

PH4211 Statistical Mechanics

Problem Sheet 2 — Answers

1 In Section 2.1 we saw that the density of free-particle states for a three dimensional volume V was shown to be

$$g(\varepsilon) = \frac{1}{4} \frac{V}{\pi^2 \hbar^3} (2m)^{3/2} \varepsilon^{1/2}.$$

This followed from counting the number of states in the octant of radius

$$R = \sqrt{n_x^2 + n_y^2 + n_z^2}.$$

By similar arguments show that in two dimensions, by counting the number of states in the quadrant of radius

$$R = \sqrt{n_x^2 + n_y^2},$$

the density of states is given by

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2}$$

where A is the area. Note in two dimensions the density of states is independent of energy.

And similarly, show that in one dimension the density of states is

$$g(\varepsilon) = \frac{L}{\pi\hbar} \left(\frac{m}{2}\right)^{1/2} \varepsilon^{-1/2}.$$

In the two-dimensional case the energy states may be specified (similar to Eq. 2.2) by

$$\varepsilon = \frac{\pi^2 \hbar^2}{2mA} (n_x^2 + n_y^2),$$

where A is the area. This may be written as

$$\varepsilon = \frac{\pi^2 \hbar^2}{2mA} R^2$$

where $R^2 = n_x^2 + n_y^2$.

Now the number of states of energy up to ε , denoted by $N(\varepsilon)$, is given by the number of points in the *quadrant* up to $\varepsilon(R)$. (A quadrant is used since n_x and n_y are restricted to being positive). And the number of points in the quadrant is approximately equal to the area of the quadrant:

$$N(\varepsilon) = \frac{1}{4} \pi R^2.$$

But since

$$R = \left(\frac{2mA}{\pi^2 \hbar^2}\right)^{1/2} \varepsilon^{1/2},$$

we then obtain

$$\begin{aligned}
 N(\varepsilon) &= \frac{1}{4} \pi \frac{2mA}{\pi^2 \hbar^2} \varepsilon \\
 &= \frac{1}{2} \frac{mA}{\pi \hbar^2} \varepsilon .
 \end{aligned}$$

Recall that the density of states $g(\varepsilon)$ is defined by saying that the number of states with energy between ε and $\varepsilon + d\varepsilon$ is $g(\varepsilon)d\varepsilon$. In other words

$$g(\varepsilon)d\varepsilon = N(\varepsilon + d\varepsilon) - N(\varepsilon)$$

or, simply

$$g(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} .$$

So differentiating $N(\varepsilon)$ we obtain, in two dimensions

$$g(\varepsilon) = \frac{mA}{2\pi \hbar^2}$$

which is the required expression for the density of states. And indeed we observe this is a constant, independent of energy.

In the one-dimensional case the energy states are specified (similar to Eq. 2.2) by

$$\varepsilon = \frac{\pi^2 \hbar^2}{2mL^2} n_x^2 ,$$

where L is the length. This may be written as

$$\varepsilon = \frac{\pi^2 \hbar^2}{2mL^2} R^2$$

where here $R^2 = n_x^2$.

Now the number of states of energy up to ε , denoted by $N(\varepsilon)$, is given by the number of points in the *positive line* up to $\varepsilon(R)$. (The positive line is used since n_x is restricted to being positive). And the number of points in the line is approximately equal to the length of the line:

$$N(\varepsilon) = R .$$

But since

$$R = \left(\frac{2mL^2}{\pi^2 \hbar^2} \right)^{1/2} \varepsilon^{1/2} ,$$

we then obtain

$$\begin{aligned}
 N(\varepsilon) &= \left(\frac{2mL^2}{\pi^2 \hbar^2} \right)^{1/2} \varepsilon^{1/2} \\
 &= \frac{L}{\pi \hbar} (2m)^{1/2} \varepsilon^{1/2} .
 \end{aligned}$$

Recall that the density of states $g(\varepsilon)$ is defined by saying that the number of states with energy between ε and $\varepsilon + d\varepsilon$ is $g(\varepsilon)d\varepsilon$. In other words

$$g(\varepsilon)d\varepsilon = N(\varepsilon + d\varepsilon) - N(\varepsilon)$$

or, simply

$$g(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon}.$$

So differentiating $N(\varepsilon)$ we obtain, in one dimension

$$g(\varepsilon) = \frac{L}{\pi\hbar} \left(\frac{m}{2}\right)^{1/2} \varepsilon^{-1/2}$$

which is the required expression for the density of states.

2 In Sections 2.3.1 and 2.3.2 the ideal gas partition function was calculated quantum-mechanically and classically. Although the calculations were quite different, they both resulted in (different) Gaussian integrals. By writing the Gaussian integral of the classical case as

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-(x^2+y^2+z^2)}$$

and transforming to spherical polar coordinates, you can perform the integration over θ and φ trivially. Show that the remaining integral can be reduced to that of the quantum case.

If we transform to spherical polar coordinates:

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

then the volume element is given by

$$dv = dx dy dz = r^2 \sin \theta dr d\theta d\varphi.$$

So the Gaussian integral is given by

$$I = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-(x^2+y^2+z^2)} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \theta d\theta \int_0^{\infty} r^2 e^{-r^2} dr.$$

We can evaluate the θ and φ integrals, to give

$$I = 4\pi \int_0^{\infty} r^2 e^{-r^2} dr.$$

Now change variables to $x = r^2$ so that $dr = dx/2x^{1/2}$. Then the integral becomes

$$I = 2\pi \int_0^{\infty} x^{1/2} e^{-x} dx$$

and this is the integral that appeared in the quantum calculation of the partition function.

3 The Sakur-Tetrode equation, discussed in Section 2.3.3,

$$S = Nk \ln V - Nk \ln N + \frac{3}{2} Nk \ln T + Nks_0$$

is often interpreted as indicating different contributions to the entropy: the volume contribution is in the first term, the number contribution in the second term and the temperature contribution in the third term. Show that such an identification is fallacious, by demonstrating that the various contributions depend on the choice of

units adopted – even though the total sum is independent. Discuss the origin of the fallacy.

The entropy of the ideal gas is given by

$$S = Nk \ln \left[\left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \frac{V}{N} e^{5/2} \right].$$

The argument of the logarithm is dimensionless. This may be checked explicitly, but note that in terms of the thermal de Broglie wavelength Λ ,

$$S = Nk \ln \left[\frac{V}{\Lambda^3 N} e^{5/2} \right]$$

so the argument is a volume divided by a volume – dimensionless.

By analogy with the Sakur-Tetrode equation, we may write the entropy as

$$S = Nk \ln V - Nk \ln N - Nk \ln \Lambda^3 + \frac{5}{2} Nk.$$

Now imagine that we decide to measure length in centimetres instead of meters. Then the numerical value of the argument in $\ln V$ will increase by a factor of 10^6 . So the calculated value of $\ln V$ will increase by $6 \ln 10 = 13.82 \dots$. So the size of the ‘volume contribution’ to the entropy will change if the unit of volume is changed.

Of course the total entropy does not change. Clearly if the length unit is changed in this way then the thermal de Broglie wavelength term will also change – and in just such a way to cancel the change from the V term. *Thus* it makes no sense to identify $Nk \ln V$ as the volume contribution to the entropy.

Another way of looking at this matter is to recognise that for any function that contains different powers of its arguments (recall the power series for the logarithm), the argument of the function should be dimensionless. If not, the analysis of dimensions becomes senseless. Thus we conclude that it is acceptable to write S as

$$S = Nk \ln \frac{V}{\Lambda^3} - Nk \ln N + \frac{5}{2} Nk,$$

but any further decomposition gives non-unique apportionments of the different contributions.

4 Show that the Fermi energy for a two-dimensional gas of Fermions is

$$\varepsilon_F = \frac{2\pi\hbar^2}{\alpha m} \frac{N}{A}$$

where A is the area of the system.

At zero temperature, where the Fermi distribution becomes a step function, the number of particles in the system may be expressed as

$$N = \alpha \int_0^{\varepsilon_F} g(\varepsilon) d\varepsilon$$

and since in 2d the density of states is

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2},$$

independent of energy, N is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2} \varepsilon_F$$

so that

$$\varepsilon_F = \frac{2\pi\hbar^2}{\alpha m} \frac{N}{A},$$

as required.

5 Show that the chemical potential of a two-dimensional gas of fermions may be expressed analytically as

$$\mu = kT \ln \left\{ e^{\varepsilon_F/kT} - 1 \right\}$$

At finite temperatures the number of particles is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2} \int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}$$

and since

$$\varepsilon_F = \frac{2\pi\hbar^2}{\alpha m} \frac{N}{A}$$

the integral for N may be expressed

$$\varepsilon_F = \int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}.$$

We shall change variables to $x = (\varepsilon - \mu)/kT$ so that

$$\varepsilon_F = kT \int_{-\mu/kT}^\infty \frac{dx}{e^x + 1}.$$

The integral may be evaluated (using *Mathematica*, for example) as

$$\int_{-\mu/kT}^\infty \frac{dx}{e^x + 1} = \frac{\mu}{kT} + \ln \left\{ 1 + e^{-\mu/kT} \right\}$$

however this may be simplified by writing the argument of the logarithm differently

$$\begin{aligned} \frac{\mu}{kT} + \ln \left\{ 1 + e^{-\mu/kT} \right\} &= \frac{\mu}{kT} + \ln \left\{ e^{-\mu/kT} (e^{\mu/kT} + 1) \right\} \\ &= \frac{\mu}{kT} + \ln e^{-\mu/kT} + \ln \left\{ e^{\mu/kT} + 1 \right\} \\ &= \ln \left\{ e^{\mu/kT} + 1 \right\}. \end{aligned}$$

Thus we conclude

$$\varepsilon_F = kT \ln \left\{ e^{\mu/kT} + 1 \right\}.$$

And this may be rearranged to give the chemical potential as

$$\mu = kT \ln \left\{ e^{\varepsilon_F/kT} - 1 \right\},$$

as required.

6 Calculate the low temperature chemical potential of a two-dimensional gas of fermions by the Sommerfeld expansion method of Section 2.4.3. Observe that the temperature series expansion terminates. Compare this result with the exact result of the previous question. Discuss the difference between the two results.

We start from the expression for the Fermi energy for a two-dimensional system, following from the expression for the total number of particles:

$$\varepsilon_F = \int_0^{\infty} n(\varepsilon) d\varepsilon .$$

This may be integrated by parts (the other factor of the integrand in this case is unity):

$$\varepsilon_F = - \int_0^{\infty} \varepsilon n'(\varepsilon) d\varepsilon .$$

We have to expand the ε factor about $\varepsilon = \mu$

$$\varepsilon = \mu + (\varepsilon - \mu)$$

so the Fermi energy is given by

$$\varepsilon_F = -\mu \int_0^{\infty} n'(\varepsilon) d\varepsilon - \int_0^{\infty} (\varepsilon - \mu) n'(\varepsilon) d\varepsilon$$

with no higher order terms. The first integral is $-I_0$ of Section 2.4.3 and the second integral is $-kTI_1$. Now $I_0 = 1$, which may be shown quite simply. And within the approximation of the Sommerfeld expansion, that the lower limit of the integral be extended to $-\infty$, I_1 vanishes, as do all the odd-order terms. Thus the Sommerfeld expansion expression for the chemical potential is simply

$$\mu = \varepsilon_F ;$$

the expansion terminates (rather early) and it completely misses the temperature dependence of μ away from $T = 0$.

The previous Problem showed in 2d the chemical potential may be expressed exactly

$$\mu = kT \ln \left\{ e^{\varepsilon_F/kT} - 1 \right\} .$$

This can also be written as

$$\begin{aligned} \mu &= kT \ln \left\{ e^{\varepsilon_F/kT} \left(1 - e^{-\varepsilon_F/kT} \right) \right\} \\ &= \varepsilon_F + \ln \left\{ 1 - e^{-\varepsilon_F/kT} \right\} \end{aligned}$$

which is more convenient at low temperatures. Then the logarithm term is the part missed by the Sommerfeld series. And at low temperatures the exponential is small and so the logarithm can be expanded in powers of this exponential

$$\ln \left\{ 1 - e^{-\varepsilon_F/kT} \right\} = -e^{-\varepsilon_F/kT} - \frac{1}{2} e^{-2\varepsilon_F/kT} - \frac{1}{3} e^{-3\varepsilon_F/kT} - \frac{1}{4} e^{-4\varepsilon_F/kT} - \dots$$

so that

$$\mu = \varepsilon_F - e^{-\varepsilon_F/kT} - \frac{1}{2} e^{-2\varepsilon_F/kT} - \frac{1}{3} e^{-3\varepsilon_F/kT} - \frac{1}{4} e^{-4\varepsilon_F/kT} - \dots$$

It is the exponential terms that are missed by the Sommerfeld expansion.

7 The general formula for the Fermi integrals I_n of Section 2.4.3 was quoted as

$$I_n = \int_{-\infty}^{\infty} \frac{e^x}{(e^x + 1)^2} x^n dx$$

$$= (2 - 2^{2-n}) \zeta(n) n!$$

Derive this result. (You might find the discussion in Landau and Lifshitz, *Statistical Physics*, helpful.)

8 Obtain the chemical potential μ , the internal energy E and the heat capacity C_v for system with general density of states $g(\varepsilon)$ as in Section 2.4.4. I.e. show that these are given in terms of the behaviour of the density of states at the Fermi surface.

9 Consider the Bose gas at low temperatures. You saw in Section 2.5.2 and 2.5.3 that when the occupation of the ground state is appreciable then the chemical potential μ is very small and it may be ignored, compared with ε in the integral for the number of excited states.

Show that when the ground state occupation N_0 is appreciable then μ may be approximated by

$$\mu \sim -kT/N_0.$$

Now consider the more stringent requirement that μ may be neglected in comparison with ε in the integral for the number of excited states. This will be satisfied if μ is much less than the energy ε_1 of the first excited state. The expression for ε_1 is

$$\varepsilon_1 \sim \frac{\pi^2 \hbar^2}{2mV^{2/3}}.$$

Where does this expression come from?

Show that the condition $\mu \ll \varepsilon_1$ is satisfied when $N \gg 10^{15}$ (approximately) when considering 1cm^3 of ^4He (molar volume 27cm^3) at a temperature of about 1K.

Thus show that the expression

$$N_0 = N \left\{ 1 - \left(\frac{T}{T_c} \right)^{3/2} \right\}$$

is then valid to temperatures below T_c right up to within $\sim 10^{-8} T_c$ of the critical temperature.

10 Liquid ^4He has a molar volume at saturated vapour pressure of 27cm^3 . Treating the liquid as an ideal gas of bosons, find the temperature at which Bose-Einstein condensation will occur. How will this temperature change as the pressure on the fluid is increased?

The condensation temperature is given, from Eq. 2.50, by

$$T_c = 3.313 \frac{\hbar^2}{mk} \left\{ \frac{N}{V} \right\}^{2/3} .$$

The 'atomic mass' of helium is 4. And one atomic mass unit weighs 1.66×10^{-27} kg so a helium atom weighs 6.64×10^{-27} kg.

The volume of one mole of liquid helium is 27 cm^3 , or $27 \times 10^{-6} \text{ m}^3$. Now N/V is Avogadro's number divided by the molar volume:

$$\begin{aligned} \frac{N}{V} &= \frac{6.02 \times 10^{23}}{27 \times 10^{-6}} \\ &= 2.23 \times 10^{28} \end{aligned}$$

We need the $2/3$ power of this:

$$\left(\frac{N}{V} \right)^{2/3} = 7.92 \times 10^{18} .$$

Then, using the values for Planck's constant and Boltzmann's constant, we obtain

$$\begin{aligned} T_c &= 3.313 \frac{(1.05 \times 10^{-34})^2}{6.64 \times 10^{-27} \times 1.38 \times 10^{-23}} \times 7.92 \times 10^{18} \\ &= 3.15 \text{ K} . \end{aligned}$$

The temperature at which Bose-Einstein condensation occurs is 3.15 K.

Since the formula for T_c indicates that it increases with density, N/V , and since increasing the pressure will increase the density, it follows that increasing the pressure will increase the Bose-Einstein condensation temperature.

11 The superfluid transition temperature of liquid helium decreases with increasing pressure. Very approximately $\partial T_c / \partial p \sim -0.015 \text{ K bar}^{-1}$. How does this compare with the behaviour predicted from the Bose-Einstein condensation?

The Bose-Einstein condensation temperature is given by

$$T_c = 3.313 \frac{\hbar^2}{mk} \left\{ \frac{N}{V} \right\}^{2/3} ;$$

this predicts that T_c will *increase* with pressure (density). So the observed *reduction* of T_c with pressure is in conflict with this model.

12 Show that below the transition temperature the entropy of a Bose gas is given by

$$S = \frac{5}{3} Nk \frac{I_{3/2}}{I_{1/2}} \left(\frac{T}{T_c} \right)^{3/2} .$$

Since the number of excited particles is given by

$$N_{\text{ex}} = N \left(\frac{T}{T_c} \right)^{3/2} ,$$

show that the entropy per excited particle is given by

$$\frac{S}{N_{ex}} = \frac{5}{3} \frac{I_{3/2}}{I_{1/2}} k \approx 1.28k .$$

Discuss the connection between this result and the two fluid model of superfluid ^4He .

The entropy is most conveniently found from the heat capacity C_V . Since

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_V$$

it follows that

$$S = \int \frac{C_V}{T} dT .$$

And since the heat capacity is given from Eq. 2.54 as

$$C_V = \frac{5}{2} Nk \frac{I_{3/2}}{I_{5/2}} \left(\frac{T}{T_c} \right)^{3/2} ,$$

upon integration we find

$$S = \frac{5}{3} Nk \frac{I_{3/2}}{I_{1/2}} \left(\frac{T}{T_c} \right)^{3/2}$$

as required.

Now the number of particles out of the ground state, the number of excited particles, is

$$N_{ex} = N \left(\frac{T}{T_c} \right)^{3/2} ,$$

so dividing the total entropy we obtain the entropy per excited particle as

$$\frac{S}{N_{ex}} = \frac{5}{3} \frac{I_{3/2}}{I_{1/2}} k \approx 1.28k .$$

This indicates that the entropy per excited particle is a constant, independent of temperature. Now the two fluid model argues that the system is comprised of two interpenetrating fluids – the ‘normal’ component and the ‘superfluid’ component. And it asserts that the superfluid component carries no entropy. Here we see that the entropy may indeed all be associated with the ‘normal’ or excited component, each excited particle carrying the same ‘unit’ of entropy.

13 Show that the Bose-Einstein transition temperature of a gas of bosons and the Fermi temperature for a gas of ‘similar’ fermions are of comparable magnitude. Discuss why this should be.

The Bose-Einstein transition temperature for a gas of spin 0 bosons ($\alpha = 1$) is given by

$$\begin{aligned} T_c &= \frac{2\pi\hbar^2}{mk} \left\{ \frac{N}{2.612V} \right\}^{2/3} \\ &= 3.313 \frac{\hbar^2}{mk} \left\{ \frac{N}{V} \right\}^{2/3} . \end{aligned}$$

The Fermi temperature of a gas of spin $\frac{1}{2}$ fermions ($\alpha = 2$), ε_F/k , is given by

$$T_F = \frac{\hbar^2}{2mk} \left\{ \frac{6\pi^2 N}{2V} \right\}^{2/3}$$

$$= 4.785 \frac{\hbar^2}{mk} \left\{ \frac{N}{V} \right\}^{2/3} .$$

Thus the Bose condensation temperature and the Fermi temperature are very similar. They certainly depend on all the system variables in the same way and the the Fermi temperature of a gas of spin $\frac{1}{2}$ fermions is approximately 1.4 times the Bose-Einstein transition temperature for a gas of spin 0 bosons.

The reason they are similar is because both temperatures are characterised by the thermal deBroglie wavelength for the particles becoming comparable with the inter-particle spacing. This is the condition for quantum effects to become important.

14 In Section 2.6 we studied a paramagnetic *solid*: a collection of essentially *distinguishable* magnetic moments. If we were to consider a (classical) gas of indistinguishable magnetic moments, how would the partition function be modified? What would the observable consequences of this modification?

A full treatment of a gas of particles would require consideration of the particles' kinetic energy. However if we neglect this then the only modification to the partition function would be the inclusion of the $N!$ term for indistinguishability. Then we would have

$$Z = \frac{\{2 \cosh \varepsilon/kT\}^N}{N!}$$

and since $F = -kT \ln Z$ the free energy is

$$F = -NkT \ln \{2 \cosh \varepsilon/kT\} + NkT \ln N - NkT .$$

The first term is the free energy calculated for the distinguishable solid case and the second and third terms are the corrections for indistinguishability.

Since $dF = -SdT - MdB$, it follows that

$$M = -\frac{\partial F}{\partial B} \quad \text{and} \quad S = -\frac{\partial F}{\partial T} .$$

This means that while the magnetisation of the system will be the same as that of the distinguishable system, there will be extra contributions to the entropy (and thus to the heat capacity etc.).

15 Show, using arguments similar to those in Section 2.1.3, that the energy levels of an ultra-relativistic or a massless particle with energy-momentum relation $E = cp$ are given by

$$\varepsilon = \frac{c\pi\hbar}{V^{1/3}} (n_x^2 + n_y^2 + n_z^2)^{1/2} .$$

Hence show that the pressure of a gas of such particles is one third of the (internal) energy density.

16 Evaluate the Fermi temperature for liquid ${}^3\text{He}$, assuming it to be a Fermi “gas”. Its molar volume is 36 cm^3 . Calculate the de Broglie wavelength at $T = T_F$ and show that it is comparable to the interparticle spacing as expected.

The Fermi temperature is given by

$$T_F = \frac{\hbar^2}{2mk} \left\{ \frac{6\pi^2 N}{V} \right\}^{2/3}$$

$$= 4.785 \frac{\hbar^2}{mk} \left\{ \frac{N}{V} \right\}^{2/3} .$$

The ‘atomic mass’ of ${}^3\text{He}$ is 3. And one atomic mass unit weighs $1.66 \times 10^{-27}\text{ kg}$ so a helium-3 atom weighs $4.98 \times 10^{-27}\text{ kg}$.

The volume of one mole of liquid helium-3 is 36 cm^3 , or $36 \times 10^{-6}\text{ m}^3$. Now N/V is Avogadro’s number divided by the molar volume:

$$\frac{N}{V} = \frac{6.02 \times 10^{23}}{36 \times 10^{-6}}$$

$$= 1.67 \times 10^{28}$$

so that

$$\left(\frac{N}{V} \right)^{2/3} = 6.53 \times 10^{18} .$$

Then, using the values for Planck’s constant and Boltzmann’s constant, we obtain

$$T_F = 4.785 \times \frac{(1.05 \times 10^{-34})^2}{4.98 \times 10^{-27} \times 1.38 \times 10^{-23}} \times 6.53 \times 10^{18}$$

$$= 5.01\text{ K} .$$

Thus the Fermi temperature is calculated as 5.01 K.

17 In Problem 2.1 we found the expression for the energy density of states $g(\varepsilon)$ for a gas of fermions confined to two dimensions and we saw that it was independent of energy. What surface density of electrons is necessary in order that $T_F = 100\text{ mK}$? Show that, for a given area, the low temperature heat capacity is linear in T and independent of the number of electrons.

At zero temperature, where the Fermi distribution becomes a step function, the number of particles in the system may be expressed as

$$N = \alpha \int_0^{\varepsilon_F} g(\varepsilon) d\varepsilon$$

and since in 2d the density of states is

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2} ,$$

independent of energy, N is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2} \varepsilon_F$$

so that

$$\varepsilon_F = \frac{2\pi\hbar^2}{\alpha m} \frac{N}{A},$$

and the Fermi temperature $T_F = \varepsilon_F/k$ is

$$T_F = \frac{2\pi\hbar^2}{\alpha mk} \frac{N}{A}.$$

The areal density corresponding to a given Fermi temperature is

$$\frac{N}{A} = \frac{\alpha mk T_F}{2\pi\hbar^2}.$$

For electrons we have $\alpha = 2$, $m = 9.11 \times 10^{-31}$ kg, so for a Fermi temperature of 0.1 K we have

$$\begin{aligned} \frac{N}{A} &= \frac{2 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 0.1}{2\pi \times (1.06 \times 10^{-34})^2} \\ &= 3.56 \times 10^{13}. \end{aligned}$$

So the number density is 3.56×10^{13} electrons per square meter.

The heat capacity is found by differentiating the internal energy.

The simple treatment of the next part is to use the argument that at low temperatures only a fraction T/T_F of the electrons will be excited so that the internal energy is essentially this fraction of the classical (equipartition) internal energy NkT in two dimensions. Thus

$$E \sim Nk \frac{T^2}{T_F}$$

and so, upon differentiation,

$$C_V \sim Nk \frac{T}{T_F}.$$

Thus in the low-temperature limit the heat capacity is linear in T , as in the 3-d case.

The point here is that in 2-d the Fermi temperature is proportional to N (over A) so this N cancels with that in the expression for C_V , making the heat capacity independent of N :

$$T_F = \frac{2\pi\hbar^2}{\alpha mk} \frac{N}{A}$$

so that

$$\begin{aligned} C_V &\sim Nk \frac{T}{T_F} \\ &\sim \frac{\alpha mk^2 A}{2\pi\hbar^2} T \end{aligned}$$

and thus we see that the low-temperature heat capacity of a 2-d electron gas is independent of the number of electrons.

A better treatment of the low-temperature heat capacity would use the Sommerfeld expansion. This gives the internal energy as

$$E = \text{const} + \frac{\pi^2}{6} N \frac{kT^2}{T_F}$$

and the heat capacity is then

$$C_V = \frac{\pi^2}{6} Nk \frac{T}{T_F}.$$

This is the correct low temperature limiting behaviour, but as we saw in Problem 2.6, the Sommerfeld method is incapable of treating the higher-order terms in 2-d.

Observe this result demonstrated that the previous approximation gives the correct result to within a numerical correction factor. Then that argument about the N from the Fermi temperature cancelling still applies and again we find that the heat capacity will be independent of N .

18 Use the Sommerfeld expansion method of Section 2.4.3 to show that the Fermi-Dirac distribution function may be approximated, at low temperatures, by

$$\frac{1}{e^{(\varepsilon-\mu)/kT} + 1} \sim \Theta(\mu - \varepsilon) - \frac{\pi^2}{6} (kT)^2 \delta'(\varepsilon - \mu) + \dots$$

where Θ is the unit step function and δ' is the first derivative of the Dirac delta function.

Can you write down the general term of the series?

According to the results of Section 2.4.3 integrals of the form

$$I = \int_0^\infty \frac{\varphi(\varepsilon)}{e^{(\varepsilon-\mu)/kT} + 1} d\varepsilon,$$

where $\varphi(\varepsilon)$ is an arbitrary function, are approximated by

$$I = \sum_{n=0}^{\infty} I_n \frac{(kT)^n}{n!} \left. \frac{d^n \psi}{d\varepsilon^n} \right|_{\varepsilon=\mu}.$$

Here ψ is the integral of φ :

$$\psi(\varepsilon) = \int \varphi(\varepsilon) d\varepsilon.$$

We have the expression for the Sommerfeld expansion as

$$\int_0^\infty \frac{\varphi(\varepsilon)}{e^{(\varepsilon-\mu)/kT} + 1} d\varepsilon = \sum_{n=0}^{\infty} I_n \frac{(kT)^n}{n!} \left. \frac{d^n \psi}{d\varepsilon^n} \right|_{\varepsilon=\mu}$$

and we can write this out term by term

$$\int_0^\infty \frac{\varphi(\varepsilon)}{e^{(\varepsilon-\mu)/kT} + 1} d\varepsilon = \int_0^\mu \varphi(\varepsilon) d\varepsilon + \frac{(kT)^2}{2} I_2 \frac{d\varphi}{d\varepsilon} + \frac{(kT)^4}{4!} I_4 \frac{d^3\varphi}{d\varepsilon^3} + \dots + \frac{(kT)^n}{n!} I_n \frac{d^{n-1}\varphi}{d\varepsilon^{n-1}} + \dots$$

The right hand side could be written as

$$\begin{aligned} & \int_0^\infty \Theta(\mu - \varepsilon) \varphi(\varepsilon) d\varepsilon + \frac{(kT)^2}{2} I_2 \int_0^\infty \delta'(\varepsilon - \mu) \varphi(\varepsilon) d\varepsilon + \\ & + \frac{(kT)^4}{4!} I_4 \int_0^\infty \delta'''(\varepsilon - \mu) \varphi(\varepsilon) d\varepsilon + \dots + \frac{(kT)^n}{n!} I_n \int_0^\infty \delta^{(n-1)}(\varepsilon - \mu) \varphi(\varepsilon) d\varepsilon + \dots \end{aligned}$$

where Θ is the step function and $\delta^{(n)}$ is the n th derivative of the delta function. Then since this result holds for arbitrary functions $\varphi(\varepsilon)$ we can equate the insides of the integral to give

$$\frac{1}{e^{(\varepsilon-\mu)/kT} + 1} = \Theta(\mu - \varepsilon) + \frac{(kT)^2}{2} I_2 \delta'(\varepsilon - \mu) + \frac{(kT)^4}{4!} I_4 \delta'''(\varepsilon - \mu) + \dots + \frac{(kT)^n}{n!} I_n \delta^{(n-1)}(\varepsilon - \mu) + \dots$$

If we substitute for the values of I_n in the first few terms we obtain

$$\frac{1}{e^{(\varepsilon-\mu)/kT} + 1} = \Theta(\mu - \varepsilon) + \frac{\pi^2}{6} (kT)^2 \delta'(\varepsilon - \mu) + \frac{7\pi^4}{360} (kT)^4 \delta'''(\varepsilon - \mu) + \dots,$$

as required. And the general term of the expansion is given by

$$\frac{I_n}{n!} (kT)^n \delta^{(n-1)}(\varepsilon - \mu) = (2 - 2^{2-n}) \zeta(n) (kT)^n \delta^{(n-1)}(\varepsilon - \mu).$$

Observe that the first term of the expansion, the step function, is the zero-temperature distribution function. The higher order terms give the finite-temperature corrections to this.