2.11 Problems for Chapter 2

2.1 In Section 2.1 we saw that the density of free-particle states for a three dimensional volume V was shown to be

$$g(\varepsilon)=\frac{1}{4}\frac{V}{\pi^2\hbar^3}(2m)^{3/2}\varepsilon^{1/2};$$

this followed from counting the number of states in the octant of radius

$$R=\sqrt{n_x^2+n_y^2+n_z^2}.$$

(a) By similar arguments show that in two dimensions, by counting the number of states in the quadrant of radius

$$R=\sqrt{n_x^2+n_y^2},$$

the density of states is given by

$$g(\varepsilon)=\frac{mA}{2\pi\hbar^2}$$

where A is the area. Note in two dimensions the density of states is independent of energy.

(b) And similarly, show that in one dimension the density of states is

$$g(\varepsilon)=\frac{L}{\pi\hbar}\left(\frac{m}{2}\right)^{1/2}\varepsilon^{-1/2}.$$

(a) **Two dimensions**

In the two-dimensional case the energy states may be specified (similar to Eq. (2.1.8)) by

$$\varepsilon=\frac{\pi^2\hbar^2}{2mA}\left(n_x^2+n_y^2\right),$$

where A is the area. This may be written as

$$\varepsilon=\frac{\pi^2\hbar^2}{2mA}R^2$$

where $R^2=n_x^2+n_y^2$.

Now the number of states of energy up to ε, denoted by $N(\varepsilon)$, is given by the number of points in the *quadrant* up to $\varepsilon(R)$. (A quadrant is used since n_x and n_y are restricted to being positive). And the number of points in the quadrant is approximately equal to the area of the quadrant:

$$N(\varepsilon) = \frac{1}{4}\pi R^2.$$

But since

$$R = \left(\frac{2mA}{\pi^2\hbar^2}\right)^{1/2}\varepsilon^{1/2},$$

we then obtain

$$\begin{aligned} N(\varepsilon) &= \frac{1}{4}\pi\frac{2mA}{\pi^2\hbar^2}\varepsilon \\ &= \frac{1}{2}\frac{mA}{\pi\hbar^2}\varepsilon. \end{aligned}$$

Recall that the density of states $g(\varepsilon)$ is defined by saying that the number of states with energy between ε and $\varepsilon + \mathrm{d}\varepsilon$ is $g(\varepsilon)\mathrm{d}\varepsilon$. In other words

$$g(\varepsilon)\mathrm{d}\varepsilon = N(\varepsilon + \mathrm{d}\varepsilon) - N(\varepsilon)$$

or, simply

$$g(\varepsilon) = \frac{\mathrm{d}N(\varepsilon)}{\mathrm{d}\varepsilon}.$$

So differentiating $N(\varepsilon)$ we obtain, in two dimensions

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2},$$

which is the required expression for the density of states. And indeed we observe this is a constant, independent of energy.

(b) **One dimension**

In the one-dimensional case the energy states are specified (similar to Eq. (2.1.8)) by

$$\varepsilon = \frac{\hbar^2\hbar^2}{2mL^2}n_x^2,$$

where L is the length. This may be written as

$$\varepsilon = \frac{\pi^2\hbar^2}{2mL^2}R^2$$

where here $R^2 = n_x^2$.

Now the number of states of energy up to ε, denoted by $N(\varepsilon)$, is given by the number of points in the *positive line* up to $g(R)$. (The positive line is used since n, is restricted to being positive). And the number of points in the line is approximately equal to the length of the line:

$$N(\varepsilon) = R.$$

But since

$$R = \left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2} \varepsilon^{1/2},$$

we then obtain

$$\begin{aligned} N(\varepsilon) &= \left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2} \varepsilon^{1/2} \\ &= \frac{L}{\pi\hbar}(2m)^{1/2}\varepsilon^{1/2}. \end{aligned}$$

Recall that the density of states $g(\varepsilon)$ is defined by saying that the number of states with energy between ε and $\varepsilon + \mathrm{d}\varepsilon$ is $g(\varepsilon)\mathrm{d}\varepsilon$. In other words

$$g(\varepsilon)\mathrm{d}\varepsilon = N(\varepsilon + \mathrm{d}\varepsilon) - N(\varepsilon)$$

of, simply

$$g(\varepsilon) = \frac{\mathrm{d}N(\varepsilon)}{\mathrm{d}\varepsilon}.$$

So differentiating $N(\varepsilon)$ we obtain, in one dimension

$$g(\varepsilon) = \frac{L}{\pi\hbar}\left(\frac{m}{2}\right)^{1/2} \varepsilon^{-1/2}$$

which is the required expression for the density of states.

2.2 Show, using arguments similar to those in Section 2.1.3, that the energy levels of an ultra-relativistic or a massless particle with energy-momentum relation $E = cp$ are given by

$$\varepsilon = \frac{c\pi\hbar}{V^{1/3}}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}.$$

Hence show that the pressure of a gas of such particles is one third of the (internal) energy density.

The energy of a state, in 3 dimensions, is

$$\varepsilon = c\left(p_x^2 + p_y^2 + p_z^2\right)^{1/2}$$

and p_x, p_y, p_z are given, as in Eq. (2.1.6):

$$p_x = \frac{\pi\hbar}{V^{1/3}} n_x, \quad p_y = \frac{\pi\hbar}{V^{1/3}} n_y, \quad p_z = \frac{\pi\hbar}{V^{1/3}} n_z \,.$$

This gives

$$\varepsilon = \frac{c\pi\hbar}{V^{1/3}} \left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}$$

as required.

We observe the energy states vary with volume as $\varepsilon \propto V^\alpha$ with an exponent $\alpha = -1/3$.

In Section 2.1.6 we saw that when $\varepsilon \propto V^\alpha$, the pressure is given by Eq. (2.1.33):

$$p = -\alpha \frac{E}{V}.$$

So in our case, for a gas of mass-less particles, we find

$$p = \frac{1}{3}\frac{E}{V},$$

as required.

2.3 In Sections 2.3.1 and 2.3.2 the ideal gas partition function was calculated quantum-mechanically and classically. Although the calculations were quite different, they both resulted in (different) gaussian integrals. By writing the gaussian integral of the classical case as

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \; e^{-\left(x^2+y^2+z^2\right)}$$

and transforming to spherical polar coordinates, you can perform the integration over θ and ϕ trivially. Show that the remaining integral can be reduced to that of the quantum case.

Upon transforming to spherical polar coordinates

$$r = x^2 + y^2 + z^2 \quad \text{and} \quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

we obtain

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z \; e^{-\left(x^2+y^2+z^2\right)} \\ &= \int_0^{\infty} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, r^2 \sin\theta e^{-r^2}. \end{aligned}$$

The ϕ and θ integrals are easily evaluated:

$$\int_0^{2\pi} \mathrm{d}\phi = 2\pi \quad \text{and} \quad \int_0^{\pi} \sin\theta \, \mathrm{d}\theta = 2.$$

Then I becomes

$$I = 4\pi \int_0^{\infty} r^2 e^{-r^2} \mathrm{d}r.$$

Now change the variable of integration to $w = r^2$ (then $\mathrm{d}r = \mathrm{d}x/2x^{1/2}$). This gives

$$I = 2\pi \int_0^{\infty} w^{1/2} e^{-w} \mathrm{d}w,$$

the gaussian integral in the quantum expression, Eq. (2.3.3).

2.4 The Sackur-Tetrode equation, discussed in Section 2.3.3, and written as

$$S = Nk \ln V - Nk \ln N + \frac{3}{2} Nk \ln T + Nks_0$$

is often interpreted as indicating different contributions to the entropy: the volume contribution is in the first term, the number contribution in the second term and the temperature contribution in the third term. Show that such an identification is fallacious, by demonstrating that the various contributions depend on the choice of units adopted – even though the total sum is independent. Discuss the origin of the fallacy.

The entropy of the ideal gas is given by

$$S = Nk \ln \left[\left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \frac{V}{N} e^{5/2} \right].$$

The argument of the logarithm is dimensionless. This may be checked explicitly, but note that in terms of the thermal de Broglie wavelength Λ

$$S = Nk \ln \left[\frac{V}{\Lambda^3 N} e^{5/2} \right]$$

so the argument is a volume divided by a volume – dimensionless.

By analogy with the Sakur-Tetrode equation, we may write the entropy as

$$S = Nk \ln V - Nk \ln N - Nk \ln \Lambda^3 + \frac{5}{2} Nk.$$

Now imagine that we decide to measure length in centimetres instead of metres. Then the numerical value of the argument in $\ln V$ will increase by a factor of 10^6. So the calculated value of $\ln V$ will increase by $6 \ln 10 = 13.82$. So the size of the "volume contribution" to the entropy will change if the unit of volume is changed.

Of course the total entropy does not change. Clearly if the length unit is changed in this way then the thermal de Broglie wavelength term will also change – and in just such a way to cancel the change from the V term. *Thus* it makes no sense to identify $Nk \ln V$ as the volume contribution to the entropy.

Another way of looking at this matter is to recognise that for any function that contains different powers of its arguments (recall the power series for the logarithm), the argument of the function should be dimensionless. If not, the analysis of dimensions becomes senseless. Thus we conclude that it is acceptable to write S as

$$S = Nk \ln \frac{V}{\Lambda^3} - Nk \ln N + \frac{5}{2} Nk,$$

but any further decomposition gives non-unique apportionments of the different contributions.

2.5 The Gibbs-Duhem relation tells us that the intensive variables T, p, μ are not independent so that, for example, μ depends on T and p. Demonstrate this by showing that

$$\mu = -kT \ln \left[\left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \frac{(kT)^{5/2}}{p} \right]$$

for an ideal gas.

We found, in Eq. (2.3.16), the chemical potential μ for an ideal gas in terms of T, V and N:

$$\mu = -kT \ln \left[\left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \frac{V}{N} \right].$$

We require μ in terms of the other *intensive* variables T and p; we must eliminate V/N. This may be done by using the ideal gas equation of state: $V/N = kT/p$. Thus we obtain

$$\mu = -kT \ln \left[\left(\frac{m}{2\pi\hbar^2} \right)^{3/2} \frac{(kT)^{5/2}}{p} \right]$$

as required.

2.6 Show that the Fermi energy for a two-dimensional gas of Fermions is

$$\varepsilon_{\mathrm{F}} = \frac{2\pi\hbar^2}{\alpha m}\frac{N}{A}$$

where A is the area of the system.

At zero temperature, where the Fermi distribution becomes a step function, the number of particles in the system may be expressed as

$$N = \alpha\int_0^{\varepsilon_{\mathrm{F}}} g(\varepsilon)\mathrm{d}\varepsilon$$

and since in 2d the density of states is

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2},$$

independent of energy, N is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2}\varepsilon_{\mathrm{F}}$$

so that

$$\varepsilon_{\mathrm{F}} = \frac{2\pi\hbar^2}{\alpha m}\frac{N}{A}$$

as required.

2.7 Show that the chemical potential of a two-dimensional gas of fermions may be expressed analytically as

$$\mu = kT\ln\left\{e^{\varepsilon_{\mathrm{F}}/kT} - 1\right\}.$$

At finite temperatures the number of particles is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2}\int_0^\infty \frac{\mathrm{d}\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}$$

and since

$$\varepsilon_{\mathrm{F}} = \frac{2\pi\hbar^2}{\alpha m}\frac{N}{A}$$

the integral for N may be expressed

$$\varepsilon_{\mathrm{F}} = \int_0^\infty \frac{\mathrm{d}\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}.$$

We shall change variables to $x = (\varepsilon - \mu)/kT$ so that

$$\varepsilon_{\mathrm{F}} = kT\int_{-\mu/kT}^\infty \frac{\mathrm{d}x}{e^{x}+1}.$$

The indefinite integral is straightforward:

$$\int \frac{\mathrm{d}x}{e^{x}+1} = -\ln(1+e^{-x}),$$

which can be checked by differentiation. Then the limits can be applied, giving

$$\int_{-\mu/kT}^\infty \frac{\mathrm{d}x}{e^{x}+1} = \ln\left\{1+e^{\mu/kT}\right\}$$

so that

$$\varepsilon_{\mathrm{F}} = kT\ln\left\{1+e^{\mu/kT}\right\}.$$

And this may be rearranged to give the chemical potential as

$$\mu = kT\ln\left\{e^{\varepsilon_{\mathrm{F}}/kT}-1\right\},$$

as required.

2.8 Calculate the low temperature chemical potential of a two-dimensional gas of fermions by the Sommerfeld expansion method of Section 2.5.3. Observe that the temperature series expansion terminates. Compare this result with the exact result of the previous question. Discuss the difference between the two results.

We start from the expression for the Fermi energy for a two-dimensional system, following from the expression for the total number of particles:

$$\varepsilon_{\mathrm{F}} = \int_0^\infty n(\varepsilon)\mathrm{d}\varepsilon.$$

This may be integrated by parts (the other factor of the integrand in this case is unity):

$$\varepsilon_{\mathrm{F}} = -\int_0^\infty \varepsilon n'(\varepsilon)\mathrm{d}\varepsilon.$$

We have to expand the ε factor about $\varepsilon = \mu$

$$\varepsilon = \mu + (\varepsilon - \mu)$$

so the Fermi energy is given by

$$\varepsilon_\mathrm{F} = -\mu \int_0^\infty n'(\varepsilon)\mathrm{d}\varepsilon - \int_0^\infty (\varepsilon - \mu)n'(\varepsilon)\mathrm{d}\varepsilon$$

with no higher order terms. The first integral is $-\mathscr{F}_0$ of Section 2.5.3 and the second integral is $-kT\mathscr{F}_1$. Now $\mathscr{F}_0 = 1$, which may be shown quite simply. And within the approximation of the Sommerfeld expansion, that the lower limit of the integral be extended to $-\infty$, $\mathscr{F}_1$ vanishes, as do all the odd-order terms. Thus the Sommerfeld expansion expression for the chemical potential is simply

$$\mu = \varepsilon_\mathrm{F};$$

the expansion terminates (rather early) and it completely misses the temperature dependence of μ away from $T = 0$.

The previous Problem showed in 2d the chemical potential may be expressed exactly

$$\mu = kT \ln\left\{e^{\varepsilon_\mathrm{r}/kT} - 1\right\}.$$

This can also be written as

$$\begin{aligned}\mu &= kT \ln\left\{e^{\varepsilon_\mathrm{F}/kT}\left(1 - e^{-\varepsilon_\mathrm{F}/kT}\right)\right\}\\ &= \varepsilon_\mathrm{F} + \ln\left\{1 - e^{-\varepsilon_\mathrm{F}/kT}\right\}\end{aligned}$$

which is more convenient at low temperatures. Then the logarithm term is the part missed by the Sommerfeld series. And at low temperatures the exponential is small and so the logarithm can be expanded in powers of this exponential

$$\ln\left\{1 - e^{-\varepsilon_\mathrm{F}/kT}\right\} = -e^{-\varepsilon_\mathrm{F}/kT} - \frac{1}{2}e^{-2\varepsilon_\mathrm{F}/kT} - \frac{1}{3}e^{-3\varepsilon_\mathrm{F}/kT} - \frac{1}{4}e^{-4\varepsilon_\mathrm{F}/kT} - \ldots$$

so that

$$\mu = \varepsilon_\mathrm{F} - e^{-\varepsilon_\mathrm{F}/kT} - \frac{1}{2}e^{-2\varepsilon_\mathrm{F}/kT} - \frac{1}{3}e^{-3\varepsilon_\mathrm{F}/kT} - \frac{1}{4}e^{-4\varepsilon_\mathrm{f}/kT} - \ldots$$

It is the exponential terms that are missed by the Sommerfeld expansion.

2.9 Obtain expressions for the chemical potential μ, the internal energy E and the heat capacity C_V for a system of fermions with general density of states $g(\varepsilon)$. That is, show that these are given in terms of the behaviour of the density of states at the Fermi surface.

These were obtained in Section 2.5.4. The steps of the derivations are given there. The results are collected:

$$\begin{aligned}
\mu &= \varepsilon_\mathrm{F} - \frac{\pi^2}{6}(kT)^2\frac{g'(\varepsilon_\mathrm{F})}{g(\varepsilon_\mathrm{F})} \\
E &= E_0 + \alpha\frac{\pi^2}{6}(kT)^2 g(\varepsilon_\mathrm{F}) \\
C_V &= \frac{1}{3}\alpha\pi^2 g(\varepsilon_\mathrm{F})k^2T.
\end{aligned}$$

2.10 Evaluate the Fermi temperature for liquid ^{3}He, assuming it to be a Fermi "gas". Its molar volume is 36 cm^3. Calculate the thermal de Broglie wavelength at $T = T_\mathrm{F}$ and show that it is comparable with the inter-particle spacing as expected.

The Fermi temperature is given by

$$\begin{aligned}
T_\mathrm{F} &= \frac{\hbar^2}{2mk}\left\{\frac{6\pi^2}{2}\frac{N}{V}\right\}^{2/3} \\
&= 4.785\frac{\hbar^2}{mk}\left\{\frac{N}{V}\right\}^{2/3}.
\end{aligned}$$

The atomic mass of ^{3}He is 3. And one atomic mass unit weighs 1.66×10^{-27}kg so a helium-3 atom weighs 4.98×10^{-27}kg.

The volume of one mole of liquid ^{3}He is 36 cm^3, or 36×10^{-6}m^3. Now N/V is Avogadro's number divided by the molar volume:

$$\begin{aligned}
\frac{N}{V} &= \frac{6.02 \times 10^{23}}{36 \times 10^{-6}} \\
&= 1.67 \times 10^{28}\,\mathrm{m}^{-3}
\end{aligned}$$

so that

$$\left(\frac{N}{V}\right)^{2/3} = 6.53 \times 10^{18}.$$

Then, using the values for Planck's constant and Boltzmann's constant, we obtain

$$\begin{aligned} T_{\mathrm{F}} &= 4.785 \times \frac{\left(1.05 \times 10^{-34}\right)^2}{4.98 \times 10^{-27} \times 1.38 \times 10^{-23}} \times 6.53 \times 10^{18} \\ &= 5.01\,\mathrm{K} \end{aligned}$$

Thus the Fermi temperature is calculated as 5.01 K.

The thermal de Broglie wavelength is

$$\Lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}$$

so at $T = T_{\mathrm{F}}$

$$\begin{aligned} \Lambda_{T_{\mathrm{F}}} &= \frac{2}{3^{1/3}\pi^{1/6}} \left(\frac{V}{N}\right)^{1/3} \\ &= 1.15 \left(\frac{V}{N}\right)^{1/3}. \end{aligned}$$

From above

$$\left(\frac{N}{V}\right)^{1/3} = 3.91 \times 10^{-10}\,\mathrm{m}$$

so

$$\Lambda_{T_{\mathrm{F}}} = 4.48 \times 10^{-10}\,\mathrm{m}.$$

The inter-particle spacing is $(N/V)^{1/3} = 3.91 \times 10^{-10}\,\mathrm{m}$. We see it is indeed comparable with $\Lambda_{T_{\mathrm{F}}}$.

2.11 In Problem 2.1 we found the expression for the energy density of states $g(\varepsilon)$ for a gas of fermions confined to two dimensions and we saw that it was independent of energy. What surface density of electrons is necessary in order that $T_{\mathrm{F}} = 100\,\mathrm{mK}$? Show that, for a given area, the heat capacity is independent of the number of electrons.

At zero temperature, where the Fermi distribution becomes a step function, the number of particles in the system may be expressed as

$$N = \alpha \int_0^{\varepsilon_{\mathrm{F}}} g(\varepsilon) \mathrm{d}\varepsilon$$

and since in 2d the density of states is

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2},$$

independent of energy, N is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2}\varepsilon_\mathrm{F}$$

so that

$$\varepsilon_\mathrm{F} = \frac{2\pi\hbar^2}{\alpha m}\frac{N}{A},$$

and the Fermi temperature $T_\mathrm{F} = \varepsilon_\mathrm{F}/k$ is

$$T_\mathrm{F} = \frac{2\pi\hbar^2}{\alpha mk}\frac{N}{A}.$$

The areal density corresponding to a given Fermi temperature is

$$\frac{N}{A} = \frac{\alpha mkT_\mathrm{F}}{2\pi\hbar^2}.$$

For electrons we have $\alpha = 2, m = 9.11 \times 10^{-31}$kg, so for a Fermi temperature of 0.1K we have

$$\begin{aligned}\frac{N}{A} &= \frac{2 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 0.1}{2\pi \times (1.06 \times 10^{-34})^2} \\ &= 3.56 \times 10^{13}\,\mathrm{m}^{-2}.\end{aligned}$$

So the number density is 3.56×10^{13} electrons per square metre.

The heat capacity is found by differentiating the internal energy.

The simple treatment of the next part is to use the argument that at low temperatures only a fraction T/T_F of the electrons will be excited so that the internal energy is essentially this fraction of the classical (equipartition) internal energy NkT in two dimensions. Thus

$$E \sim Nk\frac{T^2}{T_\mathrm{F}}$$

and so, upon differentiation,

$$C_V \sim Nk\frac{T}{T_\mathrm{F}}.$$

Thus in the low-temperature limit the heat capacity is linear in T, as in the 3d case.

The next point is that in 2d the Fermi temperature is proportional to N (over A)

$$T_{\mathrm{F}} = \frac{2\pi\hbar^2}{\alpha m k}\frac{N}{A}$$

so this N cancels with that in the expression for C_V, making the heat capacity independent of N:

$$C_V \sim \frac{\alpha m k^2 A}{2\pi\hbar^2}T$$

and thus we see that the low-temperature heat capacity of a 2d electron gas in a given area is independent of the number of electrons.

A better treatment of the low-temperature heat capacity would use the Sommerfeld expansion. This gives the internal energy as

$$E = E_0 + \frac{\pi^2}{6}N\frac{kT^2}{T_{\mathrm{F}}}$$

and the heat capacity is then

$$C_V = \frac{\pi^2}{6}Nk\frac{T}{T_{\mathrm{F}}}$$

This is the correct low temperature limiting behaviour, but as we saw in Problem 2.8, the Sommerfeld method is incapable of treating the higher-order terms in 2d (the low-order terms are OK; the series is fine *until* it terminates). Observe this result demonstrated that the previous approximation gives the correct result to within a numerical correction factor. Then that argument about the N from the Fermi temperature cancelling still applies and again we find that the heat capacity will be independent of N.

2.12 Use the Sommerfeld expansion method of Section 2.5.3 to show that the Fermi-Dirac distribution function may be approximated, at low temperatures, by

$$\frac{1}{e^{(\varepsilon-\mu)kT}+1} \sim \Theta(\mu-\varepsilon) - \frac{\pi^2}{6}(kT)^2\,\delta'(\varepsilon-\mu) + \cdots .$$

where Θ is the unit step function and δ' is the first derivative of the Dirac delta function.

Can you write down the general term of the series?

According to the results of Section 2.5.3 integrals of the form

$$I = \int_0^\infty \frac{\varphi(\varepsilon)}{e^{(\varepsilon-\mu)/kT}+1}\mathrm{d}\varepsilon$$

where $\varphi(\varepsilon)$ is an arbitrary function, are approximated by

$$I = \sum_{n=0}^{\infty} \mathscr{F}_n \frac{(kT)^n}{n!} \frac{\mathrm{d}^n\psi}{\mathrm{d}\varepsilon^n}\bigg|_{\varepsilon=\mu} .$$

Here ψ is the integral of φ :

$$\psi(\varepsilon) = \int \varphi(\varepsilon)\mathrm{d}\varepsilon.$$

We have the expression for the Sommerfeld expansion as

$$\int_0^\infty \frac{\varphi(\varepsilon)}{e^{(\varepsilon-\mu)/kT}+1}\mathrm{d}\varepsilon = \sum_{n=0}^{\infty} \mathscr{F}_n \frac{(kT)^n}{n!} \frac{\mathrm{d}^n\psi}{\mathrm{d}\varepsilon^n}\bigg|_{\varepsilon=\mu}$$

and we can write this out term by term

$$\int_0^\infty \frac{\varphi(\varepsilon)}{e^{(\varepsilon-\mu)/kT}+1}\mathrm{d}\varepsilon = \int_0^\mu \varphi(\varepsilon)\mathrm{d}\varepsilon + \frac{(kT)^2}{2}\mathscr{F}_2\frac{d\varphi}{d\varepsilon} + \frac{(kT)^4}{4!}\mathscr{F}_4\frac{d^3\varphi}{d\varepsilon^3} + \dots$$
$$\dots + \frac{(kT)^n}{n!}\mathscr{F}_n\frac{d^{n-1}\varphi}{d\varepsilon^{n-1}} + \dots$$

The right hand side could be written as

$$\int_0^\infty \Theta(\mu-\varepsilon)\varphi(\varepsilon)\mathrm{d}\varepsilon + \frac{(kT)^2}{2}\mathscr{F}_2\int_0^\infty \delta'(\varepsilon-\mu)\varphi(\varepsilon)\mathrm{d}\varepsilon +$$
$$+ \frac{(kT)^4}{4!}\mathscr{F}_4\int_0^\infty \delta'''(\varepsilon-\mu)\varphi(\varepsilon)\mathrm{d}\varepsilon + \dots$$
$$\dots + \frac{(kT)^n}{n!}\mathscr{F}_n\int_0^\infty \delta^{(n-1)}(\varepsilon-\mu)\varphi(\varepsilon)\mathrm{d}\varepsilon + \dots$$

where Θ is the step function and $\delta^{(n)}$ is the nth derivative of the delta function. Then since this result holds for arbitrary functions $\varphi(\varepsilon)$ we can equate the insides of the integral to give

$$\frac{1}{e^{(\varepsilon-\mu)/kT}+1} = \Theta(\mu-\varepsilon) + \frac{(kT)^2}{2}\mathscr{F}_2\delta'(\varepsilon-\mu) + \frac{(kT)^4}{4!}\mathscr{F}_4\delta'''(\varepsilon-\mu) + \dots +$$
$$+ \frac{(kT)^n}{n!}\mathscr{F}_n\delta^{(n-1)}(\varepsilon-\mu) + \dots .$$

If we substitute for the values of $\mathscr{F}_n$ in the first few terms we obtain

$$\frac{1}{e^{(\varepsilon-\mu)/kT}+1}=\Theta(\mu-\varepsilon)+\frac{\pi^2}{6}(kT)^2\delta'(\varepsilon-\mu)+\frac{7\pi^4}{360}(kT)^4\delta'''(\varepsilon-\mu)+\dots$$

as required. And the general term of the expansion is given by

$$\frac{I_n}{n!}(kT)^n\delta^{(n-1)}(\varepsilon-\mu)=\left(2-2^{2-n}\right)\zeta(n)(kT)^n\delta^{(n-1)}(\varepsilon-\mu).$$

Observe that the first term of the expansion, the step function, is the zero-temperature distribution function. The higher order terms give the finite-temperature corrections to this.

2.13 Liquid ^{4}He has a molar volume of 27 cm^3 at saturated vapour pressure. Treating the liquid as an ideal gas of bosons, find the temperature at which Bose-Einstein condensation will occur. How will this temperature change as the pressure on the fluid is increased?

The superfluid transition temperature of liquid helium decreases with increasing pressure; very approximately $\partial T_\mathrm{c}/\partial p \sim -0.015\,\mathrm{K\,bar}^{-1}$. How does this compare with the behaviour predicted for ideal gas Bose-Einstein condensation?

The condensation temperature is given, from Eq. (2.6.23), by

$$T_\mathrm{B}=3.313\frac{\hbar^2}{mk}\left\{\frac{N}{V}\right\}^{2/3}.$$

The atomic mass of helium is 4. And one atomic mass unit weighs 1.66×10^{-27} kg so a helium atom weighs 6.64×10^{-27} kg.

The volume of one mole of liquid helium is 27 cm^3, or 27×10^{-6}m^3. Now N/V is the Avogadro constant divided by the molar volume:

$$\begin{aligned}\frac{N}{V}&=\frac{6.02\times10^{23}}{27\times10^{-6}}\\&=2.23\times10^{28}.\end{aligned}$$

We need the ⅔ power of this:

$$\left(\frac{N}{V}\right)^{2/3}=7.92\times10^{18}.$$

Then, using the values for Planck's constant and Boltzmann's constant, we obtain

$$\begin{aligned} T_\mathrm{B} &= 3.313\frac{(1.05\times 10^{-34})^2}{6.64\times 10^{-27}\times 1.38\times 10^{-23}}\times 7.92\times 10^{18} \\ &= 3.15\,\mathrm{K}. \end{aligned}$$

The temperature at which Bose-Einstein condensation occurs is 3.15 K.

Since the formula for T_B indicates that it increases with density, N/V, and since increasing the pressure will increase the density, it follows that increasing the pressure will *increase* the Bose-Einstein condensation temperature.

By contrast the superfluid transition temperature of liquid helium *decreases* with increasing pressure; the observed reduction in T_B is in conflict with the BEC model.

2.14 Consider the Bose gas at low temperatures. You saw in Section 2.6.3 and 2.6.4 that when the occupation of the ground state is appreciable then the chemical potential μ is very small and it may be ignored, compared with ε in the integral for the number of excited states.

(a) Show that when the ground state occupation N_0 is appreciable then μ may be approximated by

$$\mu \sim -kT/N_0.$$

(b) Now consider the requirement that μ may be neglected in comparison with ε in the integral for the number of excited states. This will be satisfied if $|\mu|$ is much less than the energy ε_1 of the first excited state. The expression for ε_1 is

$$\varepsilon_1 = \frac{\pi^2\hbar^2}{2mV^{2/3}}.$$

Where does this expression come from?

(c) Show that the condition $|\mu| \ll \varepsilon_1$ is satisfied right up to T_B in the thermodynamic limit. Hint: demonstrate that $|\mu|/\varepsilon_1 \sim N^{-1/3}$.

(d) For a finite system comprising 1 cm^3 of ^{4}He (molar volume 27 cm^3) show that the expression

$$N_0 = N\left\{1-\left(\frac{T}{T_\mathrm{B}}\right)^{3/2}\right\}$$

is valid to temperatures below T_B right up to within $\sim 10^{-8}T_\mathrm{B}$ of the Bose temperature.

(a) The ground state occupation ($\varepsilon = 0$) is, Eq. (2.6.6),

$$N_0 = \frac{1}{e^{-\mu/kT} - 1}.$$

If the ground state occupation is significant then the denominator will be small, which means that the exponential will differ only slightly from unity. Then μ must be very small (and negative) so that

$$N_0 = \frac{1}{1 - \mu/kT - \cdots - 1} \sim -kT/\mu \tag{2.11.1}$$

or

$$\mu \sim -kT/N_0 \tag{2.11.2}$$

as required.

(b) The energy of a general quantum state, with quantum numbers n_x, n_y, n_z, is given by Eq. (2.1.8)

$$\varepsilon = \frac{\pi^2\hbar^2}{2mV^{2/3}}\left(n_x^2 + n_y^2 + n_z^2\right).$$

The ground state corresponds to $n_x = n_y = n_z = 0$; so $\varepsilon_0 = 0$. The first excited state has *one* $n_x, n_y, n_z = 1$. And so the energy of the first excited state is

$$\varepsilon_1 = \frac{\pi^2\hbar^2}{2mV^{2/3}}$$

as required.

(c) It is convenient to express ε_1 in terms of $T_\mathrm{B} = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{V}\frac{1}{\zeta(\frac{3}{2})}\right)^{2/3}$:

$$\varepsilon_1 = kT_\mathrm{B} \times \frac{\pi}{4}\left(\frac{\zeta(\frac{3}{2})}{N}\right)^{2/3}$$

so the ratio $|\mu|/\varepsilon_1$ is

$$\begin{aligned}\frac{|\mu|}{\varepsilon_1} &= \frac{T}{T_\mathrm{B}}\frac{N}{N_0}\frac{1}{N^{1/3}} \times \frac{4}{\pi}\zeta(\tfrac{3}{2})^{-2/3} \\ &= \frac{T}{T_\mathrm{B}}\frac{N}{N_0}\frac{1}{N^{1/3}} \times 0.671\,.\end{aligned} \tag{2.11.3}$$

Now in the vicinity of T_B we have $T/T_\mathrm{B} \sim 1$, $N/N_0 \sim 1$ and so

$$\frac{|\mu|}{\varepsilon_1} \sim N^{-1/3}$$

so that in the thermodynamic limit $(N \to \infty)$ we have

$$|\mu| \ll \varepsilon_1$$

and then μ may be neglected in comparison with ε in the Bose-Einstein integrals.

We have shown that this condition is satisfied *right up to* T_B in the thermodynamic limit. The key point is the $N^{1/3}$ factor, which will go to infinity in the thermodynamic limit.

To sharpen this conclusion we note that $|\mu| = \varepsilon_1$ indicates the temperature above which the condition breaks down. Call this $\tilde{T}$. Then, from Eq. (2.11.3)

$$\frac{\tilde{T}}{T_\mathrm{B}} \frac{N}{N_0} \frac{1}{N^{1/3}} \times 0.671 = 1. \qquad (2.11.4)$$

At this temperature

$$\frac{N}{N_0} = \left\{ 1 - \left(\frac{\tilde{T}}{T_\mathrm{B}} \right)^{3/2} \right\}^{-1}.$$

Now when $\tilde{T}$ will be close to T_B then we can do an expansion in powers of the temperature deviation $T_\mathrm{B} - \tilde{T}$:

$$\frac{\tilde{T}}{T_\mathrm{B}} \frac{N}{N_0} = \frac{\frac{\tilde{T}}{T_\mathrm{B}}}{1 - \left(\frac{\tilde{T}}{T_\mathrm{B}} \right)^{3/2}} = \frac{2}{3(1 - \frac{\tilde{T}}{T_\mathrm{B}})} - \frac{1}{2} - \frac{7(1 - \frac{\tilde{T}}{T_\mathrm{B}})}{72} + \dots .$$

We shall take the leading order term. Then $\tilde{T}$ satisfies

$$\frac{2}{3(1 - \frac{\tilde{T}}{T_\mathrm{B}})} \frac{1}{N^{1/3}} \times 0.671 = 1$$

or

$$\begin{aligned} 1 - \frac{\tilde{T}}{T_\mathrm{B}} &= \frac{2 \times 0.671}{3} \times \frac{1}{N^{1/3}} \\ &= \frac{0.447}{N^{1/3}}, \end{aligned} \qquad (2.11.5)$$

or

$$\tilde{T} = \left(1 - \frac{0.447}{N^{1/3}}\right) T_{\mathrm{B}}\,.$$

So again we see that when $N \to \infty$ then $\tilde{T}$ will approach T_{B}.

(d) If the molar volume of ^{4}He is $27\,\mathrm{cm}^3$ and we have $1\,\mathrm{cm}^3$ then this specimen will have $N = N_{\mathrm{A}}/27 = 2.23 \times 10^{22}$ particles. We require $N^{1/3}$: $N^{1/3} = 2.81 \times 10^7$. Substitute this into Eq. (2.11.5), to obtain

$$\begin{aligned}\left(1 - \frac{\tilde{T}}{T_{\mathrm{B}}}\right) &= \frac{0.447}{N^{1/3}} \\ &= 1.59 \times 10^{-8}.\end{aligned} \tag{2.11.6}$$

So the expression for N_0 will not break down until T is within $\sim 10^{-8} T_{\mathrm{B}}$ of T_{B}.

2.15 (a) Show that the pressure of a Bose gas below T_{B} depends *only* on temperature; that is, it is independent of other thermodynamic variables such as N and V.

(b) Comment on this result in the context of the Gibbs-Duhem relation (asserting that the intensive variables T, p, μ are not independent).

(a) Below T_{B} the internal energy is (Eq. (2.6.36))

$$E = \frac{3}{2} Nk \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} \frac{T^{5/2}}{T_{\mathrm{B}}^{3/2}}.$$

We know that for a free gas $pV = \frac{2}{3}E$ and so

$$p = \frac{N}{V} k \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} \frac{T^{5/2}}{T_{\mathrm{B}}^{3/2}}$$

But T_{B} depends on N/V; from Eq. (2.6.20),

$$T_{\mathrm{B}}^{3/2} = \left(\frac{2\pi\hbar^2}{mk}\right)^{3/2} \frac{N}{V} \frac{1}{\zeta(\frac{3}{2})}$$

so that N and V cancel, giving

$$p = \zeta(\tfrac{5}{2}) \left(\frac{mk}{2\pi\hbar^2}\right)^{3/2} kT^{5/2}.$$

Thus p depends only on the thermodynamic variable T.

(b) The Gibbs-Duhem relation tells us that the intensive variables T, p, μ are not independent so that, for example, we may write p as a function of T and μ only. But below T_c, $\mu = 0$. So in this case the Gibbs-Duhem relation implies that p is a function of T alone.

2.16 (a) Show that below the transition temperature the entropy of a Bose gas is given by

$$S = \frac{5}{2}Nk\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(\frac{T}{T_\mathrm{B}}\right)^{3/2}.$$

(b) Since the number of excited particles is given by

$$N_\mathrm{ex} = N\left(\frac{T}{T_\mathrm{B}}\right)^{3/2},$$

show that the entropy per excited particle is given by

$$\frac{S}{N_\mathrm{ex}} = \frac{5}{2}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}k \approx 1.28\,k.$$

(c) Discuss the connection between this result and the two fluid model of superfluid ^{4}He.

(a) The entropy is most conveniently found from the heat capacity C_V. Since

$$C_V = T\frac{\partial S}{\partial T}\bigg|_V$$

it follows that

$$S = \int \frac{C_V}{T}\,\mathrm{d}T.$$

And since the heat capacity is given from Eq. (2.6.37) as

$$C_V = \frac{15}{4}Nk\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(\frac{T}{T_\mathrm{B}}\right)^{3/2},$$

upon integration we find

$$S = \frac{5}{2}Nk\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\left(\frac{T}{T_\mathrm{B}}\right)^{3/2}$$

as required.

(b) Now the number of particles out of the ground state, the number of excited particles, is

$$N_{\mathrm{ex}} = N\left(\frac{T}{T_{\mathrm{B}}}\right)^{3/2},$$

so dividing the total entropy we obtain the entropy per excited particle as

$$\frac{S}{N_{\mathrm{ex}}} = \frac{5}{2}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}\,k \approx 1.28\,k.$$

(c) This indicates that the entropy per *excited* particle is a constant, independent of temperature. Now the two fluid model argues that the system is comprised of two interpenetrating fluids – the "normal" component and the "superfluid" component. And it asserts that the superfluid component carries no entropy. Here we see that the entropy may indeed all be associated with the "normal" or excited component, each excited particle carrying the same "unit" of entropy.

2.17 Show that the chemical potential of a two-dimensional gas of bosons may be expressed analytically as

$$\mu = kT\ln\left\{1 - e^{-\varepsilon_{\mathrm{q}}/kT}\right\}$$

where ε_{q} is the two dimensional quantum energy, the analogue of the 2d Fermi energy in Problem 2.6

This problem is the Bose analogue of the Fermi case treated in Problem 2.7.

At finite temperatures the number of particles is given by

$$N = \frac{\alpha mA}{2\pi\hbar^2}\int_0^\infty \frac{\mathrm{d}\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}$$

and since

$$\varepsilon_{\mathrm{q}} = \frac{2\pi\hbar^2}{\alpha m}\frac{N}{A}$$

the integral for N may be expressed

$$\varepsilon_{\mathrm{q}} = \int_0^\infty \frac{\mathrm{d}\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}.$$

We shall change variables to $x = (\varepsilon - \mu)/kT$ so that

$$\varepsilon_{\mathrm{q}} = kT \int_{-\mu/kT}^{\infty} \frac{\mathrm{d}x}{e^{x}-1}.$$

The indefinite integral is straightforward:

$$\int \frac{\mathrm{d}x}{e^{x}-1} = \ln(1-e^{-x}),$$

which can be checked by differentiation. Then the limits can be applied, giving

$$\int_{-\mu/kT}^{\infty} \frac{\mathrm{d}x}{e^{x}-1} = -\ln\left\{1-e^{\mu/kT}\right\}$$

so that

$$\varepsilon_{\mathrm{q}} = -kT\ln\left\{1-e^{\mu/kT}\right\}.$$

And this may be rearranged to give the chemical potential as

$$\mu = kT\ln\left\{1-e^{-\varepsilon_{\mathrm{q}}/kT}\right\},$$

as required.

2.18 Show that the Bose-Einstein transition temperature of a gas of bosons and the Fermi temperature for a gas of “similar” fermions are of comparable magnitude. Discuss why this should be.

The Bose-Einstein transition temperature for a gas of spin 0 bosons $(\alpha = 1)$ is given by

$$\begin{aligned} T_{\mathrm{B}} &= \frac{2\pi\hbar^{2}}{mk}\left\{\frac{N}{2.612V}\right\}^{2/3} \\ &= 3.313\frac{\hbar^{2}}{mk}\left\{\frac{N}{V}\right\}^{2/3}. \end{aligned}$$

The Fermi temperature of a gas of spin ½ fermions $(\alpha = 2), \varepsilon_{\mathrm{F}}/k$, is given by

$$\begin{aligned} T_{\mathrm{F}} &= \frac{\hbar^{2}}{2mk}\left\{\frac{6\pi^{2}}{2}\frac{N}{V}\right\}^{2/3} \\ &= 4.785\frac{\hbar^{2}}{mk}\left\{\frac{N}{V}\right\}^{2/3}. \end{aligned}$$

Thus the Bose condensation temperature and the Fermi temperature are very similar. They certainly depend on all the system variables in the same way and the the Fermi temperature of a gas of spin ½ fermions is approximately 1.4 times the Bose-Einstein transition temperature for a gas of spin 0 bosons.

The reason they are similar is because both temperatures are characterised by the thermal deBroglie wavelength for the particles becoming comparable with the interparticle spacing. This is the condition for quantum effects to become important.

2.19 The formula for the Bose integrals $\mathscr{B}_n$ of Sec. 2.6.4 was quoted as

$$\begin{aligned}\mathscr{B}_n &= \int_0^\infty \frac{x^n}{e^x-1}\,\mathrm{d}x \\ &= \Gamma(n+1)\zeta(n+1).\end{aligned}$$

Derive this result.

Hint: Use the sequence of transformations below

$$\frac{1}{e^x-1} = e^{-x}\frac{1}{1-e^{-x}} = e^{-x}\sum_{m=0}^{\infty} e^{-mx} = \sum_{m=0}^{\infty} e^{-(m+1)x} = \sum_{m=1}^{\infty} e^{-mx}$$

and then change the variable of integration to $y = mx$.

Performing the above transformations we obtain

$$\mathscr{B}_n = \int_0^\infty \frac{x^n}{e^x-1}\,\mathrm{d}x = \sum_{m=1}^{\infty}\int_0^\infty x^n e^{-mx}\mathrm{d}x.$$

Now do the suggested change of variables: $y = mx$, so that

$$\mathscr{B}_n = \sum_{m=1}^{\infty}\frac{1}{m^{n+1}}\int_0^\infty y^n e^{-y}\mathrm{d}y.$$

Observe this has factorized into a sum and an integral. The sum is a Reimann zeta function

$$\sum_{m=1}^{\infty}\frac{1}{m^{n+1}} = \zeta(n+1)$$

and the integral is a gamma function

$$\int_0^\infty y^n e^{-y}\mathrm{d}y = \Gamma(n+1)$$

so that

$$\mathscr{B}_n = \Gamma(n+1)\zeta(n+1)$$

as required.

2.20 The general formula for the Fermi integrals $\mathscr{F}_n$ of Section 2.5.3 was quoted as

$$\begin{aligned}\mathscr{F}_n &= \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2}\, x^n \,\mathrm{d}x \\ &= 2\left(1-2^{1-n}\right)\zeta(n)\, n!\end{aligned}$$

where $\zeta(n)$ is the Reimann zeta function. Derive this result. This is difficult; *Mathematica* v.12 can't do it!

Hint: The first step is to integrate by parts, essentially the reverse of Eq. (2.5.28), to obtain

$$\mathscr{F}_n = 2n \int_0^{\infty} \frac{x^{n-1}}{e^x+1}\,\mathrm{d}x,$$

noting that this holds for even n, while $\mathscr{F}_n = 0$ for odd n. Next, follow a sequence of transformations *similar* to those for the Fermi calculation of Problem 2.19.

Since $e^x/(e^x+1)^2$ is an even function of x the integral is non-zero only for even n. Then we can put the lower limit of the integral to zero and double it:

$$\mathscr{F}_n = 2\int_0^{\infty} \frac{e^x}{(e^x+1)^2}\, x^n \,\mathrm{d}x.$$

As suggested, we integrate this by parts so that

$$\mathscr{F}_n = 2\left\{ \left.\frac{-x^n}{e^x+1}\right|_0^{\infty} + \int_0^{\infty} \frac{n x^{n-1}}{e^x+1}\,\mathrm{d}x \right\}.$$

The first term vanishes, so that

$$\mathscr{F}_n = 2n \int_0^{\infty} \frac{x^{n-1}}{e^x+1}\,\mathrm{d}x,$$

rather similar to the Fermi case. The corresponding sequence of transformations is

$$\begin{aligned}\frac{1}{e^x+1} &= e^{-x}\frac{1}{1+e^{-x}} = e^{-x}\sum_{m=0}^{\infty}(-1)^m e^{-mx} = \\ &= \sum_{m=0}^{\infty}(-1)^m e^{-(m+1)x} = \sum_{m=1}^{\infty}(-1)^{m-1} e^{-mx}\end{aligned}$$

so that

$$\mathscr{F}_n = 2n \int_0^\infty \frac{x^{n-1}}{e^x+1}\mathrm{d}x = 2n \sum_{m=1}^\infty (-1)^{m-1} \int_0^\infty x^{n-1} e^{-mx}\mathrm{d}x.$$

Now do the suggested change of variables: $y = mx$, so that

$$\mathscr{F}_n = 2n \sum_{m=1}^\infty \frac{(-1)^{m-1}}{m^n} \int_0^\infty y^{n-1} e^{-y}\mathrm{d}y.$$

Observe this has factorized into a sum and an integral. The integral is a gamma function

$$\int_0^\infty y^{n-1} e^{-y}\mathrm{d}y = \Gamma(n).$$

The sum, the Dirichlet eta function $\eta(z)$, is related to the Reimann zeta function through (see Problem 2.21 below):

$$\eta(n) = \sum_{m=1}^\infty \frac{(-1)^{(m-1)}}{m^n} = (1-2^{1-n})\zeta(n).$$

So we obtain

$$\mathscr{F}_n = 2n\Gamma(n)(1-2^{1-n})\zeta(n).$$

Since n is an integer, $n\Gamma(n) = n!$, giving

$$\mathscr{F}_n = 2(1-2^{1-n})\zeta(n)n!.$$

as required.

2.21 The Reimann zeta function and the Dirichlet eta function are defines as infinite sums:

$$\zeta(n) = \sum_{m=1}^\infty \frac{1}{m^n}, \quad \eta(n) = \sum_{m=1}^\infty \frac{(-1)^{m-1}}{m^n}.$$

Show that $\zeta(n)-\eta(n) = 2^{1-n}\zeta(n)$, and hence that $\eta(n) = (1-2^{1-n})\zeta(n)$.

We have

$$\begin{aligned}\zeta(n)-\eta(n) &= \sum_{m=1}^\infty \frac{1}{m^n} - \sum_{m=1}^\infty \frac{(-1)^{m-1}}{m^n}\\ &= \sum_{m=1}^\infty \frac{1}{m^n} + \sum_{m=1}^\infty \frac{(-1)^{m}}{m^n}\\ &= \sum_{m=1}^\infty \left(\frac{1}{m^n} + \frac{(-1)^{m}}{m^n}\right)\end{aligned}$$

Here the odd terms cancel so that

$$\begin{aligned}\zeta(n) - \eta(n) &= 2\sum_{m=1}^{\infty}\frac{1}{(2m)^n} \\ &= 2^{1-n}\sum_{m=1}^{\infty}\frac{1}{m^n} \\ &= 2^{1-n}\zeta(n)\end{aligned}$$

as required.

Then from the result

$$\zeta(n) - \eta(n) = 2^{1-n}\zeta(n)$$

we can re-arrange, to obtain the required relation

$$\eta(n) = (1 - 2^{1-n})\zeta(n).$$

2.22 A dilute gas of 10^6 rubidium atoms (relative atomic mass 86.9) is confined to an isotropic harmonic trap of frequency 150 Hz.

(a) Calculate the Bose-Einstein condensation (BEC) temperature T_{B}.

(b) Estimate the width of the thermal cloud at T_{B}.

(c) Estimate the width of the condensate cloud.

(a) T_{B} is given in Eq. (2.8.20)

$$T_{\mathrm{B}} = 0.940\frac{hf}{k}N^{1/3}$$

(writing hf instead of $\hbar\omega$). We have $f = 150\,\mathrm{Hz}$ and $N = 10^6$, or $N^{1/3} = 100$. So

$$\begin{aligned}T_{\mathrm{B}} &= \frac{0.940 \times 6.63 \times 10^{-34} \times 150 \times 100}{1.38 \times 10^{-23}} \\ &= 6.77 \times 10^{-7}\,\mathrm{K}\,.\end{aligned}$$

(b) The mean square extent of the thermal cloud was given in Eq. (2.8.29)

$$\langle x^2\rangle_T = \frac{kT}{m\omega^2}.$$

The mass of an atom is the relative atomic mass (86.9) multiplied by the atomic mass unit $m_\mathrm{u} = 1.66 \times 10^{-27}\,\mathrm{kg}$. So then at $T = T_\mathrm{B}$

$$\begin{aligned}\langle x^2\rangle_T &= \frac{1.38 \times 10^{-23} \times 6.77 \times 10^{-7}}{86.9 \times 1.66 \times 10^{-27} \times (2\pi \times 150)^2} \\ &= 7.29 \times 10^{-11}\,\mathrm{m}^2\end{aligned}$$

The "width" of the thermal cloud is estimated from the square root of this, thus

$$\sqrt{\langle x^2\rangle_T} = 8.54 \times 10^{-6}\,\mathrm{m}.$$

(c) The mean square width of the condensate cloud was given by Eq. (2.8.32)

$$\langle x^2\rangle_0 = \frac{\hbar}{m\omega}$$

so that

$$\begin{aligned}\langle x^2\rangle_0 &= \frac{1.05 \times 10^{-34}}{86.9 \times 1.66 \times 10^{-27} \times 2\pi \times 150} \\ &= 7.72 \times 10^{-13}\,\mathrm{m}^2.\end{aligned}$$

The "width" of the condensate cloud is estimated from the square root of this, thus

$$\sqrt{\langle x^2\rangle_0} = 8.79 \times 10^{-7}\,\mathrm{m}.$$

2.23 This is the trapped-gas analogue of Problem 2.14. Consider a Bose gas confined to an isotropic harmonic trap at low temperatures, as in the previous Problem. You know that when the occupation of the ground state is appreciable then the chemical potential μ is very small and it may be ignored, compared with ε in the integral for the number of excited states. Moreover you know that when the ground state occupation N_0 is appreciable, then $\mu \sim -kT/N_0$.

(a) The requirement that μ may be neglected in comparison with ε in the integral for the number of excited states will be satisfied if $|\mu|$ is much less than the energy ε_1 of the first excited state. The expression for ε_1 is

$$\varepsilon_1 = \hbar\omega$$

where ω is the (angular) frequency of the trap. Where does this expression come from?

(b) Show that the condition $|\mu| \ll \varepsilon_1$ is satisfied right up to T_B in the thermodynamic limit. Hint: demonstrate that $|\mu|/\varepsilon_1 \sim N^{-2/3}$.

(c) For a finite system, as in the previous Problem, comprising 10^6 atoms, show that the expression

$$N_0 = N\left\{1 - \left(\frac{T}{T_\mathrm{B}}\right)^3\right\}$$

is valid to temperatures below T_B right up to within $\sim 10^{-5}T_\mathrm{B}$ of the Bose temperature.

(a) The energy of a general quantum state, with quantum numbers n_x, n_y, n_z, is given by Eq. (2.8.3)

$$\varepsilon = (n_x + n_y + n_z)\hbar\omega\,.$$

The first excited state has *one* $n_x, n_y, n_z = 1$. And so the energy of the first excited state is

$$\varepsilon_1 = \hbar\omega$$

as required.

(b) It is convenient to express ε_1 in terms of $T_\mathrm{B} = \frac{\hbar\omega}{k}(\frac{N}{\zeta(3)})^{1/3}$:

$$\varepsilon_1 = kT_\mathrm{B}\left(\frac{\zeta(3)}{N}\right)^{1/3}$$

so the ratio $|\mu|/\varepsilon_1$ is

$$\begin{aligned}\frac{|\mu|}{\varepsilon_1} &= \frac{T}{T_\mathrm{B}}\frac{N}{N_0}\frac{1}{N^{2/3}}\frac{1}{\zeta(3)^{1/3}} \\ &= \frac{T}{T_\mathrm{B}}\frac{N}{N_0}\frac{1}{N^{2/3}} \times 0.940\,.\end{aligned} \tag{2.11.7}$$

The key point is that

$$\frac{|\mu|}{\varepsilon_1} \sim N^{-2/3}$$

so that in the thermodynamic limit ($N \to \infty$) we have

$$|\mu| \ll \varepsilon_1$$

and then μ may be neglected in comparison with ε in the Bose-Einstein integrals *right up to* T_B

To sharpen this conclusion we note that $|\mu| = \varepsilon_1$ indicates the temperature above which the condition breaks down. Call this $\tilde{T}$. Then from Eq. (2.11.7)

$$\frac{\tilde{T}}{T_{\mathrm{B}}}\frac{N}{N_0}\frac{1}{N^{2/3}} \times 0.940 = 1.$$

At this temperature

$$\frac{N}{N_0} = \left\{1 - \left(\frac{\tilde{T}}{T_{\mathrm{B}}}\right)^3\right\}^{-1}.$$

Now when $\tilde{T}$ will be close to T_{B} then we can do an expansion in powers of the temperature deviation $T_{\mathrm{B}} - \tilde{T}$:

$$\frac{\tilde{T}}{T_{\mathrm{B}}}\frac{N}{N_0} = \frac{\frac{\tilde{T}}{T_{\mathrm{B}}}}{1 - \left(\frac{\tilde{T}}{T_{\mathrm{B}}}\right)^3} = \frac{1}{3(1 - \frac{\tilde{T}}{T_{\mathrm{B}}})} - \frac{1}{9}(1 - \tfrac{\tilde{T}}{T_{\mathrm{B}}}) + \dots .$$

We shall take the leading order term. Then $\tilde{T}$ satisfies

$$\frac{1}{3(1 - \frac{\tilde{T}}{T_{\mathrm{B}}})}\frac{1}{N^{2/3}} \times 0.940 = 1$$

or

$$\begin{aligned} 1 - \frac{\tilde{T}}{T_{\mathrm{B}}} &= \frac{0.940}{3} \times \frac{1}{N^{2/23}} \\ &= \frac{0.314}{N^{2/3}}, \end{aligned} \tag{2.11.8}$$

or

$$\tilde{T} = \left(1 - \frac{0.940}{N^{2/3}}\right) T_{\mathrm{B}}.$$

So again we see that when $N \to \infty$ then $\tilde{T}$ will approach T_{B}.

(c) The finite system of the previous problem had $N = 10^6$. Substitute this into Eq. (2.11.8)

$$\begin{aligned} 1 - \frac{\tilde{T}}{T_{\mathrm{B}}} &= \frac{0.314}{N^{2/3}} \\ &= 3.14 \times 10^{-5}. \end{aligned}$$

So the expression for N_0 will not break down until T is within $\sim 10^{-5} T_{\mathrm{B}}$ of T_{B}.

2.24 In Section 2.10 we studied a paramagnetic *solid*: a collection of essentially *distinguishable* magnetic moments. If we were to consider a (classical) gas of indistinguishable magnetic moments, how would the partition function be modified? What would be the observable consequences of this modification?

A full treatment of a gas of particles would require consideration of the particles' kinetic energy. However if we neglect this then the only modification to the partition function would be the inclusion of the $N!$ term for indistinguishability. Then we would have

$$Z = \frac{\{2\cosh \varepsilon/kT\}^N}{N!}$$

and since $F = -kT \ln Z$ the free energy is

$$F = -NkT \ln\{2\cosh \varepsilon/kT\} + NkT \ln N - NkT.$$

The first term is the free energy calculated for the distinguishable solid case and the second and third terms are the corrections for indistinguishability.

Since $\mathrm{d}F = -S\mathrm{d}T - M\mathrm{d}B$, it follows that

$$M = -\frac{\partial F}{\partial B} \quad \text{and} \quad S = -\frac{\partial F}{\partial T}.$$

This means that while the magnetisation of the system will be the same as that of the distinguishable system, there will be extra contributions to the entropy (and thus to the heat capacity etc.).

2.25 According to Curie's law, the magnetic susceptibility of a paramagnet is inversely proportional to temperature.

(a) Show that a different behaviour is implied by the Third Law.

(b) What is wrong with the Curie law model?

(a) Start from the magnetic Maxwell relation

$$\left.\frac{\partial M}{\partial T}\right|_V = \left.\frac{\partial S}{\partial B}\right|_T.$$

The Third Law tells us that the right hand side goes to zero as $T \to 0$, so

$$\left.\frac{\partial M}{\partial T}\right|_V \to 0 \qquad \text{as} \qquad T \to 0.$$

Now $\chi \propto \partial M/\partial B$ so

$$\left.\frac{\partial \chi}{\partial T}\right|_V \to 0 \qquad \text{as} \qquad T \to 0.$$

or

$$\chi \to \text{const} \qquad \text{as} \qquad T \to 0.$$

But Curie's law $\chi \propto 1/T$ says that $\chi \to \infty$ as $T \to 0$.
These are contradictory.

(b) Curie's law is a high temperature approximation. The full expression for the paramagnet susceptibility is (Eq. (1.7.33))

$$\chi = \frac{N\mu^2}{VkT}\operatorname{sech}^2\left(\frac{\mu B}{kT}\right).$$

and this goes to zero as $T \to 0$. This satisfies the third law – so long as B is finite. And there will always be internal fields to ensure this is so.