producers maximize $\bar{p}^1 \cdot y - (1+r)\bar{p}^0 \cdot x - wl$ s.t. $g(y, x, l) \leqq 0$. An *equilibrium* is a

6. Since Walras' law no longer implies that one of the market-clearing conditions is redundant in models of steady states, the added supply equals demand equation ensures generic determinacy in the $n = 1$ case as well.

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$(\bar{\pi}, y, x, l) \geqq 0$ such that (y, x, l) is a solution to this problem and such that $z(\bar{\pi}) \leqq y$, $x \leqq \omega$, $l \leqq \omega_l$.

Suppose, in the manner of our earlier argument, that we fix the aggregate quantities (y, x, l) and that both $\bar{\pi}$ and $\bar{\pi}^*$ are equilibrium price vectors for (y, x, l). The Kuhn–Tucker theorem implies that there exist $\lambda > 0$ and $\lambda^* > 0$ such that $(\bar{p}^1, (1+r)\bar{p}^0, w) = \lambda Dg(y, x, l)$ and $(\bar{p}^1, (1+r)\bar{p}^0, w)^* = \lambda^* Dg(y, x, l)$. Hence, $(1/\lambda)(\bar{p}^1, (1+r)\bar{p}^0, w) = (1/\lambda^*)(\bar{p}^1, (1+r)\bar{p}^0, w)^*$. Since $\bar{p}_1^0 = \bar{p}_1^{0*}$, $\lambda = \lambda^*$ and $\bar{\pi} = \bar{\pi}^*$.

This does *not*, it should be emphasized, show that equilibria are locally unique. The above argument only shows that there cannot be multiple equilibrium price vectors $\bar{\pi}$ for a single vector of aggregate quantities (y, x, l). Local uniqueness also requires that for each equilibrium $(\bar{\pi}, y, x, l)$ there are no other equilibria with distinct aggregate quantities arbitrarily close to $(\bar{\pi}, y, x, l)$. Such a result is only true generically, and, for reasons already indicated, genericity theorems ought not to take contemporaneous parameters to be arbitrary and must be cast in a properly intertemporal framework.[7]

Smooth production sets are highly stylized models for factor substitution; they assume an infinite number of ways to combine factors together. Consider instead a general model of linear activities that allows an arbitrary number of activities, say m, and joint production. For each activity j, let $b_j \geqq 0$ be the n-vector of outputs produced when j is run at the unit level. As before, a_j and ℓ_j represent the material inputs and labour required by activity j. An equilibrium is a $(y, \pi) \geqq 0$ such that

$$z(\pi) \leqq \textstyle\sum_{j=1}^m b_j y_j, \qquad \sum_{j=1}^m a_j y_j \leqq \omega, \qquad \sum_{j=1}^m \ell_j y_j \leqq \omega_l, \tag{5.1}$$

$$b_j \cdot p^1 \leqq (1+r) a_j \cdot p^0 + w\ell_j, \qquad j = 1, \ldots, m. \tag{5.2}$$

We restrict ourselves to normalized equilibria $(y, \bar{\pi})$ in which all outputs are produced $(\sum_{j=1}^m b_j y_j \gg 0)$ and where $\bar{\pi} \gg 0$.

Let v denote the number of activities in use, *i.e.* activities with $y_j > 0$. Unlike the model of Sections 2 and 3, v can now be either greater than or less than n, the number of outputs. Joint production can reduce v below n, while the choice of technique allows v larger than n. Given an equilibrium, let the v activities in use have the indices $1, \ldots, v$. Analogously to the model of Section 2, all of the inequalities in (5.1) and the first v inequalities in (5.2) hold with equality.

To mimic the indeterminacy argument of Section 3, recall that one market-clearing condition is redundant, and consider the $n + v - 1$ equations

$$z_i(\pi) = \textstyle\sum_{j=1}^m b_{ij} y_j, \qquad i = 2, \ldots, n, \tag{5.3}$$

$$b_j \cdot p^1 = (1+r) a_j \cdot p^0 + w\ell_j, \qquad j = 1, \ldots, v. \tag{5.4}$$

If we fix y and if the inequalities in (5.2) corresponding to activities *not* in use are strict, the omitted conditions from (5.1) and (5.2) remain satisfied for small variations in $\bar{\pi}$. Hence, subject to satisfying the appropriate independence condition, indeterminacy obtains if the number of variables in $\bar{\pi}$, $2n$, is greater than $n - 1 + v$, *i.e.* if the number of factors $(n+1)$ is greater than the number of activities in use (v).

Introducing a general model of linear activities therefore has contradictory effects. If, due to the choice of technique, the number of activities in use rises from n in the basic model to $n + 1$, our earlier indeterminacy argument can no longer be applied. (But note that the generalized indeterminacy argument introduced in the next section, which applies

7. See Mandler (1997) for a proof of local uniqueness when technology is smooth and endowments are endogenously determined by past economic decisions.

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to subsectors of an economy and is independent of the total number of activities in use, is unaffected.) If, on the other hand, v falls below n, multiple dimensions of indeterminacy can appear: more than one price can be independently varied without violating any equilibrium condition.

It is interesting to compare our account of the dimension of indeterminacy with Sraffa's analysis of joint production (1960, Chapter 7). As in our model, Sraffa permitted multiple production processes and processes yielding more than one output. He recognized the potential for additional indeterminacy beyond the single dimension claimed for the equations in (2.1), if there are "more [goods] prices to be ascertained than there are processes" in use. Since Sraffa constrained input and output prices to be equal, the number of goods prices is n. Sraffa's condition for there to be more than one dimension of indeterminacy thus coincides with ours.

This correspondence is due to a broader parallel between Sraffa's method of counting equations and unknowns and the method used in Section 3, a fact that has been obscured by Sraffa's assumption that $\bar{p}^0 = \bar{p}^1$. To see the parallel more clearly, consider again Sraffa's original argument for a single dimension of indeterminacy. Sraffa observed that the number of relative prices in (2.1), $n+1$, is one larger than the number of equations, n. Similarly, in (2.6), *if we ignore* $\bar{p}^1$, the $n+1$ remaining variables ($\bar{p}^0$, w, and r) also comprise one more variable than the number of equations, n. In Section 3, of course, we also took into account the effect of prices on demand, *via* (2.3). Since this step introduced as many equations ($n-1$) as there are variables in $\bar{p}^1$, Sraffa's counting procedure, although formally incorrect, leads him to the right conclusion. By assumption he suppressed the $n-1$ output price variables $\bar{p}^1$, but conveniently he also suppressed the same number of equations setting the demand for output equal to supply.

Returning to the general linear activities model, the parallels between the two counting procedures remain. Sraffa examined an equation system analogous to (5.4), except that he assumed that $\bar{p}^0 = \bar{p}^1$, thus eliminating $n-1$ price variables. Sraffa concluded that there will be multiple dimension of indeterminacy if $n > v$. The dimension of indeterminacy, $n+1-v$, is the same as in our model since the suppression of the $n-1$ price variables again exactly matches the $n-1$ missing supply-equals-demand equations, (5.3).

Surprisingly, Sraffa did not exploit the appearance of extra dimensions of indeterminacy. After noting the mathematical possibility of greater indeterminacy, he simply assumed that $n = v$. Nor did Sraffa mention that the presence of multiple techniques of production can *eliminate* indeterminacy when $v \geqq n+1$.[8]

6. STEADY-STATE EQUILIBRIA

This section shows that Sraffian indeterminacy typically does not arise in steady-state equilibrium, even in the presence of linear activities and inelastically supplied factors. Although this conclusion holds quite broadly, I stick as closely as possible to the basic model and suppose there is only one activity for each of the n produced goods. In the spirit of Sraffa's book, I assume that all of the material inputs are produced, rather than being natural resources. Since at least one of the n goods is consumed, the number of inelastically supplied factors is, counting labour, at most n. With the number of inelastic factors no larger than the number of activities in use (n), the argument of Theorem 3.1

8. In fact, Sraffa systematically set v to ensure that there is exactly one dimension of indeterminacy. See, for example, the chapter on land, where Sraffa assumed that the number of activities in use equals the number of produced goods plus the number of varieties of scarce land, thus introducing as many additional activities (and hence zero-profit conditions) to the basic model as there are rental prices for land varieties.

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does not directly apply. Therefore, to confirm that it is the steady-state requirement that leads to determinacy, we must show that Sraffian indeterminacy can still arise when some of the initial-period factors are supplied elastically and relative prices are *not* held constant through time.

We accomplish this by applying the condition that the number of factors is larger than the number of activities in use to a subset of factors. To see an example and set up the notation for steady-state equilibria, return to the basic model and allow the supply of the first $n-k$ factors to be a function of π, $\omega_1(\pi)$. Let the last k material inputs and labour continue to be supplied inelastically. Let A_1 and A_2 be the first $n-k$ rows of A and the last k rows of A, respectively, and let ω_2 be the supply of last k factors. Normalized equilibria (with positive prices and all outputs produced) are then defined by the equations

$$\bar{z}(\bar{\pi}) = \bar{y}, \qquad (n-1) \tag{6.1}$$

$$A_1 y = \omega_1(\pi), \qquad (n-k) \tag{6.2}$$

$$A_2 y = \omega_2, \ell \cdot y = \omega_\ell, \qquad (k+1) \tag{6.3}$$

$$\bar{p}^1 = (1+r)A'\bar{p}^0 + w\ell. \qquad (n) \tag{6.4}$$

Let v indicate the number of activities that use the $k+1$ inelastically supplied factors (activity j uses i if $a_{ij} > 0$). To repeat the by-now-standard argument, suppose we fix, at their equilibrium values, these v activity levels. Then (6.3) remains satisfied independent of the values of $\bar{\pi}$ and the other $n-v$ activity levels. The remaining equations—(6.1), (6.2), and (6.4)—comprise $3n-k-1$ equations in the remaining $3n-v$ variables. The difference between the number of variables and equations, $k+1-v$, if positive, is the dimension of indeterminacy. Thus, Sraffian indeterminacy can still occur if some factors are supplied elastically, or, equivalently, if some of the period 0 commodities are consumption goods. Of course, a complete proof requires that an independence condition is satisfied.

It is clear that the current argument can be applied to an arbitrary subset of inelastically supplied factors. The following definition therefore provides the key prerequisite for indeterminacy.

Definition D6.1. The indeterminacy condition for a 2-period model is satisfied if there is a set of s inelastically supplied factors with positive prices such that the number of activities using these factors is strictly less than s.

We turn to the steady-state model. There are n produced goods per period, which use labour and some subset of the previous period's n goods as inputs. To construct a demand side, we use the simplest option of having overlapping generations of agents with 2-period lives. Agents face prices $p^0 \geqq 0$ and $w^0 \geqq 0$ when young and $p^1 \geqq 0$ and $w^1 \geqq 0$ when old and can borrow or lend at the interest rate r. Let $\rho = (p^0, w^0, p^1, w^1, 1+r)$.

Let labour and the final k material goods again be the commodities with inelastic excess-demand (the "factors"). Let $\omega_l^0 > 0$ and $\omega_l^1 \geqq 0$ indicate the supply of labour when young and old, respectively, and let $\omega_l = \omega_l^0 + \omega_l^1$. As mentioned, there are no exogenous endowments of the k material factors. Aggregate demand for the first $n-k$ produced goods (the "consumption goods") is given by a pair of continuously differentiable aggregate demand functions, $z^0(\rho)$ and $z^1(\rho)$, indicating demand when young and old, respectively. We assume that z^0 and z^1 both obey homogeneity, *i.e.* for any $\lambda > 0$, $z^i(p^0, w^0, p^1, w^1, 1+r) = z^i(\lambda p^0, \lambda w^0, \lambda p^1, \lambda w^1, 1+r) = z^i(p^0, w^0, \lambda p^1, \lambda w^1, \lambda(1+r))$, and together obey Walras' law, $(1+r)(p^0 \cdot (z^0(\rho), 0) - w^0\omega_l^0) + p^1 \cdot (z^1(\rho), 0) - w^1\omega_l^1 = 0$. When z^i is

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evaluated at a p such that $p^0 = p^1 = p$ and $w^0 = w^1 = w$, we use the notation $z^i(p, w, 1+r)$. Let $z(p, w, 1+r) = \sum_{i=0}^{1} z^i(p, w, 1+r)$.

The matrix A and vector ℓ meet the assumptions of Section 2. We also assume that each of the n produced goods uses labour, either directly or indirectly, as an input (thus bounding the set of goods that can be produced) and that it is possible to produce all of the goods in strictly positive amounts, *i.e.* there exists $y \geqq 0$ such that $y \gg Ay$. As usual, let $\bar{p}$ denote $(1, p_2, \ldots, p_n)$. Let y_1 and y_2 be the first $n-k$ and last k components of y, respectively.

Definition D6.2. A (steady-state, normalized) equilibrium is a $y > 0$, $(\bar{p}, w, 1+r) \geqq 0$ such that

$$z(\bar{p}, w, 1+r) + A_1 y \leqq y_1, \tag{6.5}$$

$$A_2 y \leqq y_2, \tag{6.6}$$

$$\ell \cdot y \leqq \omega_l, \tag{6.7}$$

$$\bar{p} \leqq (1+r)A'\bar{p} + w\ell.^{9} \tag{6.8}$$

Equilibria proceed through time as follows. During each period, capital goods invested in the previous period and owned by the old—$(y_1 - z(\bar{p}, w, 1+r), y_2)$—are combined with the aggregate labour supply, ω_l, to produce y_1 and y_2. A portion of y_1, $z(\bar{p}, w, 1+r)$, is purchased for use as the terminal consumption of the old and the initial consumption of the young, while the remainder of y_1 and all of y_2 are purchased by the young and invested in the next period's production.

We assume henceforth that (1) $z(p, w, 1+r) \gg 0$, for all $(p, w, 1+r) \gg 0$, (2) if the price of any of the consumption goods equals 0, then demand is infeasible,[10] and (3) each of the $k+1$ factors is used, either directly or indirectly, as an input to some consumption good. Under these assumptions, it is not difficult to see that $(y, \bar{p}, w, 1+r) \gg 0$ and that (6.5)–(6.8) hold with equality.

A comparison of (6.5)–(6.8) and (2.3)–(2.6) reveals that we have the steady-state parallel to the basic model. Outside of the fact that $p = p^0 = p^1$, the only substantial differences are that a portion of consumption, $A_1 y$, is now invested in production (though the reader is free to suppose $A_1 = 0$) and that the endowment of the inelastically supplied resources (ω in Section 2, ω_2 at the beginning of this section) is now the endogenous variable, y_2. Due to the way in which $1+r$ appears in Walras' law, the standard argument that one of the equilibrium conditions is redundant is not valid in the present model.

It is entirely possible that s of the inelastically demanded goods are used by fewer than s activities; all that is needed is that the corresponding s rows of $\binom{A}{\ell'}$ have sufficiently few non-zero entries. Furthermore, this case is compatible with the fact that the endowments that allow D6.1 to be satisfied are contained in a measure-zero set; equilibrium ensures that the stock of inelastically demanded resources, y_2, satisfies $A_2 y = y_2$. As argued informally in Section 3, such endowments therefore cannot be dismissed out of hand.

9. Observe that if $r = 0$, the steady-state equilibria are Walrasian equilibria for a standard, finite-horizon model with n goods and labour. Hence, steady-state equilibria exist if we drop the normalization requirement that $p_1 > 0$. Using different arguments (not relying on setting $r = 0$), existence of equilibrium, steady-state and otherwise, can be established under wide-ranging conditions. See Muller and Woodford (1984) for a discussion.

10. That is, there is no $(y, p) \geqq 0$ such that one of the first $n-k$ coordinates of p equals 0 and such that $z(p, w, 1+r) + A_1 y \leqq y_1$, $\ell \cdot y \leqq \omega_l$, $A_2 y \leqq y_2$.

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Despite the systematic appearance of the troublesome endowment points, the theorem below shows that the steady-state equilibria are generically determinate. Of course, just as in a standard general equilibrium model, indeterminacy may occur in exceptional circumstances. We therefore parameterize the model and show that at almost all parameters, equilibria are determinate.

For an arbitrary pair of demand functions z^0 and z^1 and labour endowment ω_l^0 meeting our assumptions, and for $h^0=(h_1^0,\ldots,h_{n-k}^0)\gg 0$ and $h^1>0$, define the perturbed demand functions, $z_i^0(h)=z_i^0+(w^0/p_i^0)h_i^0$, $z_i^1(h)=z_i^1+(1+r)(w^0/p_i^1)h^1$, $i=1,\ldots,n-k$, and perturbed labour endowment, $\omega_l^0(h)=\omega_l^0+\sum_{i=1}^{n-k}h_i^0+(n-k)h^1$. The perturbed functions still obey Walras' law and homogeneity, and the demand for consumption goods remains positive.[11] As for technology, we must make sure that our perturbations do not eliminate cases where goods are used as inputs in only a strict subset of the activities; otherwise, any potential for satisfying D6.1 would be eliminated. Beginning with an arbitrary A and ℓ, let A^* and ℓ^* be admissible technology parameters, if and only if $a_{ij}^*>0$ implies $a_{ij}>0$ and $l_i^*>0$ implies $\ell_i>0$.

An *economy* is a choice of an admissible A and ℓ and an h; given these parameters, we can let the set of economies be an open subset of a finite dimensional Euclidean space. An equilibrium $(y,\bar{p},w,1+r)$ is *determinate* if there does not exist a sequence of equilibrium prices $(\bar{p},w,1+r)_t\neq(\bar{p},w,1+r)$ converging to $(\bar{p},w,1+r)$.

Theorem 6.1. *There is an open, full-measure subset of parameters containing only economies whose equilibria are determinate.*

Proof. See Appendix. ||

7. CONCLUSION

Sraffian indeterminacy is a coherent claim that can be translated into the language of general equilibrium theory, but it cannot survive in Sraffa's preferred framework of constant relative prices. There are substantive and telling objections that can be made to the constant relative price model of Section 6, but internal inconsistency and indeterminacy are not among them.

This paper has not addressed the *size* of Sraffian indeterminacy. Consider an equilibrium $(y,\bar{\pi})^*$ of the general linear activities model of Section 5. As we know, multiple price vectors may well support a single vector of aggregate quantities. But, with enough variety among the activities, the maximal distance between any two supporting price vectors may be small. A solution $\bar{\pi}$ to (5.3) and (5.4) that is far from $\bar{\pi}^*$ may cause some unused activity to make positive profits; hence the solution would not be an equilibrium price vector. The purely local argument demonstrating indeterminacy would not be threatened, of course, but the indeterminacy would not be as distressing. My purpose, however, has been to clarify the logic of what the general equilibrium model allows and what it precludes, not to evaluate which cases are likely.

Sraffa's comment that his analysis does not presuppose "constant returns" has provoked considerable debate over the years. See, for example, the exchange by Levine (1974), Burmeister (1975), and Eatwell (1977). The models in this paper have certainly assumed constant returns to scale in the sense that if *all* inputs are multiplied by some scalar λ,

11. Note that our parameterization allows ω_l^0 to vary. This is legitimate, despite the arguments of Section 3, since ω_l^0 does not vary independently of the equilibrium value of y_2.

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then the output (or vector of outputs), produced from these inputs is also multiplied by λ. Such an assumption is essential to prevent price-taking firms from expanding boundlessly (when there is increasing returns to scale) or from making profits beyond the amount earned on invested capital at the common profit rate (when there is decreasing returns to scale). To avoid these outcomes, Sraffa had little choice but to accept constant returns to scale in the above sense. But this requirement is consistent with Sraffa's comment about the irrelevance of constant returns. In contrast to current usage, Sraffa used "constant returns" to refer to industries where there is constant returns to scale *and* where increases in scale are feasible and do not change the cost of any input. "Diminishing returns" refers to industries where there is some input in fixed supply (or that cannot be produced at constant cost) whose shadow or market price rises as output increases, *e.g.* a mining industry with limited quantities of each type of mine. Diminishing returns *in this sense* is obviously compatible with the contemporary meaning of constant returns to scale.[12] It seems, therefore, that the mystery of Sraffa's constant-returns assumption is an artifact of terminology.

APPENDIX

I state and prove a genericity result for regularity as defined in D3.1. The n^2 coefficients in the matrix A and the n coefficients in the vector ℓ will be the technology parameters. To parameterize demand, suppose we begin with a $z(\pi)$ satisfying A2.2. For $h = (h_2, \ldots, h_n) \in R^{n-1}_{++}$, define the perturbed demand function

$$z^h(\pi) = \left(z_1(\pi) + \frac{\sum_{i=2}^n z_i(\pi) p_i^1 h_i}{\sum_{i=2}^n p_i^1 h_i},\ z_2(\pi) - \frac{z_2(\pi) h_2}{\sum_{i=2}^n h_i}, \ldots, z_n(\pi) - \frac{z_n(\pi) h_n}{\sum_{i=2}^n h_i} \right).$$

It is readily confirmed that any resulting z_h satisfies A2.2 and that $z_h(\pi) \gg 0$ for all $\pi \geqq 0$. Let the set of economies, Γ, be an arbitrary open subset of $R^{n^2+2n-1}_{++}$ with typical element $(A, \ell, h) \in \Gamma$. I have taken the simplest route here. There would be no difficulty in constraining arbitrary subsets of technology coefficients to equal 0, as in Section 6.

Theorem A.1. *The set of economies such that each equilibrium is regular is open and has full measure in* Γ.

Proof. The proof of the full-measure claim is an application of the transversality theorem (see Guillemin–Pollack (1974)). Let $\Gamma^* = \Gamma \backslash \{(A, \ell, h) \in \Gamma : \sum_{i=2}^n h_i = 1\}$. Define $F: R^{2n}_{++} \times \Gamma^* \to R^{2n-1}$ by $F(\bar{\pi}, A, \ell, h) = (\bar{z}_h(\bar{p}^1, \bar{p}^0, w, 1+r) - \bar{y}, \bar{p}^1 - (1+r)A'\bar{p}^0 - w\ell)$. Given the exclusion of the h such that $\sum_{i=2}^n h_i = 1$, $D_{h, a_{2,1}, \ldots, a_{i,i-1}, \ldots, a_{n,n-1}, \ell_n} F$

$$= \begin{bmatrix} z_2((\sum_i h_i)^{-1} - h_2(\sum_i h_i)^{-2}) & z_2 h_2(\sum_i h_i)^{-2} & \cdots & z_2 h_2(\sum_i h_i)^{-2} & & & & \\ z_3 h_3(\sum_i h_i)^{-2} & z_3((\sum_i h_i)^{-1} - h_3(\sum_i h_i)^{-2}) & & z_3 h_3(\sum_i h_i)^{-2} & & & & \\ & & \ddots & & 0 & \cdots & 0 & 0 \\ \cdot & & & & & & & \\ z_n h_n(\sum_i h_i)^{-2} & z_n((h_n(\sum_i h_i)^{-2} & & z_n(\sum_i h_i)^{-1} - h_n(\sum_i h_i)^{-2}) & & & & \\ & 0 & & & (1+r)p_2 & & 0 & 0 \\ & \vdots & & & & \ddots & & \vdots \\ & 0 & & & 0 & & (1+r)p_n & 0 \\ & 0 & & & 0 & \cdots & 0 & w \end{bmatrix}$$

12. Sraffa (1960) explicitly introduced fixed factors (and used "diminishing returns" to describe the situation) in his discussion of land and rent. Fixed factors could be added to our model as well with no difficulty. See Sraffa (1926) for more examples of how the concepts of constant, diminishing, and increasing returns are used. Sraffa (1926) also indicates acceptance of the principle that if increasing returns does not occur as an externality, then perfect competition will be impossible.

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has rank $2n-1$. Consequently, the subset of parameters such that

$$D_{\bar{\pi}}F(\bar{\pi}, A, \ell, h) = \begin{pmatrix} D_{\bar{p}^1}\bar{z}_h(\bar{\pi}) & D_{\bar{p}^0}\bar{z}_h(\bar{\pi}) & D_w\bar{z}_h(\bar{\pi}) & D_r\bar{z}(\bar{\pi}) \\ \bar{I}_n & -(1+r)\bar{A}' & -\ell & -A'\bar{p}^0 \end{pmatrix}, \tag{A.2}$$

(*i.e.* the matrix in D3.1) has rank $2n-1$ at all equilibria has full measure in Γ.

The proof of openness is essentially standard. Suppose, to the contrary, that the set of economies such that all equilibria are regular is not open. There would then be a sequence $(A, \ell, h)_t \to (A, \ell, h)$ where each $(A, \ell, h)_t$ has an equilibrium $(\bar{\pi}_t, y_t)$ that is not regular and where all equilibria of (A, ℓ, h) are regular. It is clear, given A2.1, that there is a compact subset of R^n_+ that contains the set $\{y_t\}$. To place the equilibrium price vectors in a compact set, identify each $\bar{\pi}_t$ with

$$\rho_t = (\bar{p}^1_t, (1+r_t)\bar{p}^0_t, w_t)(1/(\textstyle\sum_i \bar{p}^1_{i,t} + \sum_i (1+r_t)\bar{p}^0_{i,t} + w_t)) \in \Delta^{2n},$$

where Δ^{2n} is the $2n$-dimensional price simplex. Hence there is a subsequence of (ρ_t, y_t) converging to an equilibrium (ρ^*, y^*) of (A, l, h).

For each ρ_t, the matrix

$$\begin{pmatrix} D_{\bar{p}^1}\bar{z}_h(\rho_t, 1) & D_{p^0}\bar{z}_h(\rho_t, 1) & D_w\bar{z}_h(\rho_t, 1) \\ \bar{I}_n & -\bar{A}' & -\ell \end{pmatrix}, \tag{A.3}$$

has rank less than $2n-1$ since (A.3) can be derived from (A.2) by rank-preserving operations. To see this, first use A2.2(a) to obtain

$$\begin{pmatrix} D_{\bar{p}^1}\bar{z}_h(\bar{\pi}_t) & D_{p^0}\bar{z}_h(\bar{\pi}_t) & D_w\bar{z}_h(\bar{\pi}_t) \\ \bar{I}_n & -(1+r_t)A' & -\ell \end{pmatrix} \tag{A.4}$$

by postmultiplying the second block of columns in (A.2) by $(p^0_{2,t}, \ldots, p^0_{n,t})$ and subtracting the result from the final column multiplied by $1+r_t$. Next, multiply the second block of columns of (A.4) by $1/(1+r_t)$ and then multiply the first $n-1$ rows of the resulting matrix by $\sum_i \bar{p}^1_{i,t} + \sum_i (1+r_t)\bar{p}^0_{i,t} + w_t$. Using A2.2(a) again, we have (A.3).

Since there is a subsequence of (ρ_t, y_t) converging to an equilibrium (ρ^*, y^*) of (A, ℓ, h), there is a subsequence of corresponding matrices (A.3) converging to a matrix that also has rank less than $2n-1$. Since the operations above can be made in reverse order and are rank-preserving, we have that (A, ℓ, h) has an equilibrium that is not regular, a contradiction. ||

Note that the above theorem uses generic demand functions and technology matrices, but (ω, ω_l) is not restricted to be an element of a generic set. The reason is that there is a generic set of endowments at which Sraffa equilibria do not exist (see Section 3). Since the regularity of these economies is therefore irrelevant for the indeterminacy theorem, we need to include the economies with non-generic endowments. On the other hand, there is a potential limitation to Theorem A.1. As discussed in Section 3, the measure-zero set of endowments at which Sraffa equilibria exist can arise systematically in a dynamic setting. Since demand functions and their derivatives vary with consumption and endowments, one could therefore imagine that the zero-measure set of demand functions such that some equilibria are not regular could also arise systematically. See Mandler (1995), however, for a demonstration that when the demand derivatives are endogenously determined in a dynamic setting, the analogous regularity condition will in fact be satisfied generically.

Proof of Theorem 6.1.

The proof of openness, which is standard and similar to that given for Theorem A.1, is omitted. The proof of the full-measure claim combines existing techniques in the theory of regular production economies and the theory of overlapping generations economies. See Mas-Colell (1985, Proposition 8.7.3), from which I draw several steps of the argument establishing transversality, and Kehoe and Levine (1984), whose separate treatment of two types of equilibria I follow.

We begin by asserting that, for an open and full-measure subset of the perturbable technology parameters, if $[A_2 - [0 \;\; I_k]]y = 0$ for some $y \gg 0$ then $[A_2 - [0 \;\; I_k]]$ has full row rank. We omit the routine proof of the openness part of this assertion. To prove the full-measure part, note that if $[A_2 - [0 \;\; I_k]]y = 0$ for some $y \gg 0$ then there is a perturbable entry in each row of A_2. The requirement that $y \gg 0$ also implies that the derivative of $[A_2 - [0 \;\; I_k]]y$

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with respect to the perturbable entries has full row rank, and the therefore full-measure conclusion follows from the transversality theorem. Let Λ indicate this open and full-measure set of technology parameters.

(i) First, consider equilibria with $r \neq 0$. Conditions (6.5)–(6.8) consist of $2n+1$ equations in the $2n+1$ variables $(y, \bar{p}, w, 1+r)$. Removing $r=0$ from the domain, define a function F from the space of these $2n+1$ variables (*i.e.* $R^n_{++} \times R^{n-1}_{++} \times R_{++} \times R_{++} \backslash \{1\}$) and, adding p_1, h, and (A, ℓ), from $R_{++} \times R^{n-k+1}_{++} \times \Lambda$ to R^{2n+1} by $F(h, A, \ell, y, p, w, 1+r) = (z(\bar{p}, w, 1+r) + A_1 y - y_1, \ell \cdot y - \omega_l, A_2 y - y_2, \bar{p} - (1+r)A'\bar{p} - w\ell)$. The transversality theorem implies that if $D_{h,y,A,\ell}F$ has full row rank at all values of $(h, A, \ell, y, p, w, 1+r)$ such that $F=0$ then, for almost all values of h, A, ℓ, and p_1, $D_{y,\bar{p},w,r}F_{h,A,\ell,p_1}$ also has full row rank when $F=0$. Due to the degree 0 homogeneity of F with respect to (p, w), the rank of $D_{y,\bar{p},w,r}F_{h,A,\ell,p_1}$ is not affected by multiplying (p, w) by any $\lambda > 0$; thus, if $D_{y,\bar{p},w,r}F_{h,A,\ell,p_1}$ is nonsingular when evaluated at $(y, p, w, 1+r)$, then $D_{y,\bar{p},w,r}F_{h,A,\ell,1}$ is nonsingular when evaluated at $(y, (1/p_1)(p, w), 1+r)$. The implicit function theorem then implies that equilibria with $r \neq 0$ are locally isolated.

We have $D_{h,y,A,\ell}F =$

$$\begin{array}{cc}
 & \begin{array}{ccc} h & y & (A,\ell) \end{array} \\
\begin{array}{c} n-k \\ \\ \\ 1 \\ k \\ n \end{array} &
\begin{bmatrix}
w/p_1 & & 0 & (1+r)(w/p_1) & & \\
 & \ddots & & \vdots & & \\
0 & & w/p_{n-k} & (1+r)(w/p_{n-k}) & & \\
1 & \cdots & 1 & n-k & & \\
 & & 0 & & A_2-[0\;I_k] & \\
 & & 0 & & 0 & G
\end{bmatrix}
\end{array} \qquad \text{(A.5)}$$

Given the removal of $r=0$ from the domain of F, the upper left $(n-k+1) \times (n-k+1)$ block in (A.5) is nonsingular. Given the restriction of the domain of y to the strictly positive orthant and the definition of Λ, $[A_2 - [0\;\; I_k]]$ has full row rank when $F=0$. To see that G has full row rank, observe that each of n entries in $\bar{p} - (1+r)A'\bar{p} - w\ell$ has a distinct perturbation parameter since, for each activity j, either $a_{ij} > 0$ for some i or $\ell_j > 0$. Given that $((1+r)\bar{p}, w) \gg 0$, G has full row rank. Therefore, (A.5) has full row rank when $F=0$, proving that generically the equilibria with $r \neq 0$ are determinate.

(ii) In the case $r=0$, we are dealing with the standard general equilibrium model and therefore we could just appeal to preexisting generic regularity theorems. It is simple enough to argue directly, however. Due to Walras' law, one of the demand equals supply equations is now redundant. Therefore, construct a new function F, but eliminate $\ell \cdot y - \omega_l$ from the range, remove h^1 from the domain and set $r=0$. We now have $D_{h^0,y,A,\ell}F =$

$$\begin{array}{cc}
 & \begin{array}{ccc} h^0 & y & (A,\ell) \end{array} \\
\begin{array}{c} n-k \\ \\ \\ k \\ n \end{array} &
\begin{bmatrix}
w/p_1 & & 0 & & \\
 & \ddots & & & \\
0 & & w/p_{n-k} & & \\
 & 0 & & A_2-[0\;I_k] & \\
 & 0 & & 0 & G
\end{bmatrix}
\end{array}$$

Arguing just as before, $D_{h^0,y,A,\ell}F$ has full row rank when $F=0$ and hence for a full measure subset of parameters the equilibria with $r=0$ are locally isolated from each other.

(iii) *It remains to show that there is no sequence $(y, \bar{p}, w, 1+r)_t$ of equilibria with $r \neq 0$ converging to an* equilibrium $(y, \bar{p}, w, 1)^*$. Let $\bar{p}_1$ and p_2 indicate the first $n-k$ and last k coordinates of $\bar{p}$, respectively, and note that at any equilibrium, Walras' law implies $\bar{p}_1 \cdot ((1+r)z^0 + z^1) - w((1+r)\omega_l^0 + \omega_l^1) = 0$. Multiply (6.5)–(6.8) (as equalities) by $\bar{p}_1, w, p_2$, and y, respectively, and sum to get $\bar{p}_1 \cdot (z^0 + z^1) - w(\omega_l^0 + \omega_l^1) - r\bar{p} \cdot Ay = 0$. Combining, we have $r(\bar{p}_1 \cdot z^0 - w\omega_l^0 + \bar{p} \cdot Ay) = 0$. Therefore, at each $(y, \bar{p}, w, 1+r)_t$, $\bar{p}_1 \cdot z^0 - w\omega_l^0 + \bar{p} \cdot Ay = 0$. By continuity, the equilibrium $(y, \bar{p}, w, 1)^*$ also has $\bar{p}_1 \cdot z^0 - w\omega_l^0 + \bar{p} \cdot Ay = 0$. Therefore it is sufficient to show that $\bar{p}_1 \cdot z^0 - w\omega_l^0 + \bar{p} \cdot Ay = 0$ does not occur with $r=0$ for a full-measure subset of parameters. To that end, again construct a function F, omitting $\ell \cdot y - \omega_l$ and setting $r=0$ (as in the last paragraph), but including the equation

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$\bar{p}_1 \cdot z^0 - w\omega_l^0 + \bar{p} \cdot Ay = 0$ and the variable h^1. We then have $D_{h,y,A,\ell}F=$

$$\begin{array}{cc}
 & \begin{array}{cccccc} & \phantom{w/p_{n-k}} & h & & y & (A,\ell) \end{array} \\
\begin{array}{c} \\ n-k \\ \\ 1 \\ k \\ n \end{array} & \left[\begin{array}{cccccc}
w/p_1 & & 0 & w/p_1 & & \\
 & \ddots & & \vdots & & \\
0 & w/p_{n-k} & w/p_{n-k} & & & \\
0 & \cdots & 0 & -w(n-k) & & \\
 & & 0 & & A_2 - [0\ I_k] & \\
 & & 0 & & 0 & G
\end{array}\right]
\end{array}.$$

This matrix has full row rank when $F=0$, and therefore, for a full-measure set of h and (A,ℓ), $D_{y,\bar{p},w}F_{h,1}$ has full row rank when $F=0$. Since $D_{y,\bar{p},w}F_{h,1}$ has more rows than columns, however, it must be that with these values of h and (A,ℓ), $F=0$ does not occur; that is, there are no equilibria with $\bar{p}_1 \cdot z^0 - w\omega_l^0 + \bar{p} \cdot Ay = 0$ and $r=0$.

So far we have shown that each equilibrium $(y, \bar{p}, w, 1+r)$ of a full-measure set of parameters is locally unique. We show that equilibrium prices are locally unique to conclude the proof. If equilibrium prices were not locally unique, there would exist a sequence of equilibria $(y, \bar{p}, w, 1+r)_t$ such that $(\bar{p}, w, 1+r)_t \to (\bar{p}, w, 1+r)$ but where y_t does not converge. Observe, however, that for the set of parameters constructed so far the left-hand matrix of the equilibrium conditions,

$$\begin{pmatrix} I_n - A \\ \ell' \end{pmatrix} y = \begin{pmatrix} z(\bar{p}, w, 1+r) \\ 0 \\ \omega_l \end{pmatrix},$$

has rank n—as follows from the fact that $D_{y,\bar{p},w,r}F_{h,A,\ell,1}$ in (i) and $D_{y,\bar{p},w}F_{h,A,\ell,1}$ in (ii) are nonsingular. Hence there are n rows of

$$\begin{pmatrix} I_n - A \\ \ell' \end{pmatrix}$$

that form a nonsingular matrix, and, using these rows to calculate y_t, it is clear that y_t converges. ||

The theorem may be extended in several directions. For instance, we could introduce joint production and an arbitrary number of activities and permit some prices to equal zero. More relevant to the indeterminacy under study, if there are multiple inelastically supplied primary goods (rather than just labour) and if the endowments of those goods are included as perturbation parameters, then the above proof will work with only minor changes.

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