PH2420 Outline Solutions 2006

Question 1

(a) EMF given by minus rate of change of flux: $EMF = -\mathrm{d}\Phi/\mathrm{d}t$.
Then

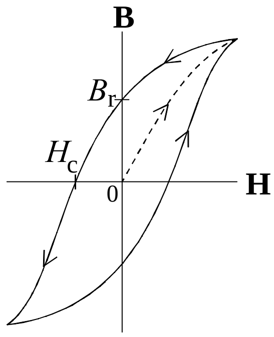
$$EMF = \oint \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \mathbf{B}\cdot \mathrm{d}\mathbf{a}.$$

Use Stokes's theorem / definition of curl to get $\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$. [4]

(b) The ampere is the current which, when flowing through two long wires 1m apart in a vacuum, produced a force of 2×10^{-7} newtons per unit length.

Description of current balance. [4]

(c) Sketch



To measure – wind a coil around the specimen. *H* is determined by current in the coil. *B* is determined from integrating up the time-dependence of the voltage. [4]

(d) $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$. Flux penetrating an area is given by the line integral of the vector potential around the perimeter. [4]

(e) Equation not valid in presence of varying *B* fields – EM induction. Correct equation is $\mathbf{E} = -\mathrm{grad}\,V - \partial\mathbf{A}/\partial t$. [4]

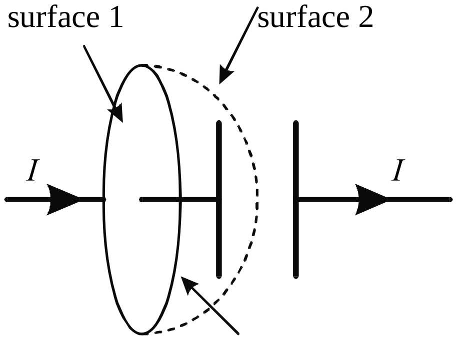
(f) Current law: sum of currents at a node is zero – conservation of charge
Voltage law: sum of voltages around a loop is zero – *E* field is conservative; when it is. [4]

Question 2

(a) Line integral of **B** around a closed loop is equal to the current penetrating the loop multiplied by μ_0. [2]

(b) Integrate over area – use Stokes's theorem / definition of curl $\rightarrow$ $\mathrm{curl}\,\mathbf{B} = \mu_0 \mathbf{J}$ [2]

(c)



Line integral around loop must be independent of the surface bounded by the loop. But for surface 1 there is a current I penetrating it, but there is no current penetrating surface 2. So there is an inconsistency. [5]

(d) Current into capacitor is $I = \mathrm{d}Q/\mathrm{d}t$ so since $Q = \varepsilon_0 AE$, then it follows that $I = \varepsilon_0 A\,\partial E/\partial t$. So inside the capacitor we need the same fictitious current, and so

$$i = \varepsilon_0 A\,\partial E/\partial t$$ [4]

(e) This is equivalent to a current density $\mathbf{j} = \varepsilon_0\,\partial \mathbf{E}/\partial t$, so add to the curl **B** equation to get

$$\mathrm{curl}\,\mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$ [4]

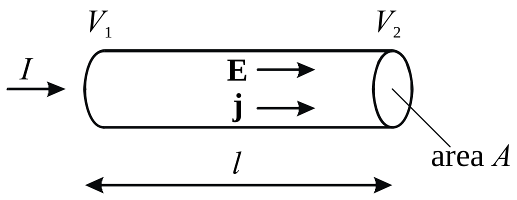
(f) In vacuum the equation becomes $\mathrm{curl}\,\mathbf{B} \sim \partial \mathbf{E}/\partial t$. When combined with $\mathrm{curl}\,E = -\partial \mathbf{B}/\partial t$, this says that time varying **E** causes time and space varying **B**, and vice versa. So each causes the other and waves thus propagate. [2]

Total [20]

Question 3

(a) $\mathbf{J}$ is the current flowing per unit area – can be expressed as $\mathbf{J} = nq\mathbf{v}$; symbols have usual meanings. [2]

(b)



Since $E = (V_2 - V_1)/l$ and $j = I/A$, it then follows that the Ohm's law expression $I = (V_2 - V_1)/R$ is equivalent to the expression $J = \sigma E$, where the conductivity σ is given by $\sigma = l/RA$. [4]

(c) Description of Drude model: Collisions cause randomisation of motion. Collisions with impurities, imperfections, lattice vibrations. So charge will accelerate for a typical time τ, achieving a velocity $v = a\tau = \frac{qE}{m}\tau$ in an electric field E. Then the mean velocity will be half of this. And since $\mathbf{J} = nq\mathbf{v}$, it then follows that

$$\sigma = \frac{Nq^2\tau}{2m} .$$ [10]

(d) Linearity means that

$$\begin{aligned} j_x &= \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z \\ j_y &= \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z \\ j_z &= \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z \end{aligned}$$

or, in matrix form

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} .$$

The matrix of σs is the conductivity tensor.

When using the principle axes the tensor is diagonal. This means that fields in these directions produce currents in these directions; in general the current is not in the same direction as the field. [4]

Total [20]

Question 4

(a) Electric dipole is equal and opposite electric charges separated by a small displacement. Induced moments occur when an originally neutral system has + and – charges forced in opposite directions by an applied electric field. Permanent moments exist in the absence of an external field. Oxygen molecule in an *E* field is an example of an induced moment. Na^+Cl^- 'molecule' is an example of a permanent dipole.

[5]

(b) Linearity means the potential is the sum of the potentials from the two charges. Thus

$$V = \frac{Q}{4\pi\varepsilon_0 \left(x - a/2\right)} - \frac{Q}{4\pi\varepsilon_0 \left(x + a/2\right)}$$
$$= \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{x - a/2} - \frac{1}{x + a/2}\right\}$$

[3]

(c) Express potential as

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{x}\left\{\left[1 - \frac{a}{2x}\right]^{-1} - \left[1 + \frac{a}{2x}\right]^{-1}\right\}$$

and then expand in powers of a/x:

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{x}\left\{1 + \frac{a}{2x} + \ldots - 1 + \frac{a}{2x} + \ldots\right\},$$

which, to leading order is

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{a}{x^2}.$$

Then p is the dipole moment Qa, so that

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p}{x^2}.$$ [3]

(d) **E** = – grad V, so that electric field in the x direction is

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\varepsilon_0}\frac{2p}{x^3}.$$ [3]

(e)

$$\text{force} = QE = \frac{Q^2}{4\pi\varepsilon_0 x^2}$$

so that

$$x = \left(\frac{Q}{4\pi\varepsilon_0 E}\right)^{1/2}$$

and then the dipole moment is

$$p = Qx = \frac{Q^{3/2}}{\left(4\pi\varepsilon_0 E\right)^{1/2}}.$$ [3]

(f) This is a stupid model. The equilibrium is unstable, the dipole moment increases with decreasing field. In reality the – charge is spread in space, not localised. [3]

Total [20]

Question 5

(a) $\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ so that

$$\frac{\partial \mathbf{B}}{\partial t}=-\mathrm{i}\omega\mathbf{B}_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}=-\mathrm{i}\omega\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial x}=\mathrm{i}k_x\mathbf{B}_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}=\mathrm{i}k_x\mathbf{B} \quad \text{etc.}$$

so that

$$\frac{\partial}{\partial t}\to -\mathrm{i}\omega,\quad \frac{\partial}{\partial x}\to \mathrm{i}k_x,\quad \frac{\partial}{\partial y}\to \mathrm{i}k_y,\quad \frac{\partial}{\partial z}\to \mathrm{i}k_z.$$

Then

$$\operatorname{div}\mathbf{B}=\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=\mathrm{i}\left(k_xB_x+k_yB_y+k_zB_z\right)=\mathrm{i}\mathbf{k}\cdot\mathbf{B}$$

and

$$\operatorname{curl}\mathbf{B}=\begin{pmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_x & B_y & B_z\end{pmatrix}=\mathrm{i}\begin{pmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ k_x & k_y & k_z\\ B_x & B_y & B_z\end{pmatrix}=\mathrm{i}\mathbf{k}\times\mathbf{B}$$

[6]

(b) By substitution

$$\operatorname{div}\mathbf{B}=0\to \mathrm{i}\mathbf{k}\cdot\mathbf{B}=0\to\mathbf{k}\cdot\mathbf{B}=0,$$
$$\operatorname{div}\mathbf{E}=0\to \mathrm{i}\mathbf{k}\cdot\mathbf{E}=0\to\mathbf{k}\cdot\mathbf{E}=0,$$
$$\operatorname{curl}\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\to \mathrm{i}\mathbf{k}\times\mathbf{E}=\mathrm{i}\omega\mathbf{B}\to\mathbf{k}\times\mathbf{E}=\omega\mathbf{B},$$
$$\operatorname{curl}\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\to \mathrm{i}\mathbf{k}\times\mathbf{B}=-\frac{1}{c^2}\mathrm{i}\omega\mathbf{E}\to\mathbf{k}\times\mathbf{B}=-\frac{1}{c^2}\omega\mathbf{E}.$$

[4]

(c) Direction of propagation is along **k**. So $\mathbf{k}\cdot\mathbf{B}=0$ means that **B** is perpendicular to the direction of propagation.
Similarly $\mathbf{k}\cdot\mathbf{E}=0$ means that **E** is perpendicular to the direction of propagation.
So **E** and **B** lie in the plane perpendicular to the direction of propagation.
But $\mathbf{k}\times\mathbf{E}=\omega\mathbf{B}$ means that **B** is perpendicular to **E** (as does $\mathbf{k}\times\mathbf{B}=-\omega\mathbf{E}/c^2$).
Thus **E**, **B** and direction of propagation are mutually perpendicular.

[5]

(d) By cyclic permutation $\mathbf{k}\times\mathbf{E}=\omega\mathbf{B}$ gives $\mathbf{E}\times\mathbf{B}=\mathbf{k}/\omega$ (or can demonstrate this by drawing the rectangular triad **k**, **E**, **B**). Then

$$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}=\frac{1}{\mu_0}\frac{\mathbf{k}}{\omega}$$

and so **S** points along the propagation of the wave.

S represents the energy flux density. [5]

Total [20]