A Quick Proof of the Order-Extension Principle

Szpilrajn [2] proved that any partial order can be extended to a linear order. The standard proof (e.g., Fishburn [1]) relies on Zorn’s lemma and can be difficult to grasp. A more straightforward and quicker proof lets the well-ordering theorem assume the technical place of Zorn’s lemma. The axiom of choice is still invoked, but in a different way.

**Theorem.** Suppose \( \preceq \) is a partial order on a set \( X \). Then \( \preceq \) can be extended to a linear order on \( X \).

**Proof.** Let \( \preceq \) be a well ordering of \( X \). Let \( S \) be the set of functions from \( X \) to \( \{0, 1\} \), and let \( \leq_{\text{lex}} \) be the lexicographic order on \( S \). In other words, for \( f, g \in S \), \( f \leq_{\text{lex}} g \) if and only if either \( f = g \) or \( f(z) \leq g(z) \), where \( z \) is the \( \preceq \)-least element of \( \{ w \in X : f(w) \neq g(w) \} \). It is well known (and easy to verify) that \( \leq_{\text{lex}} \) is a linear order on \( S \).

For each \( x \in X \) let \( f_x \in S \) be defined by

\[
f_x(y) = \begin{cases} 
0, & \text{if } x \preceq y, \\
1, & \text{otherwise}.
\end{cases}
\]

Suppose that \( f_x = f_y \) for some \( x, y \in X \). Then, since \( f_x(y) = 0 \), \( f_y(y) = 0 \) and so \( x \preceq y \), and a similar argument shows that \( y \preceq x \). By the antisymmetry of \( \preceq \), \( x = y \). Thus, if \( x \neq y \) then \( f_x \neq f_y \). It follows that if we define a relation \( \preceq' \) on \( X \) by \( x \preceq' y \) if and only if \( f_x \leq_{\text{lex}} f_y \), then \( \preceq' \) is a linear order on \( X \).

We claim now that \( \preceq' \) extends \( \preceq \). To prove this, suppose that \( x \preceq y \) for some \( x, y \in X \). Then by the transitivity of \( \preceq \), for all \( z \in X \), if \( y \preceq z \) then \( x \preceq z \); in other words, if \( f_y(z) = 0 \) then \( f_x(z) = 0 \). It follows that for all \( z \in X \), \( f_x(z) \leq f_y(z) \), so \( f_x \leq_{\text{lex}} f_y \) and therefore \( x \preceq' y \).

The final paragraph of the proof shows that the map that takes \( x \) to \( f_x \) embeds the order \((X, \preceq)\) into \( S \) endowed with the product order: \( x \preceq y \iff \) (for all \( z \in X \), \( f_x(z) \leq f_y(z) \)). This conclusion is independent of the rest of the proof. The role of the well-ordering theorem is only to furnish a convenient linear order on \( S \), namely \( \leq_{\text{lex}} \), that extends the product order. Given the identification of each \( x \in X \) with \( f_x \), the corresponding extension \( \preceq' \) of \( \preceq \) is also linear. But any linear extension of the product order on \( S \) would do equally well. The product order also does not have to be defined on or extended to a linear order on the entirety of \( S \); only the \( f \in S \) such that \( f = f_x \) for some \( x \in X \) are relevant.

REFERENCES


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doi.org/10.1080/00029890.2020.1801081
MSC: Primary 06A06, Secondary 06A05; 03E25