PH2420 Outline Solutions 2005

Question 1

(a) $F_{\mathrm{e}} = \dfrac{e^2}{4\pi\varepsilon_0 r^2}$ and $F_{\mathrm{g}} = G\dfrac{m_{\mathrm{p}}m_{\mathrm{e}}}{r^2}$, so that $\dfrac{F_{\mathrm{e}}}{F_{\mathrm{g}}} = \dfrac{e^2}{4\pi\varepsilon_0}\dfrac{1}{Gm_{\mathrm{p}}m_{\mathrm{e}}}$, independent of separation r.

Use $1/4\pi\varepsilon_0 = 9.0\times10^9$

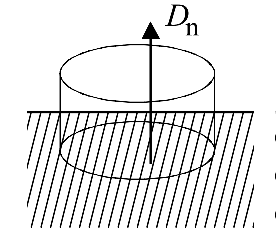
$$\frac{F_{\mathrm{e}}}{F_{\mathrm{g}}} = \frac{\left(1.6\times10^{-19}\right)^2\times9.0\times10^9}{6.7\times10^{-11}\times1.7\times10^{-27}\times9.1\times10^{-31}}$$ [4]

$$= 2.2\times10^{39}.$$

(b) Work done in moving a charge is independent of path; just depends on end points, work done in moving around a closed loop is zero, $\oint \mathbf{E}.\mathrm{d}\mathbf{l} = 0$, $\mathrm{curl}\,\mathbf{E} = 0$.
Not conservative for em induction, varying **B** field, since $\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$. [4]

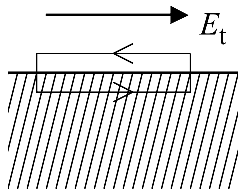
(c) $\mathbf{F} = q\left\{\mathbf{E} + \mathbf{v}\times\mathbf{B}\right\}$. Zero force if $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$. If **E**, **v**, **B** mutually perpendicular then $v = E/B$ (magnitudes – ignore sign.) [4]

(d) Construct Gaussian pillbox. Shrink height to zero. Then div **D** = 0 means that flud of **D** is zero.



Flux of D is from normal component of D through top and bottom faces. Thus D normal is continuous. [2]

Construct loop for line integral. Then curl **E** = 0 means that line integral of **E** around the loop is zero.



Shrink height to zero. Then only contributions come from the horizontal lines. These line integrals involve tangential components of E. Total line integral is zero so inside and outside parts cancel, so this means that E tangential is continuous. [2]

(e) Collisions between carriers and impurities, dislocations, phonons etc. These tend to randomise the motion of the carriers. They accelerate, under the influence of E, between collisions. But at a collision the motion is randomised and the velocity starts, effectively, from zero. Thus there is a 'terminal velocity' effect. [4]

(f) Poisson's equation: $\nabla^2 V = -\rho/\varepsilon_0$. But in the absence of any free charge, $\rho = 0$ so we have Laplace's equation $\nabla^2 V = 0$. Now V = const in a conductor. So V = const on surface of the cavity. Now inside the cavity we must solve Laplace's equation subject to the boundary condition of constant V. Earnshaw's theorem (extreme value theorem) says that there cannot be an extremum away from the boundary. So if V is const on the boundary it must have the same value everywhere inside the cavity. Then if V is const, we have that $\mathbf{E} = 0$. [4]

Question 2

(a) Gauss's law:
Total electric flux through a closed surface is total charge enclosed / ε_0.

$$\Phi_E = \underset{\substack{\text{closed}\\\text{surface}}}{\oiint} \mathbf{E}.\mathbf{ds} = \sum Q\big/\varepsilon_0 = \underset{\text{volume}}{\iiint} \rho \mathrm{d}v / \varepsilon_0 .$$ [3]

(b) Charge is distributed uniformly so it scales with the volume

$$q = Q\left(\frac{r}{a}\right)^3 .$$ [3]

(c) Field given by

$$\begin{aligned} E &= \frac{q}{4\pi\varepsilon_0 r^2} \\ &= \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r}{a}\right)^3 \\ &= \frac{Qr}{4\pi\varepsilon_0 a^3} . \end{aligned}$$ [4]

(d) Field will point radially (inwards for negative charge and outwards for positive charge). [3]

(e) Force given by

$$\begin{aligned} F &= QE \\ &= \frac{Q^2 r}{4\pi\varepsilon_0 a^3} . \end{aligned}$$ [3]

(f) Balance this force with electric force from external applied field

$$\frac{Q^2 r}{4\pi\varepsilon_0 a^3} = QE$$

so that

$$Qr = 4\pi\varepsilon_0 a^3 E .$$

But Qr is the dipole moment p, thus

$$p = 4\pi\varepsilon_0 a^3 E .$$ [3]

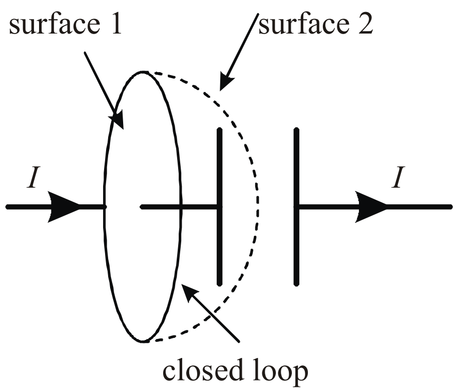
(g) Linearity follows from assumption of uniform distribution of charge. So if the distribution of charge is not uniform then the relation between p and E will not be linear (for large-ish r) [2]

Question 3

(a) The displacement current (density) is the term $\varepsilon_0\,\partial\mathbf{E}/\partial t$ which occurs in the curl **B** equation. [1]

Since it has the same dimensions as **J** and it occurs together with **J** in the equation it may be termed a current density; although it is not due to the passage of real physical charges, its magnetic effect is equivalent to that of a flowing charge density. [4]

(b) Consider charging the capacitor:



Now $\oint_{\text{loop}} \mathbf{B}.\mathbf{dl}$ is the same for surface 1 and surface 2 since the loop is the same in both cases. But there is no physical charge passing through surface 2 which would imply an inconsistency. This is overcome by saying that inside the capacitor there is a 'fictitious' current – the displacement current – that also can produce magnetic fields. [5]

(c) In absence of ρ and **J**, $\mathrm{curl}\,\mathbf{B} = \mu_0\varepsilon_0\,\partial\mathbf{E}/\partial t$ - just the displacement current term.

Take the curl of the curl**E** equation:

$$\mathrm{curl\;curl}\,\mathbf{E} = -\frac{\partial}{\partial t}\mathrm{curl}\,\mathbf{B}\,,$$

interchanging order of differentiating.

But curl**B** is given by the displacement current expression above, so

$$\mathrm{curl\;curl}\,\mathbf{E} = -\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\,.$$

We have the identity for curl curl, so

$$\mathrm{grad\;div}\,\mathbf{E} - \nabla^2\mathbf{E} = -\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\,.$$

But div **E** = 0 in free space

$$\nabla^2\mathbf{E} - \mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0\,;$$

this is the wave equation.

So the displacement current bit is crucial. [5]

(d) Assumption that E is uniform between the plates. Then a circular loop of radius r encloses a fraction $(r/a)^2$ of the current I.

Ampere's law: $\oint_{\text{loop}} \mathbf{B}.\mathbf{dl} = \mu_0 I_{\text{enclosed}}$.

By symmetry, for a concentric loop the **B** will be circular, so $B \times 2\pi r = \mu_0 I \left(r/a\right)^2$. Thus we obtain

$$B = \frac{\mu_0 I r}{2\pi a^2}. \qquad [5]$$

Question 4

(a) Lines of **B** are concentric circles around the wire, 2 marks for sketch. [4]

(b) Start from $\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times\mathbf{r}}{r^3}$ and use $\frac{\mathbf{r}}{r^3} = -\,\mathrm{grad}\frac{1}{r}$ to express as

$$\mathrm{d}\mathbf{B} = -\frac{\mu_0 I}{4\pi}\mathbf{dl}\times\mathrm{grad}\frac{1}{r}. \qquad (3)$$

Now use curl(a**b**) identity to write

$$\mathbf{dl}\times\mathrm{grad}\frac{1}{r} = \frac{1}{r}\mathrm{curl}\,\mathbf{dl} - \mathrm{curl}\frac{\mathbf{dl}}{r}. \qquad (3)$$

Now curl d**l** = 0 so that

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\mathrm{curl}\frac{\mathbf{dl}}{r}. \qquad (2)$$

Thus we may write **B** = curl **A** where

$$\mathrm{d}\mathbf{A} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}}{r} \qquad (2) \qquad [10]$$

(c) Lines of **A** are lines parallel to the wire, 1 mark for sketch. [2]

(d) Since **K** has only a component in the x direction,

$$\mathbf{K}\times\mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ K_x & 0 & 0 \\ x & y & z \end{vmatrix} = K_x\left(-z\hat{\mathbf{y}} + y\hat{\mathbf{z}}\right).$$

Then

$$\mathrm{curl}\left(\mathbf{K}\times\mathbf{r}\right) = K_x \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix} = 2K_x\hat{\mathbf{x}}.$$

And so the **B** field from $\mathbf{A} = \frac{1}{2}(\mathbf{K}\times\mathbf{r})$ is simply **K**. [4]

Question 5

(a) Magnetisation **M** is the total magnetic moment per unit volume. [1]

Magnetic susceptibility χ is defined by $\mathbf{M} = \chi\mathbf{H}$ [1]

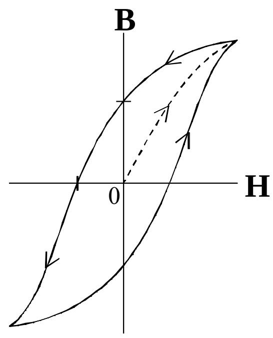
$\mathbf{B} = \mu_0\left(\mathbf{H}+\mathbf{M}\right)$ where μ_0 is the permeability of free space. [1]

(b) Diamagnetic materials have negative χ. Diamagnetism results from materials with no permanent dipoles and it occurs as a consequence of distortion of electron orbits – induced dipole moments – Lenz's law. [2]

Paramagnetic materials have positive χ. Paramagnetism results from materials with permanent magnetic moments and it occurs as a consequence of orientation of magnetic moments in a magnetic field. – Free energy consideration, balance between energy minimum and entropy maximum. [2]

Ferromagnetic materials can exhibit a magnetisation in the absence of an applied magnetic field. Different domains are spontaneously magnetised in different directions. Domains can be aligned by external field – can give large magnetisation, and hysteretic effects. [2]

(c)



[2]

When **H** = 0, can have **B** ≠ 0 so **B** can exist without an external current → permanent magnet.

The area $\oint \mathbf{B}.\mathrm{d}\mathbf{H}$ is the energy dissipated in one cycle around the loop. It will appear as heat in the transformer core → inefficiency. This area must be minimised in order to maximise the transformer efficiency. [4]

(d) General relation is $\mathbf{B} = \mu_0\left(\mathbf{H}+\mathbf{M}\right)$ and **H** = 0, so $\mathbf{M} = \mathbf{B}/\mu_0$. [2]

(e) Magnetic moment $m = M \times$ volume $= B_{\text{magnet}} Al/\mu_0$ where A is area and l length of needle.

Torque = $\mathbf{m}\times\mathbf{B}_{\text{earth}}$, magnitude = $m\, B_{\text{earth}} \sin\theta$.

$\sin\theta = ½$

$$\begin{aligned}\text{Torque} &= \frac{1}{2} \times \frac{B_{\text{magnet}} A l}{\mu_0} \times B_{\text{earth}} \\ &= \frac{1}{2} \times \frac{0.01 \times \left(1 \times 10^{-6}\right) \times 10^{-2}}{4\pi \times 10^{-7}} \times 4 \times 10^{-5}\,\text{Nm} \qquad [3] \\ &= \frac{10^{-8}}{2\pi}\,\text{Nm} = 1.6 \times 10^{-9}\,\text{Nm.}\end{aligned}$$