Dynamic Optimisation

$t = 1, 2$ i.e. two time periods

K_t - capital stock of economy at time t

C_t - consumption at time t

Dynamic Constraint

$$K_2 = g(K_1, C_1)$$

Simple Example

$$K_2 = \underbrace{K_1^{\alpha}}_{\text{production}} - \underbrace{C_1}_{\text{consumption}} \qquad 0 < \alpha < 1$$

Utility

$$U(C_1, C_2)$$

Simple Example

$$U(C_1, C_2) = \ln(C_1) + B\ln(C_2)$$

where $0 < B < 1$ is a <u>discount factor</u>

Typical Economic Problem

$$\max_{C_1, C_2} U(C_1, C_2)$$

s.t. $C_1 \leq f(K_1)$ - <u>Consumption less than production</u>

$C_2 \leq f(K_2)$ ‖

$K_2 = g(K_1, C_1)$

K_1 given

Known as - "Optimal Growth" "Consumption-Saving"

"Neoclassical Growth"

In special Example

$*$ $\max_{C_1, C_2} \ \ln(C_1) + \beta \ln(C_2)$

s.t. $0 \leq C_1 \leq K_1^{\alpha}$

$C_2 = K_2^{\alpha}$ note

$K_2 = K_1^{\alpha} - C_1$

Equivalent formulation

$\max_{C_1} \ \ln(C_1) + \beta \ln\left[\left(K_1^{2} - C_1\right)^{2}\right]$

s.t. $C_1 \leq K_1^{2}$

But $C_1 = K_1^{2}$ cannot be optimal

(Also $C_1 \leq 0$ does not make sense)

Can treat as unconstrained optimisation in C_1

Have students solve in class

$$C_1^* = \frac{K_1^{\alpha}}{\alpha B + 1}$$

$$C_2^* = K_1^{\alpha \cdot \alpha} \left(\frac{\alpha B}{\alpha B + 1} \right)^{\alpha}$$

Second Order Condition ?

What have we done ?

1. Eliminated constraints through substitution
2. Eliminated a constraint by verifying that it will not be binding.

Let's proceed more formally

Consider $*$ again but maintain the (correct) assumptions that at a solution (c_1^*, c_2^*):

1. $0 < c_1^* < K_1^\alpha$
2. $c_2^* = (K_2^*)^\alpha$

Form the Lagrangian

$$L(c_1, K_2, \lambda) = \ln(c_1) + B\alpha \ln(K_2) - \lambda(K_2 - K_1^\alpha + c_1)$$

a) $\dfrac{1}{c_1} = \lambda$ (c_1 derivative)

b) $\dfrac{B\alpha}{K_2} = \lambda$ (K_2 derivative)

c) $K_2 = K_1^\alpha - c_1$ (λ derivative)

Check solution is the same as before

Why do a, b & c characterise a solution to * ?

1. C is simply the dynamic constraint
2. For any $\lambda^* > 0$ such that $(c_1^*, K_2^*, \lambda^*)$ solves a, b & c (c_1^*, K_2^*) maximize $L(c_1, K_2, \lambda^*)$ for this fixed λ^*
3. 1 + 2 $\Rightarrow$ (c_1^*, K_2^*) solves *

What is λ ?

The value of relaxing the dynamic constraint

Why ?

Consider

$$L(c_1, k_2, \lambda; a) = \ln(c_1) + \beta\lambda \ln(k_2) - \lambda(k_2 - k_1 + c_1 - a)$$

i.e. the economy gets a free units of capital in period 2

How much extra utility does this yield per unit?

$$L(c_1^*(a), k_2^*(a), \lambda^*(a); a) - L(c_1^*(0), k_2^*(0), \lambda^*(0); 0)$$

$$\frac{dL}{da} = \underbrace{\frac{\partial L}{\partial c_1}}_{=0}\frac{dc_1^*}{da} + \underbrace{\frac{\partial L}{\partial k_2}}_{=0}\frac{dk_2^*}{da} + \underbrace{\frac{\partial L}{\partial \lambda}}_{=0}\frac{d\lambda^*}{da} + \frac{\partial L}{\partial a}$$

$$= \frac{\partial L}{\partial a} = \lambda$$ (envelope theorem)

Cryptic <u>note</u>

we can replace λ with $B\lambda_2$ without changing anything