ECONOMICS DEPT

ASSESSED TEST

For Internal Students of Royal Holloway

COURSE UNIT: EC5040
TITLE: Econometrics (Mid-Term Exam 1)

Date of Test 12th November 2008

Time Allowed: 1 hour

Instructions to candidates:

ANSWER BOTH QUESTIONS

WRITE ALL YOUR ANSWERS (INCLUDING ROUGH WORKING) ON THIS ANSWER BOOK

STATISTICAL TABLES ARE PROVIDED

SILENT NON-PROGRAMMABLE CALCULATORS MAY BE USED

DO NOT TURN OVER UNTIL TOLD TO BEGIN
Given the general linear model
\[ y = X\beta + u \]
where \( y \) is an \( n \times 1 \) vector of observations on the dependent variable, \( X \) is an \( n \times k \) matrix of observations on a set of explanatory variables, \( \beta \) is a \( k \times 1 \) vector of parameters and \( u \) is an \( n \times 1 \) vector of residuals.

a) Derive, from first principles, an expression for the ordinary least squares (OLS) estimate of \( \beta \).

Minimising the sum of squared residuals implies
\[
\min_{\beta} u' u = \left[ u_1, u_2, \ldots, u_n \right]^T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1^2 + u_2^2 + \cdots + u_n^2
\]
\[ = (y - X\beta)'(y - X\beta) = y'y - \beta'X'y - y'X\beta + \beta'X'X\beta \]
Since all terms are scalars (1,1) can add middle two terms (one is transpose of the other)
\[ = y'y - 2\beta'X'y + \beta'X'X\beta \]
F.O.C. minimum
\[
\frac{\partial u' u}{\partial \beta} = -2X'y + 2X'X\hat{\beta} = 0
\]
which gives k normal equations
\[ X'X\hat{\beta} = X'y \]
and the k variable OLS solution
\[ \hat{\beta} = (X'X)^{-1}X'y \]

b) Derive an expression for the variance of the OLS estimator.

\[
\text{Var}(\hat{\beta}) = \text{E}\left((\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))'\right)
\]
\[
\text{Var}(\hat{\beta}) = \text{E}\left((\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right) = \text{E}(XX)'uu'XX(XX)' = (XX)'E(uu')X(XX)'
\]
\[ = (XX)'\sigma^2 I X(XX)' \]
\[ \sigma^2(XX)' \]

(c) Show that if the Gauss-Markov conditions are satisfied then OLS has the minimum variance of all linear unbiased estimators.

(15 marks)
Consider another linear unbiased estimator \( \hat{\beta} = C y \)

If \( \hat{\beta} \) is unbiased then \( E(\hat{\beta}) = E(C y) = E[CX\beta + Cu] = \beta \)

Hence \( CX = I \) and \( \hat{\beta} = \beta + Cu \)

So \( \text{Var}(\hat{\beta}) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \] 

\[= E[Cuu' C'] \]

\[= \sigma^2 CC' \]

Let \( D \) be the difference between the OLS and alternative estimated explanatory component ie

\[ D = C - (XX)'X' \]

So \( \text{Var}(\hat{\beta}) = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')' \] 

Since \( CX = I = DX + (X'X)^{-1}X'X \) the \( DX = 0 \)

Cross product terms vanish and

\[ \text{Var}(\hat{\beta}) = \sigma^2 DD' + \sigma^2 (XX)' = \sigma^2 DD' + \text{Var}(\hat{\beta})_{OLS} \]

ie variance of alternative estimator equals that of OLS plus a non-negative definite matrix (see problem set 0)

Hence OLS estimate has minimum variance property (BLUE – Best Linear Unbiased Estimator). Main reason for widespread use of OLS, will always provide estimators with smaller standard errors

e) Suppose that one of the independent variables is subject to a linear transformation, (multiplied by a constant \( \lambda \)) such that \( Z = X\Lambda \) where \( \Lambda \) is a diagonal matrix containing the transformation constant. Show the effect of this transformation on the OLS estimates of the parameters

(10 marks)

Given \( y = Z\gamma + \nu \)

OLS implies \( \hat{\gamma} = (Z'Z)^{-1}Z'y \)

Sub. in \( Z = X\Lambda \)

\[ \hat{\gamma} = (\Lambda'X'X\Lambda)^{-1}\Lambda'X'y \]

Using rules on inverse of a matrix product

\[ \hat{\gamma} = \Lambda^{-1}(X'X)^{-1}\Lambda^{-1}\Lambda'X'y \]
\[ \gamma = A^{-1}(X'X)^{-1}X'y \]
\[ \gamma = A^{-1} \hat{\beta} \]

If the variable to be transformed is \( X_j \) then the transformation matrix looks like

\[
\Lambda = \begin{bmatrix}
1 & 0 \\
0 & \lambda_j \\
1 & 1
\end{bmatrix}
\]

ie a diagonal matrix with ones down the main diagonal except for the \( j \)th element which contains the constant of multiplication for the \( j \)th variable.

Since the inverse of a diagonal matrix is also diagonal with the reciprocal of each original element on the new main diagonal then

\[
\Lambda^{-1} = \begin{bmatrix}
1 & 0 \\
0 & 1/\lambda_j \\
1 & 1
\end{bmatrix}
\]

So using the result in (3) it follows that then the corresponding regression coefficient is multiplied by \( 1/\lambda \) and all other coefficients are unchanged.

f) Show the consequences for OLS estimation of omitting relevant variables from your model specification (11 marks)

True: \( y = X_1\beta_1 + X_2\beta_2 + e \)
Estimate: \( y = X_1\beta_1 + u \)

\[ \hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y \]
\[ \hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y + (X_1'X_1)^{-1}(X_1'X_2)\beta_2 + (X_1'X_1)^{-1}X_1'e \]
\[ \hat{\beta}_1 = \beta_1 + (X_1'X_1)^{-1}(X_1'X_2)\beta_2 + (X_1'X_1)^{-1}X_1'e \]
so

\[ E(\hat{\beta}_1) = \beta_1 + (X_1'X_1)^{-1}(X_1'X_2)\beta_2 \neq \beta_1 \]

OLS estimates of the coefficients on the set of \( X_1 \) variables are biased in the presence of omitted variables and sign of bias depends on

a) the effect of the omitted variables on \( y, \beta_2 \),
b) the covariance of \( X_1 \) and \( X_2 \)

Not only is mean biased so is OLS estimates of parameter variances in an unknown way (If \( \sigma^2 \) known variance estimate is biased down. But estimate of \( \sigma^2 \) is biased up, so hard to sign direction of bias)
2. The following regression output is taken from a regression of the log of hourly pay \((lhpay)\) on the years of work experience \((xper)\), years of education \((yearsed)\), years of job tenure, \((tenure)\) a dummy variable for being female, \((female)\).

Some of the regression output has been obscured.

\[
\text{reg lnhpay xper yearsed tenure female}
\]

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs = 6005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>4</td>
<td></td>
<td></td>
<td>F( 7, 6000) =</td>
</tr>
<tr>
<td>Residual</td>
<td>1600.00000</td>
<td>6000</td>
<td>.266666666</td>
<td>Prob &gt; F =</td>
</tr>
<tr>
<td>Total</td>
<td>2400.00000</td>
<td>6004</td>
<td>.399733510</td>
<td>R-squared =</td>
</tr>
</tbody>
</table>

\[
\begin{array}{lcccc}
\text{lnhpay} & \text{Coef.} & \text{Std. Err.} & t & P>|t| & [95\% \text{ Conf. Interval}] \\
\hline
xper & 0.1100000 & 0.0900000 & 1.222 & 0.212 & -0.0664000 - 0.2864000 \\
yearsed & 0.0700000 & 0.0030894 & 22.861 & 0.000 & 0.0645705 - 0.076683 \\
tenure & -0.0100000 & 0.030000 & 0.333 & 0.515 & -0.0688000 - 0.0488000 \\
female & -0.1948365 & 0.0723197 & -2.694 & 0.007 & -0.3366081 - 0.0530649 \\
_cons & 0.7426208 & 0.0496538 & 14.956 & 0.000 & 0.6452822 - 0.8399593 \\
\end{array}
\]

a) Interpret the meaning of the coefficient on the female dummy variable

(8 marks)

This is a “semi-log” equation so the impact of being female relative to being male (net of differences in mean values of control variables) is equal to the % difference /100 in hourly pay of being female relative to being male, (since in the continuous variable case \(dLnw/d(x) = b_i = dw_i/dw/x\) = % change in w /100 with respect to a unit change in x)

However since the coefficient is a dummy variable this is only an approximation to the proportionate difference and the true effect is closer to \(\exp(\beta_{female}) - 1\)

So other things equal women earn \(\exp(-.195) - 1 = .177 = 17.7\%\) less than men.

b) Find the estimate of \(R^2\) and hence test the hypothesis that the model as a whole is a good fit

(10 marks)

\[R^2 (the \ coefficient \ of \ determination) = ESS/TSS = 1-(RSS/TSS)\]

From information in the regression output (highlighted in yellow)

\[R^2 = ESS/TSS = 1- (1600/2400) = 1 -.666 = .333\]

Test of goodness of fit of the model is given by

\[F = \frac{ESS/q}{RSS/N-k} = \frac{R^2/q}{(1-R^2)/N-k} \sim F[q,N-k]\]

So

\[F = \frac{800/4}{1600/600} = \frac{0.333/4}{0.666/600} \sim F[4,6000]\]

So estimated F is greater than 5% critical value \((F(4,\infty) = 2.37)\) so reject null that model as a whole has no explanatory power.
d) The variance/covariance matrix of the OLS parameter estimates (excluding the constant) is given by

<table>
<thead>
<tr>
<th></th>
<th>xper</th>
<th>yrsed</th>
<th>female</th>
<th>tenure</th>
</tr>
</thead>
<tbody>
<tr>
<td>xper</td>
<td>0.0081</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>yrsed</td>
<td>0.0002</td>
<td>0.0046</td>
<td></td>
<td></td>
</tr>
<tr>
<td>female</td>
<td>0.0006</td>
<td>0.0013</td>
<td>1.34</td>
<td></td>
</tr>
<tr>
<td>tenure</td>
<td>-0.0005</td>
<td>-0.0067</td>
<td>-0.0057</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

Given this information test the hypothesis that the returns to experience (xper) equal the returns to job tenure (tenure) in the model above

\[ (\hat{\beta}_{xper} - \hat{\beta}_{tenure})^2 / \text{Var}(\hat{\beta}_{xper} - \hat{\beta}_{tenure}) = F(1, N-k) \]

which can check from the variance/covariance matrix of the OLS estimates since the square root of the i\textsuperscript{th} element on the main diagonal should equal the standard error on the i\textsuperscript{th} variable in the regression and the off diagonal terms are the covariances of the parameter estimates. The relevant covariance is highlighted (in green)

\[ \text{Var}(\hat{\beta}_{xper} - \hat{\beta}_{tenure}) = \text{Var}(\hat{\beta}_{xper}) + \text{Var}(\hat{\beta}_{tenure}) - 2 \text{Cov}(\hat{\beta}_{xper}, \hat{\beta}_{tenure}) \]

\[ = 0.0081 + 0.0009 - 2(-0.0005) = 0.01 \]

\[ F = \frac{(0.11 - 0.01)^2}{0.01} = 1.44 \sim F[1, 6000] \]

Since 95% critical value = 3.84 then \( F < F_{\text{critical}} \)
So can not reject null that coefficients are equal (the relatively large standard errors ensure that the confidence intervals overlap)

d) Consider a simple model of 204 observations split equally into two sub-samples such that

\[ y_i = a_1 + b_1 X_i + u_i \quad i=1..N_1 \quad \text{in sub-sample 1} \]

and

\[ y_i = a_2 + b_2 X_i + u_i \quad i=N_1+1..N \quad \text{in sub-sample 2} \]

Suppose that RSS\(_1\) = 8 and RSS\(_2\) = 2 and that the RSS from the pooled regression is 12. Test the hypothesis of no structural change across the two sub-samples at the 5% level.

(10 marks)
The unrestricted form of the model (intercepts and the slopes vary in two periods) in (partitioned) matrix form is given by

\[
y = \begin{bmatrix} y_1 \\ \vdots \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ \vdots & \vdots \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ \vdots \\ a_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_2 \end{bmatrix} = X\beta + u
\]

(1)

where \( X_1 \) is an \( N_1 \) by 2 matrix of observations from the 1st sub-sample and \( X_2 \) is an \( N_2 \) by 2 matrix of observations from the 2nd sub-sample with \( N = N_1 + N_2 \).

ie stacking the data from the second period below that of the observations from the 1st period in a way that allows the coefficients to differ between the periods

Compare this with estimates from the restricted (pooled) model based on

\[
y = \begin{bmatrix} y_1 \\ \vdots \\ y_2 \end{bmatrix} = \begin{bmatrix} iX_1 \\
iX_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_2 \end{bmatrix} = X\beta + u
\]

To test formally use

\[
F = \frac{(RSS_{\text{restricted}} - RSS_{\text{unrestricted}})/q}{RSS_{\text{unrestricted}}/N - k} \sim F[q, N - k]
\]

which in this case becomes the Chow test

\[
F = \frac{(RSS_{\text{restricted}} - RSS_1 + RSS_2)/q}{RSS_1 + RSS_2 / N - 2k} \sim F[q, N - 2k]
\]

(remember that there are 4 parameters in the unrestricted model so \( k=4 \) and \( q=2 \) restrictions)

hence \( F = \frac{(12 - (8 + 2))/2}{(8 + 2)/204 - 2*2} = 20 \)

From Tables the 5% critical value given the degrees of freedom \( F_{0.05}[2, \infty] = 3.0 \)

\( ^\wedge \)

\( F > F_{\text{critical}} \) so reject null (of no structural change)

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\( F > F_{\text{critical}} \) so reject null (of no structural change)

d) Outline the form of a technique that could be used to test for functional form error (10 marks)

Either

Ramsey RESET Test
- if model is good fit then addition of extra variables should not be statistically significant

rather than add higher order terms of original variables a more parsimonious alternative is to use fact that

\( \hat{y} = \hat{X} \hat{\beta} \)

so predicted values are linear function of all the X variables (weighted by their estimated coefficients)

and hence

\( (\hat{y})^j = (X \hat{\beta})^j \)
are linear functions of higher powers of all the $X$ variables

$$y = X\beta + \delta_2 y^2 + \delta_3 y^3 + \ldots + \delta_j y^j + u$$

and test null $H_0$: $\delta_2 = \delta_3 = \ldots = \delta_j = 0$

If estimated $F$ value greater than critical value reject null that functional form is acceptable.

OR

**LM Test of Omitted Variables**

1. Run restricted regression (no higher order terms)
2. save residuals
3. Regress residuals on unrestricted model (containing higher order values of $X$ (or the $y^j$) - the auxiliary regression

Can show

$$NR^2_{\text{aux}} \sim \chi^2_{(\text{No of restrictions})}$$

If estimated Chi-squared value greater than critical value reject null that functional form is acceptable.