PH4211 Statistical Mechanics 2023
Outline Answers

Question 1

a)

Λ is the de Broglie wavelength corresponding to the free particle momentum p corresponding to the kinetic energy E corresponding to a thermal energy kT.
(To within a factor of $\sqrt{\pi}$). part a: [[4]]

b)

Since Z is the sum of Boltzmann factors – i.e. it is proportional to sums of probabilities, then since probabilities for independent events multiply, so the zs will multiply. [4]

Since the free energy comes from the *logarithm* of Z then to be consistent with the extensivity of energy, the zs must multiply. [4]

So generally, for N independent systems $Z = z_1 z_2 z_3 \ldots z_N$. And so if all N systems have the same form for z then $Z = z^N$. [2]

part b: [[10]]

c)

$$\begin{aligned} F &= -kT \ln Z \\ &= -NkT \ln z \\ &= -NkT \ln \left\{ V \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \right\} \end{aligned}$$

part c: [[4]]

d)

Although the N prefactor is needed for extensivity – since it ensures the total free energy F is proportional to the number of particles N – this is not enough. The problem is that the argument of the logarithm is proportional to V. True: the argument is dimensionless but this is not enough.

Extensivity requires V/N goes to a constant limit as $N, V \to \infty$ while N/V remains constant.

With the above form for F you *don't* get a constant value; instead $F \to \infty$. part d: [[8]]

e)

Gibbs's resolution. *Mathematically* the problem would be solved by dividing the (extensive) V in the logarithm by a *dimensionless, extensive* quantity: something proportional to N.

If we had some factor aN then the a would contribute an additive constant to F, so this would not be a problem classically. [4]

part e: [[4]]

f)

From the quantum perspective the particle indistinguishability leads to a factor $1/N!$. [4]

This gives the required correction to the argument of the logarithm, since $1/N! \sim N\ln(N/e)$, giving

$$F = -NkT \ln\left\{ \frac{V}{N} \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} e \right\} \qquad [2]$$

Extensivity is is related to the thermodynamic limit (used throughout the argument) – indeed extensivity only emerges in the thermodynamic limit. Also, Stirling's approximation holds only in the thermodynamic limit. [4]

part f: [[10]]

Total Q1: [[[40]]]

Question 2

a)

The origin of the field is the exchange interaction contributing $\lambda\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2$ to the energy. Since the magnetic energy is $\sum_i \boldsymbol{\mu}_i \cdot \mathbf{B}$ where $\mathbf{B}$ is an externally applied field, upon including the exchange energy we have

$$\begin{aligned} E &= \sum_i \boldsymbol{\mu}_i \cdot \mathbf{B} + \lambda \sum_{i,j} \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j \\ &= \sum_i \left\{ \boldsymbol{\mu}_i \cdot \left(\mathbf{B} + \lambda \sum_j \boldsymbol{\mu}_j \right) \right\}. \end{aligned} \qquad [4]$$

And since $\sum_j \boldsymbol{\mu}_j$ is the magnetization $\mathbf{M}$ we take the average giving

$$E = \sum_i \boldsymbol{\mu} \cdot (\mathbf{B} + \mathbf{b}) \qquad [4]$$

where the mean field $\mathbf{b}$ is

$$\mathbf{b} = \lambda \mathbf{M}. \qquad \text{part a: [[8]]}$$

b)

Given

$$M = M_0 \tanh\left(\frac{\mu B}{kT}\right),$$

put $B \to B_0 + b$, where B_0 so that

$$M = M_0 \tanh\left(\frac{\mu(B_0 + b)}{kT}\right). \qquad [4]$$

Spontaneous magnetization is when $B_0 = 0$. Then

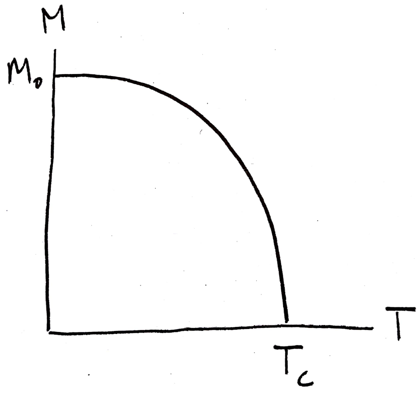
$$\begin{aligned} \frac{M}{M_0} = \tanh\left(\frac{\mu B}{kT}\right) &= \tanh\left(\frac{M_0}{N}\frac{\lambda M}{kT}\right) \\ &= \tanh\left(\frac{M}{M_0}\frac{\lambda M_0^2}{NkT}\right) \\ &= \tanh\left(\frac{M}{M_0}\frac{T_c}{T}\right) \end{aligned}$$

where $T_c = \lambda M_0^2 / Nk$. [4]

part b: [[8]]

c)

i. Sketch [4]



ii. T_c is the temperature at which the (spontaneous) magnetization vanishes (the critical temperature) [2]

iii. Transition is *2nd order* as M goes to zero *continuously* as $T \to T_c$. [2]

part c: [[8]]

d)

Start from

$$\frac{M}{M_0} = \tanh\left(\frac{M}{M_0}\frac{T_c}{T}\right)$$

and expand the tanh:

$$\frac{M}{M_0} \sim \frac{M}{M_0}\frac{T_c}{T} - \frac{1}{3}\left(\frac{M}{M_0}\frac{T_c}{T}\right)^3$$

or

$$1 \sim \frac{T_c}{T} - \frac{1}{3}\left(\frac{M}{M_0}\right)^2\left(\frac{T_c}{T}\right)^3.$$

Then

$$\begin{aligned}\frac{M}{M_0} &\sim \sqrt{3}\frac{T}{T_c}\left(1-\frac{T}{T_c}\right)^{1/2}\\ &\sim \sqrt{3}\left(1-\frac{T}{T_c}\right)^{1/2} \quad \text{since } T \sim T_c\end{aligned}$$

part d: [[8]]

e)

$$\frac{M}{M_0} \sim \left(\frac{T_c - T}{T_c}\right)^{\beta} \text{ when } T \text{ close to } T_c$$

[2]

The above result gives $\beta = 1/2$. Real (Ising) magnets have β around $1/3$. [3]

The difference is due to the neglect of fluctuations at the critical point (in mean field models). [3]

part e: [[8]]

Total Q2: [[[40]]]

Question 3

a)

The order parameter is a measure of the order in a system. In the ordered phase the order parameter will be non-zero; in the disordered phase it will be zero. part a: [[4]]

b)

The Landau expansion *must* be terminated in order to describe the *generic* features of the transition (in the vicinity of the critical point). Inclusion of higher-order terms only introduces model-dependent *specific* features. [3]

One needs terms at least up to 4th order. The highest-order term must be even order with positive coefficient for stability. So if the 4th order coefficient is not positive then must go to 6th order. And so on ... [3]

part b: [[6]]

c)

Conserved order parameter – the mean value of the order parameter (density) over the system is fixed. Example: density of a fluid. [2]

Non-conserved order parameter – there is no restriction on the mean value of the order parameter. Example: magnetization of a ferromagnet. [2]

Equilibrium state:

Conserved order parameter: double tangent construction.

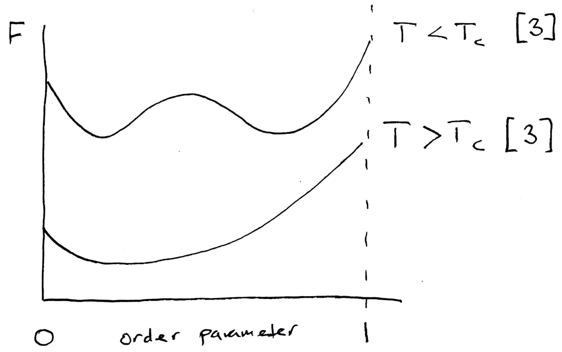
Non-conserved order parameter: minimum of free energy curve. [3]

part c: [[7]]

d)

The order parameter is the concentration of one of the species (or $x - x_c$). [2]

Sketch [4]



part d: [[6]]

e)

First term is the *energy* – depends on $x(1-x)$ and is proportional to T_c. [3]

Second term is *entropy* term – has standard form for a 2-state system. [3]

part e: [[6]]

f)

Series is terminated *after* the 4th term since $\frac{2}{3}T_c$ is positive. [4]

At the critical point $x = \frac{1}{2}$. We write the order parameter as the deviation from the value at the critical point – i.e. we use $x - \frac{1}{2}$. [3]

part f: [[7]]

g)

$$f = (T - T_c)\phi^2 + \frac{2}{3}T_c\phi^4$$

where $f = (F_m - F_0)/2Nk$ and $\phi = x - \frac{1}{2}$.

Equilibrium is specified by $\frac{\partial f}{\partial \phi} = 0$; that is

$$\frac{\partial f}{\partial \phi} = 0 = 2(T - T_c)\phi + \frac{8}{3}T_c\phi^3$$

so $\phi = 0$ or

$$\phi^2 = \frac{2(T_c - T)}{\frac{8}{3}T_c} = \frac{3}{4}\left(1 - \frac{T}{T_c}\right)$$

giving

$$\phi = \sqrt{\frac{3}{4}}\left(1 - \frac{T}{T_c}\right)^{1/2}.$$

Here the exponent $\frac{1}{2}$ is the critical exponent β. [4]

part g: [[4]]

Total Q3: [[[40]]]

Question 4

Essay question

- Distinction between *microstates* ans *macrostates.* [4]
- Fundamental postulate of SM: all microstates of an isolated system are equally likely. [4]
- Thermodynamics deals with macrostates. [4]
- So thermodynamic states are understood (in SM) probabilistically. [4]
- Boltzmann entropy expression $S = k \ln \Omega$ makes the macro-micro connection for isolated systems. [4]
- So entropy maximum principle gives the macrostate with highest probability. [4]
- The equilibrium state is understood at the macrostate with highest probability. [4]
- Temperature emerges as the common property of two systems when they are in equilibrium (even when no work is done or particles interchanged). [4]
- Similarly pressure and chemical potential emerge as equilibrium conditions when work and particles are exchanged. [4]
- In the thermodynamic limit (fractional) fluctuations vanish. So in this case calculated mean values are useful (steady) quantities to use. [4]
- Discussion of the equality of the *mean*, *median* and *mode* in the thermodynamic limit. [4]

Although discussion in terms of quantum states is simplest, connection with / discussion in terms of the classical approach is acceptable.

Total Q4: [[[40]]]

Question 5

a)

A Brownian particle, stationary in the centre of mass frame of the fluid, experiences random impacts from all directions; it experiences a random force of mean value zero. A Brownian particle moving with respect to the centre of mass frame of the fluid experiences impacts from the front which are more energetic than the ones from behind. These have a greater momentum transfer. So in addition to the random force there will be a mean force depending on the velocity of the Brownian particle (with respect to the centre of mass frame of the fluid). Simple argument then shows force proportional to the velocity. [10]

b)

Simple application of Newton's law $M\dot{v} = F$ and using $F = f - v/\mu$ gives

$$M\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{\mu}v(t) = f(t). \qquad [4]$$

c) (i)

$$\begin{aligned} v^2(t) &= \left\{ v(0)e^{-t/M\mu} + \frac{1}{M}\int_0^t e^{(s-t)/M\mu} f(s)\,\mathrm{d}s \right\}^2 \\ &= v^2(0)e^{-2t/M\mu} + \frac{2}{M}v(0)e^{-t/M\mu}\int_0^t e^{(s-t)/M\mu} f(s)\,\mathrm{d}s \\ &+ \frac{1}{M^2}\int_0^t \mathrm{d}s_1 \int_0^t \mathrm{d}s_2\, e^{(s_1-t)/M\mu} e^{(s_2-t)/M\mu} f(s_1)f(s_2). \end{aligned}$$

Now take the average

$$\begin{aligned} \left\langle v^2(t)\right\rangle &= \left\langle v^2(0)\right\rangle e^{-2t/M\mu} + \frac{2}{M}e^{-t/M\mu}\int_0^t e^{(s-t)/M\mu}\left\langle v(0)f(s)\right\rangle\,\mathrm{d}s \\ &+ \frac{1}{M^2}\int_0^t \mathrm{d}s_1 \int_0^t \mathrm{d}s_2\, e^{(s_1-t)/M\mu} e^{(s_2-t)/M\mu}\left\langle f(s_1)f(s_2)\right\rangle. \end{aligned}$$

[4]

First term is mean transient response; this dies to zero at long times. In second term $v(0)$ is uncorrelated with $f(t)$ so this term vanishes. So we are left with

$$\left\langle v^2(t)\right\rangle = \frac{1}{M^2}e^{-2t/M\mu}\int_0^t \mathrm{d}s_1 \int_0^t \mathrm{d}s_2\, e^{(s_1+s_2)/M\mu}\left\langle f(s_1)f(s_2)\right\rangle.$$

Assumption of short correlation time: $\langle f(s_1)f(s_2)\rangle = f^2\delta(s_1 - s_2)$ where $f^2 = \int_{\infty}^{\infty} \langle f(0)f(t)\rangle \,\mathrm{d}t$. This assumption forces $s_1 = s_2$ when the integral is done so that

$$\begin{aligned}\left\langle v^2(t)\right\rangle &= \frac{1}{M^2}f^2 e^{-2t/M\mu}\int_0^t e^{2s/M\mu}\mathrm{d}s \\ &= \frac{\mu}{2M}f^2 e^{-2t/M\mu}\left(e^{2t/M\mu} - 1\right) \\ &= \frac{\mu}{2M}f^2\left(1 - e^{-2t/M\mu}\right).\end{aligned}$$

So at long times the equilibrium value is

$$\left\langle v^2\right\rangle = \frac{\mu}{2M}f^2$$

or

$$\left\langle v^2\right\rangle = \frac{\mu}{2M}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle \,\mathrm{d}t. \qquad [4]$$

c) (ii)

Now equipartition gives $\langle v^2\rangle = kT/M$ so that

$$\frac{kT}{M} = \frac{\mu}{2M}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle \,\mathrm{d}t$$

or

$$\frac{1}{\mu} = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle \,\mathrm{d}t. \qquad [3]$$

part c: [[11]]

d)

Left hand side is a dissipation coefficient. Right hand side involves fluctuations. So this is an equation relating fluctuations and dissipation. Generalisation called the fluctuation-dissipation theorem. part d: [[5]]

e)

By direct analogy $M \to L$ (not needed), $v(t) \to I(t)$, $1/\mu \to R$, $f(t) \to V(t)$. [4]

Then

$$R = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle V(0)V(t)\rangle \,\mathrm{d}t. \qquad [2]$$

part e: [[6]]

f)

Can observe as the hiss (as opposed to the hum) from a loudspeaker when there is no input connected to a hi-fi amplifier. part f: [[4]]

Total Q5: [[[40]]]