UNIVERSITY OF LONDON

MSc/MSci EXAMINATION 2023

For Students of the University of London

DO NOT TURN OVER UNTIL TOLD TO BEGIN

PH4211: STATISTICAL MECHANICS
PH5211: STATISTICAL MECHANICS
PH5911: STATISTICAL MECHANICS
PAPER FOR FIRST SIT/RESIT CANDIDATES

Time Allowed: **TWO AND A HALF hours**

Answer **THREE** questions
No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

The total available marks add up to 120

All College-approved Calculators are permitted

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	$=$	4π	$\times$	10^{-7}	H m^{-1}
Permittivity of vacuum	ε_0	$=$	8.85	$\times$	10^{-12}	F m^{-1}
	$1/4\pi\varepsilon_0$	$=$	9.0	$\times$	10^{9}	m F^{-1}
Speed of light in vacuum	c	$=$	3.00	$\times$	10^{8}	m s^{-1}
Elementary charge	e	$=$	1.60	$\times$	10^{-19}	C
Electron (rest) mass	m_{e}	$=$	9.11	$\times$	10^{-31}	kg
Unified atomic mass constant	m_{u}	$=$	1.66	$\times$	10^{-27}	kg
Proton rest mass	m_{p}	$=$	1.67	$\times$	10^{-27}	kg
Neutron rest mass	m_{n}	$=$	1.67	$\times$	10^{-27}	kg
Ratio of electronic charge to mass	e/m_{e}	$=$	1.76	$\times$	10^{11}	C kg^{-1}
Planck constant	h	$=$	6.63	$\times$	10^{-34}	J s
	$\hbar = h/2\pi$	$=$	1.05	$\times$	10^{-34}	J s
Boltzmann constant	k_{B}	$=$	1.38	$\times$	10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	$=$	5.67	$\times$	10^{-8}	W m^{-2} K^{-4}
Gas constant	R	$=$	8.31			J mol^{-1} K^{-1}
Avogadro constant	N_{A}	$=$	6.02	$\times$	10^{23}	mol $^{-1}$
Gravitational constant	G	$=$	6.67	$\times$	10^{-11}	N m^{2} kg^{-2}
Acceleration due to gravity	g	$=$	9.81			m s^{-2}
Volume of one mole of an ideal gas at STP		$=$	2.24	$\times$	10^{-2}	m^{3}
One standard atmosphere	P_0	$=$	1.01	$\times$	10^{5}	N m^{-2}

NEXT PAGE

1. (a) The partition function for a single particle of mass m at temperature T, in a box of volume V is given by

$$z = V\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} = \frac{V}{\Lambda^3}.$$

Why is the quantity Λ referred to as the thermal de Broglie wavelength? [4]

(b) Carefully explain why the partition function for a collection of N similar but *distinguishable* objects, each with partition function z, is expressed as

$$Z = z^N.$$

[10]

(c) On the assumption that $Z = z^N$ is correct for a gas of particles, show that the Helmholtz free energy for such a gas would be given by

$$F = -NkT\ln\left\{V\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right\}.$$

[4]

(d) This expression gives a Helmholtz free energy that is *not extensive*. Explain what this means, and explain why this is a problem. [8]

(e) Briefly discuss Gibbs's proposal, before the quantum era, for how this problem could be resolved. [4]

(f) In the quantum view the partition function is

$$Z = \frac{1}{N!}z^N.$$

What is the reason for the $1/N!$ here? Explain why no problem appears with the Helmholtz free energy in this case and discuss why this resolution relies on the *thermodynamic limit*. [10]

[You may need Stirling's approximation: $\ln n! \approx n\ln(n/e)$.]

2. (a) In the Weiss model of ferromagnetism it is assumed that the magnetic moments are subject to an additional "mean" magnetic field $\mathbf{b}$

$$\mathbf{b} = \lambda \mathbf{M}$$

where $\mathbf{M}$ is the magnetization and λ is a constant.

Explain briefly the origin of this field. [8]

(b) The magnetization of a collection of N non-interacting spin $\frac{1}{2}$ magnetic moments μ at a temperature T is given by

$$M = M_0 \tanh\left(\frac{\mu B}{k_B T}\right)$$

where M_0 is the saturation magnetization $M_0 = N\mu$ and $\mathbf{M}$ is parallel to the applied magnetic field $\mathbf{B}$.

Show that the Weiss model leads to a spontaneous magnetization, in the absence of an external magnetic field, given by

$$\frac{M}{M_0} = \tanh\left(\frac{M}{M_0}\frac{T_\mathrm{c}}{T}\right)$$

where $T_\mathrm{c} = \lambda M_0^2/Nk_B$. [8]

(c) i. Sketch the behaviour of the spontaneous magnetization as a function of temperature. [4]

ii. What is the interpretation of T_c? [2]

iii. Discuss the order of the transition. [2]

(d) When T is very close to T_c then M/M_0 is very small. By expanding the hyperbolic tangent ($\tanh x \approx x - x^3/3 + \dots$) show that, in the vicinity of T_c, the magnetization behaves as

$$\frac{M}{M_0} \propto \left(1 - \frac{T}{T_\mathrm{c}}\right)^{1/2}.$$

[8]

(e) The critical exponent β describes the behaviour of the order parameter in the vicinity of the critical point. How is β defined?

How does the result in (d) compare with the behaviour of real systems? Discuss the difference. [8]

 NEXT PAGE

3. In the Landau theory of phase transitions the free energy is expanded in powers of the order parameter.

   (a) In the context of phase transitions, what is meant by the term *order parameter*? [4]

   (b) Explain why only a limited number of terms are needed. What determines the highest power in the expansion? [6]

   (c) What is the distinction between a *conserved* order parameter and a *non-conserved* order parameter? Give an example of each. Describe how the equilibrium state of the system is determined in these two cases. [7]

   (d) What is the order parameter of a binary alloy? Sketch the free energy for a temperature *above* and *below* the critical temperature. [6]

   (e) The free energy of mixing for the binary alloy may be written

$$F_{\mathrm{m}} = 2NkT_{\mathrm{c}}x(1-x) + NkT\left[x\ln x + (1-x)\ln(1-x)\right]$$

   where x is the order parameter.
   Explain the origin of the two terms in this expression. [6]

   (f) The expansion of F_{m} may be written as

$$F_{\mathrm{m}} = F_0 + 2Nk\left\{(T-T_{\mathrm{c}})\left(x-\frac{1}{2}\right)^2 + \frac{2}{3}T_{\mathrm{c}}\left(x-\frac{1}{2}\right)^4 + \frac{16}{15}T_{\mathrm{c}}\left(x-\frac{1}{2}\right)^6 + \cdots\right. .$$

   Discuss the Landau truncation of this expression; in particular explain at which term the series should be terminated. Why is the expansion written in powers of $x-\frac{1}{2}$? [7]

   (g) Within this model the order parameter critical exponent β has the value $\frac{1}{2}$. Show how this follows from the Landau free energy. [4]

4. Write an essay on the logical foundations of Statistical Mechanics and its connection with Thermodynamics. You should include a discussion of the nature of *equilibrium states*, the emergence of the concept of *temperature*, and the importance of the *thermodynamic limit*. [40]

5. The force on a Brownian particle in one dimension may be written as $F(t) = f(t) - v/\mu$ where $f(t)$ is the randomly fluctuating force experienced by the particle, v is the velocity and μ the mobility of the particle.

   (a) Discuss the separation of the force into the two parts. In particular, explain qualitatively how the damping force, proportional to the velocity, arises. [10]

   (b) Show that the equation of motion for the Brownian particle of mass M (the Langevin equation) may be written as [4]

   $$M\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{\mu}v(t) = f(t).$$

   (c) The solution to the Langevin equation is given by

   $$v(t) = v(0)e^{-t/M\mu} + \frac{1}{M}\int_0^t e^{(s-t)/M\mu} f(s)\,\mathrm{d}s.$$

      i. Show, using appropriate approximations, that the equilibrium mean square velocity may be expressed as $\langle v^2\rangle = \frac{\mu}{2M}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle \mathrm{d}t$. [8]

      ii. By invoking equipartition , show it follows that the mobility may be expressed $\frac{1}{\mu} = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle\,\mathrm{d}t$. [3]

   (d) Discuss how this may be regarded as an example of the *fluctuation-dissipation theorem*. [5]

   (e) The voltage across an $L - R$ circuit is given in terms of the current $I(t)$ by

   $$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) = V(t)$$

   where $V(t)$ may be regarded as a fluctuating voltage. By making the appropriate identifications, show that the resistance may be related to the voltage fluctuations through [6]

   $$R = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle V(0)V(t)\rangle\,\mathrm{d}t.$$

   (f) How may such voltage fluctuations be observed? [4]

END