A Quick Proof of the Order-Extension Principle

Szpilrajn [2] proved that any partial order can be extended to a linear order. The standard proof (e.g., Fishburn [1]) relies on Zorn's lemma and can be difficult to grasp. A more straightforward and quicker proof lets the well-ordering theorem assume the technical place of Zorn's lemma. The axiom of choice is still invoked, but in a different way.

Theorem. *Suppose $\preccurlyeq$ is a partial order on a set X. Then $\preccurlyeq$ can be extended to a linear order on X.*

Proof. Let $\trianglelefteq$ be a well ordering of X. Let $\mathcal{S}$ be the set of functions from X to $\{0, 1\}$, and let $\leq_{\text{lex}}$ be the lexicographic order on $\mathcal{S}$. In other words, for $f, g \in \mathcal{S}$, $f \leq_{\text{lex}} g$ if and only if either $f = g$ or $f(z) \leq g(z)$, where z is the $\trianglelefteq$-least element of $\{w \in X : f(w) \neq g(w)\}$. It is well known (and easy to verify) that $\leq_{\text{lex}}$ is a linear order on $\mathcal{S}$.

For each $x \in X$ let $f_x \in \mathcal{S}$ be defined by

$$f_x(y) = \begin{cases} 0, & \text{if } x \preccurlyeq y, \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that $f_x = f_y$ for some $x, y \in X$. Then, since $f_y(y) = 0$, $f_x(y) = 0$ and so $x \preccurlyeq y$, and a similar argument shows that $y \preccurlyeq x$. By the antisymmetry of $\preccurlyeq$, $x = y$. Thus, if $x \neq y$ then $f_x \neq f_y$. It follows that if we define a relation $\preccurlyeq'$ on X by $x \preccurlyeq' y$ if and only if $f_x \leq_{\text{lex}} f_y$, then $\preccurlyeq'$ is a linear order on X.

We claim now that $\preccurlyeq'$ extends $\preccurlyeq$. To prove this, suppose that $x \preccurlyeq y$ for some $x, y \in X$. Then by the transitivity of $\preccurlyeq$, for all $z \in X$, if $y \preccurlyeq z$ then $x \preccurlyeq z$; in other words, if $f_y(z) = 0$ then $f_x(z) = 0$. It follows that for all $z \in X$, $f_x(z) \leq f_y(z)$, so $f_x \leq_{\text{lex}} f_y$ and therefore $x \preccurlyeq' y$. ∎

The final paragraph of the proof shows that the map that takes x to f_x embeds the order $(X, \preccurlyeq)$ into $\mathcal{S}$ endowed with the product order: $x \preccurlyeq y \Leftrightarrow$ (for all $z \in X$, $f_x(z) \leq f_y(z)$). This conclusion is independent of the rest of the proof. The role of the well-ordering theorem is only to furnish a convenient linear order on $\mathcal{S}$, namely $\leq_{\text{lex}}$, that extends the product order. Given the identification of each x in X with f_x, the corresponding extension $\preccurlyeq'$ of $\preccurlyeq$ is also linear. But any linear extension of the product order on $\mathcal{S}$ would do equally well. The product order also does not have to be defined on or extended to a linear order on the entirety of $\mathcal{S}$; only the $f \in \mathcal{S}$ such that $f = f_x$ for some $x \in X$ are relevant.

REFERENCES

[1] Fishburn, P. (1970). *Utility Theory for Decision Making*. New York: Wiley.

[2] Szpilrajn, E. (1930). Sur l'extension de l'ordre partiel. *Fundam. Math.* 16: 386–389. doi.org/10.4064/fm-16-1-386-389

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