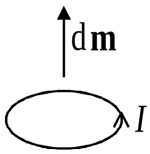
XI MAGNETIC FIELDS AND MATTER

11.1 Magnetic media and (magnetic) dipoles

The magnetic properties of matter are due to the motion of the electric charges that constitute matter (mainly electrons). This idea originated with Ampère. Spin magnetic moments are also important, but for the purposes of the present discussion they also can be understood as circulating electric currents.

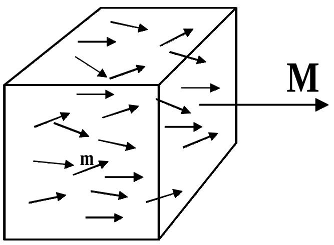
A general distribution of currents will produce a very complicated magnetic field. As in the electric case, however, most of the magnetic properties of matter can be explained in terms of *dipoles*. Here case they are magnetic dipoles.



Consider a small loop of area d**a** with a current I flowing around it. The magnetic dipole moment dm is defined by

$$d\mathbf{m} = I d\mathbf{a}. \tag{11.1}$$

One can calculate the magnetic field away from the loop. When the distance is large compared with the dimension of the loop this gives the dipole approximation to the field. The result is similar to that for the electric field of an electric dipole. At large distances the field varies as $B \sim 1/r^2$, as in the electric case.



The bulk magnetic behaviour of matter is described in terms of the magnetisation vector **M**. This is defined as the total dipole moment per unit volume.

$$\mathbf{M} = N\mathbf{m} \tag{11.2}$$

where N is the number of dipoles of moment **m**, per unit volume.

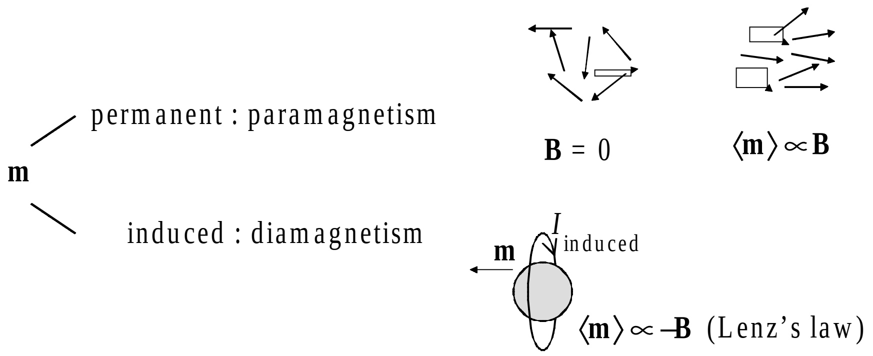
11.2 Diamagnetism and paramagnetism

The electric properties of matter were quantified in terms of the change in the capacitance of a capacitor when the material was placed between the plates. By analogy with this we can describe the magnetic properties of matter in terms of the change in the value of inductance when the material is placed in a coil. In the electrical case the capacitance was increased. In contrast to this, in the magnetic case the inductance can *increase* or *decrease*.

Restricting ourselves (for the present) to linear systems, where $M \propto B$ we refer to these two effects as *paramagnetism* and *diamagnetism*.

Paramagnetism occurs when permanent magnetic dipoles – from spin magnetic moments and/or unfilled electron shells – orient themselves in a magnetic field. In zero field the moments are randomly oriented, with zero net magnetisation. However in a magnetic field it is energetically preferable for the moments to point along the field. So the dipole field *adds* to the external magnetic field.

Diamagnetism results from induced magnetic moments. The applied field modifies the electron orbits, inducing net circulating currents and thus a dipole moment. In this case Lenz's law tells us that the magnetic moment is in the *opposite* direction to the applied field; the dipole field tends to cancel the applied field.



In paramagnetism the alignment of magnetic moments is a consequence of the competition between energy minimisation and entropy maximisation. At temperature T the equilibrium state is given by minimising the free energy $E - TS$ where S is the entropy. At low temperatures one tends to minimise the energy so $\langle \mathbf{m} \rangle$ is large, while at high temperatures the entropy term wins out. Entropy is maximised when the orientations of the moments are random. Thus at high temperatures $\langle \mathbf{m} \rangle$ is small. We see that paramagnetism is temperature-dependent.

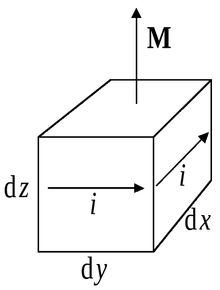
In general there will be both paramagnetic and diamagnetic effects present together. According to classical mechanics the diamagnetism and paramagnetism cancel exactly; there is no resultant magnetic effect. The fact that magnetic effects do exist is a consequence of quantum mechanics.

11.3 Ferromagnetism

In some materials it is possible to find a magnetisation in the absence of an applied magnetic field. This is known as ferromagnetism, since it is mostly observed in iron-rich substances. Ferromagnetism is a non-linear effect. It may be regarded as an extreme form of paramagnetism. It is a consequence of the existence of large-scale permanent magnetic dipole moments.

Depending on the arrangement of the magnetic moments there is a variety of sub-classes of ferromagnetism: antiferromagnetism, ferrimagnetism, helimagnetism, etc. The properties of ferromagnets will be considered in Section (11.11).

11.4 Bound surface current



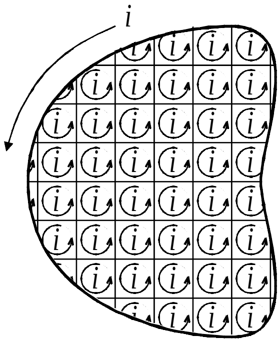
In a uniformly magnetised substance the magnetisation may be interpreted as a bound current circulating around the surface of the object. We shall show this by considering a cube dxdydz in the material that has a current i circulating around its surface. The magnetic moment of the cube is given by the current i times the area dxdy:

$$m = i \, \mathrm{d}x\mathrm{d}y$$

so dividing by the volume dxdydz, the magnetic moment per unit volume, the magnetisation M (pointing in the z direction) is given by

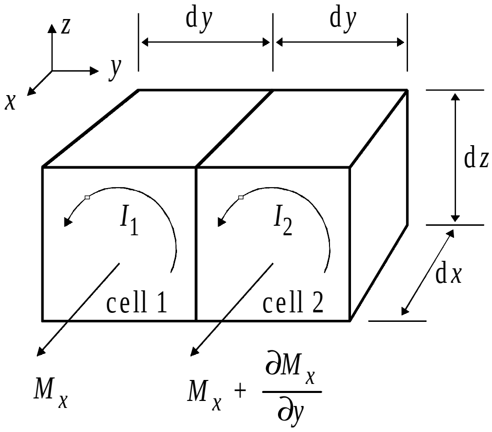
$$M_z = i/\mathrm{d}z \tag{11.3}$$

so the magnetisation can be regarded as the bound surface current per unit length.



Considering now a slice through a magnetised specimen, each cell will have a circulating bound current. When the magnetisation is uniform the bound current in each cell will be the same. Then the currents at the boundaries between the cells will cancel. As a result there will be a resultant current flowing around the periphery of the specimen but *no* current flowing internally. And the bound surface current per unit length is equal to the magnetisation of the specimen.

11.5 Bound volume current



When the magnetisation is not uniform there is no longer a cancellation of the internal bound currents. The magnetisation of cell 1 is denoted by $\mathbf{M}_1$. The magnetisation of cell 2, a distance dy away is given by a Taylor expansion:

$$\mathbf{M}_2 = \mathbf{M}_1 + \frac{\partial \mathbf{M}}{\partial y}\mathrm{d}y + \ldots$$

The x component of the magnetic moment of cell 1, $m_x^{(1)}$ is then

$$m_x^{(1)} = M_x^{(1)}\mathrm{d}x\mathrm{d}y\mathrm{d}z = I_1\mathrm{d}y\mathrm{d}z\,.$$

The x component of the magnetic moment of cell 2, $m_x^{(2)}$, is then

$$m_x^{(2)} = \left(M_x^{(1)} + \frac{\partial M_x}{\partial y}\mathrm{d}y\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = I_2\mathrm{d}y\mathrm{d}z\,.$$

Taking the difference of these expressions we find $I_1 - I_2$, the resultant current flowing at the cells' interface, is

$$I_1 - I_2 = -\frac{\partial M_x}{\partial y}\mathrm{d}x\mathrm{d}y$$

so that the current density in the z direction j_z is

$$j_z = -\frac{\partial M_x}{\partial y}\,.$$

There will also be a contribution to the current in the z direction from the y component of magnetisation. By a similar argument one obtains

$$j_z = +\frac{\partial M_y}{\partial x}.$$

so that adding these two contributions gives

$$j_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y},$$

which we recognise as the curl of **M**.

Thus we conclude that when the magnetisation of a body is non-uniform, as well as a bound surface current, there is a bound volume current given by

$$j_\mathrm{b} = \mathrm{curl}\,\mathbf{M}. \qquad (11.4)$$

11.6 Ampère's law including magnetics

The bound currents within matter will produce magnetic fields, according to Ampère's law. Thus in the presence of matter we can re-express Ampère's law by writing the current density as the sum of the *free* current density $\mathbf{j}_\mathrm{f}$ and the *bound* current density $\mathbf{j}_\mathrm{b}$

$$\mathbf{j} = \mathbf{j}_\mathrm{f} + \mathbf{j}_\mathrm{b}.$$

We then write Ampère's law

$$\mathrm{curl}\,\mathbf{B} = \mu_0\,\mathbf{j}$$

as

$$\mathrm{curl}\,\mathbf{B} = \mu_0\,(\mathbf{j}_\mathrm{f} + \mathbf{j}_\mathrm{b}). \qquad (11.5)$$

Now from Equation (11.4) we know what $\mathbf{j}_\mathrm{b}$ is, in terms of the macroscopic magnetisation **M**, $\mathbf{j}_\mathrm{b} = \mathrm{curl}\,\mathbf{M}$. Thus we have

$$\mathrm{curl}\,\mathbf{B} = \mu_0\,\mathbf{j}_\mathrm{f} + \mu_0\,\mathrm{curl}\,\mathbf{M}.$$

Regarding the free current as the source, the equation can be written as

$$\mathrm{curl}\,(\mathbf{B}/\mu_0 - \mathbf{M}) = \mathbf{j}_\mathrm{f}. \qquad (11.6)$$

This leads us to introduce a new magnetic field vector **H**

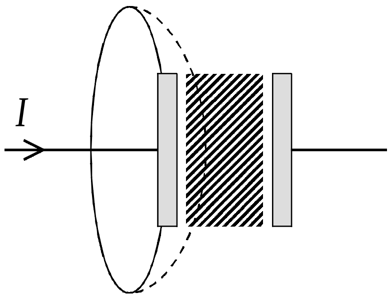
$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \qquad (11.7)$$

so that Ampère's law in the presence of magnetics becomes

$$\mathrm{curl}\,\mathbf{H} = \mathbf{j}_\mathrm{f}. \qquad (11.8)$$

We see that whereas the *total* current is the source of the magnetic **B** field, the *free* current is the source of the **H** field. And in practice one can only control the free current.

11.7 Displacement current



The final step in the 'derivation' of the microscopic Maxwell equations was the introduction of the displacement current. This was introduced by a plausibility argument when considering the magnetic field in the vicinity of a charging capacitor. Some modification to this is required if the capacitor incorporates a dielectric.

The current in the wire, I, is related to the free charge on the plates of the capacitor. Now in the case of dielectrics the charge is given in terms of the electric **D** vector by

$$D = Q/A.$$

The current is the derivative of the charge:

$$I = \frac{\partial Q}{\partial t} = A\frac{\partial D}{\partial t}$$

or

$$j = \frac{\partial D}{\partial t}. \qquad (11.9)$$

This is a generalisation of the originally-introduced displacement current. Now the expression for the 'fictitious' current holds in the presence of matter. We see that the fictitious current density is the derivative of the electric displacement vector. *Thus* the designation *displacement current*.

When this term is added to the curl**H** equation one obtains the expression

$$\mathrm{curl}\,\mathbf{H} = \mathbf{j}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}, \qquad (11.10)$$

which is the full modification of the curl**H** equation in the presence of matter.

11.8 Macroscopic Maxwell equations

The full set of Maxwell equations, in their macroscopic form, can now be written as

$$\mathrm{div}\,\mathbf{D} = \rho_\mathrm{f}$$

$$\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathrm{div}\,\mathbf{B} = 0$$

$$\mathrm{curl}\,\mathbf{H} = \mathbf{j}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t}.$$

The sources here are solely the *free* currents and charges, over which one has control. However the equations are in terms of *four* field vectors. Solutions can only be found once one has *constitutive relations*, that is, a relationship between **D** and **E**, as we saw in Section (10.11), and a relationship between **B** and **H**, as we shall examine in the following sections.

11.9 Magnetic susceptibility and permeability

In this section we shall restrict consideration to linear isotropic homogeneous systems (LIH). Recall that we have already encountered diamagnetism and paramagnetism:

$$\mathbf{M} \propto -\mathbf{B} \text{ for diamagnets}$$
$$\mathbf{M} \propto +\mathbf{B} \text{ for paramagnets.}$$

Following the rules of the S.I. system, the magnetic susceptibility χ_m is defined as the constant of proportionality between **M** and **H** (yes, it is **H**!),

$$\mathbf{M} = \chi_m \mathbf{H} \tag{11.11}$$

⌈Since $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ it follows that **M** and **H** have the same dimensions, so χ_m is dimensionless. ⌋

This does appear to be a strange definition for the magnetic susceptibility, but it is convenient experimentally; **H** is much easier to measure than **B**. This will be seen in Section (11.10). Some text book writers, however, choose to work in terms of a *B*-field susceptibility $\chi_B = \mu_0 M/B$ but this is non-standard.

Typical values of χ for diamagnetic and paramagnetic systems are

$$\chi_m \approx -10^{-6} \text{ diamagnetism}$$
$$\chi_m \approx +10^{-3} \text{ paramagnetism.}$$

Paramagnetism, when it exists, is much larger: a consequence of the spin magnetic moments of unpaired electrons in unfilled electron shells. The paramagnetism of orbital magnetic moments will be comparable with the diamagnetism.

Since

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(\mathbf{H} + \chi_m\mathbf{H})$$
$$= \mu_0(1 + \chi_m)\mathbf{H}$$

we see that **B** is proportional to **H** and we write

$$\mathbf{B} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H}$$

where μ is the absolute permeability
and μ_r is the relative permeability.

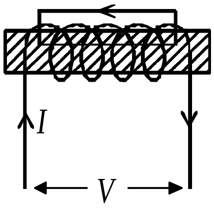
The relative permeability is related to the magnetic susceptibility by

$$\mu_r = 1 + \chi_m.$$

The electric and magnetic cases are not quite similar: we may make the following comparisons:

Electric	Magnetic
$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$	$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$	$\mathbf{M} = \chi_m\mathbf{H}$
$\varepsilon_r = 1 + \chi_e$	$\mu_r = 1 + \chi_m$
$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$	$\mathbf{B} = \mu_0\mu_r\mathbf{H}$.

11.10 Measuring B and H



Consider a coil of wire wound around a specimen whose magnetic properties we wish to investigate. The easiest things to measure are the current in the coil and the voltage across the ends of the coil. Considering the current first, here we can make a connection with Ampère's law

$$\oint \mathbf{H}.\mathrm{d}\mathbf{l} = I_{\text{free}} .$$

So applying this to the indicated path, of length l in the specimen, and recalling that the field outside a long solenoid is zero, we see that

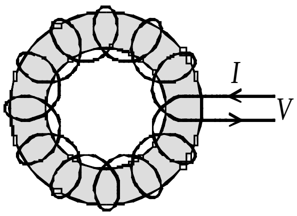
$$H\,l = N\,I$$

where N is the number of turns of the coil in the length l. Denoting the number of turns per unit length by n we can write H as

$$H = n\,I.$$

At this point we can appreciate that the units to measure **H** are conventionally specified in amperes per metre, since H is indeed simply the current flowing per metre.

The validity of the above expressions relies on having a very long coil on a very long specimen; only then is the field zero outside. In other words, for a long coil the effects of the ends can be ignored. But practically, the end effects can be eliminated in another way: by removing the ends! This can be achieved by having the coil wound round a toroidal specimen. This ensures that all the field is in the specimen. Thus when the number of turns per unit length and the current flowing are known, then the **H** field is known.



Now consider what may be inferred from measuring the voltage across the coil. The emf is given by Faraday's law

$$V = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

where (for a uniform field) the flux Φ is given by

$$\begin{aligned}\Phi &= B\,a\,N \\ &= B\,a\,n\,l\end{aligned}$$

(neglecting the sign) where a is the cross section area of the coil, l is the length and n is the number of turns per unit length. Now $a\,l$ is the volume of the coil v, so that since

$$V = -\dot{B}vn$$

it follows that the **B** field may be found from the *time integral* of the voltage

$$B = \frac{1}{vn}\int V(t)\,\mathrm{d}t\,.$$

Thus from the *current* in the coil we find the **H** field, while from the (integral of the) *voltage* across the coil we find the **B** field.

These arguments provide an easy way of discussing the energy in magnetic systems. Since the voltage and the current of the coil are given by

$$V = \dot{B}vn, \qquad I = H/n$$

it follows that the power, the rate of doing work, is given by the product

$$P = IV = \dot{B}Hv$$

since the n cancels out.

Now the power is the derivative of the energy. So in terms of the energy density U (energy per unit volume)

$$\frac{\partial U}{\partial t} = H\frac{\partial B}{\partial t}$$

or

$$\mathrm{d}U = H\,\mathrm{d}B\,.$$

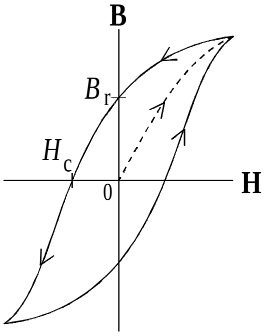
For simplicity we have been ignoring the vector character of the fields, which we could do since **B** and **H** were parallel in the model system considered. In the general case the product of the fields must be the dot product:

$$\mathrm{dU} = \mathbf{H}\,.\,\mathrm{d}\mathbf{B}\,.$$

From this we conclude that the work done on a magnetic system may be understood in terms of the area under the *B–H* curve.

11.11 Properties of Ferromagnets

In a ferromagnet there can be a magnetisation in the absence of an applied magnetic field. A ferromagnet can, however, *appear* to be un-magnetised, when different domains have their magnetisation pointing in different directions. Application of a magnetic field then changes the size and the orientation of the domains. Such an effect is usually nonlinear and hysteretic.



It is conventional to represent the behaviour of a ferromagnet in terms of the relationship between **B** and **H**. If one imagines a coil of wire wound around the material then **H** is determined by the current in the coil; this is the generalised ‘force’.

If the specimen is imagined to start from a state of no magnetisation then for small applied **H** the initial behaviour is linear and reversible. This is indicated by the dotted line on the diagram.

Gradually **B** fails to keep up with the applied **H** field and the curve flattens off. If the **H** field is then reduced, by reducing the current in the coil, there remains some **B** field when **H** has gone to zero. This value, B_r is the *remnant field* or *remnance*. The value to which the **H** field must be taken for the **B** field to go to zero, H_c is known as the *coercivity*.

$$B_r : \text{ remnance}$$
$$H_c : \text{ coercivity}$$

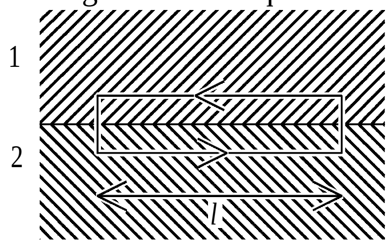
When the system is taken through a cycle around the *hysteresis* curve and returned to its original *B*–*H* state, the work done is given by the area of the curve.

11.12 Boundary conditions

At the boundary between two different media there are certain restrictions on the behaviour of the **B** and **H** fields. In the static case, and in the absence of any free currents the magnetic fields obey the equations

$$\mathrm{curl}\,\mathbf{H} = 0$$
$$\mathrm{div}\,\mathbf{B} = 0\,.$$

Taking the curl **H** equation first, we see that the line integral of **H** around the indicated loop



must be zero. The relevant component of **H** here is that *tangential* to the interface surface. If we measure **H** in the positive *x* direction then the line integral of **H** around the closed loop is

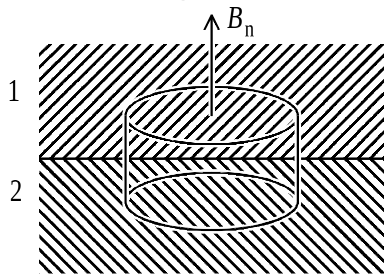
$$\oint \mathbf{H}.\mathrm{d}\mathbf{l} = \left(H_{\tan}^{(2)} - H_{\tan}^{(1)}\right) l = 0$$

or

$$H_{\tan}^{(2)} = H_{\tan}^{(1)}.$$

In other words, the tangential component of **H** at a boundary is continuous.

Now let us consider the div **B** equation, with the aid of the diagram containing the familiar Gaussian pill box. The div **B** equation is telling us that the total flux of **B** out of the pill box is zero. And in the limit that the height of the box becomes small this means that only the flux through the top and bottom need be considered. Measuring **B** in the upward



direction, and if *a* is the cross section area of the box then we have the expression

$$\oint\!\!\oint_{\text{surface}} \mathbf{B}.\mathrm{d}\mathbf{a} = \left(B_{\mathrm{n}}^{(1)} - B_{\mathrm{n}}^{(2)}\right) a = 0$$

or

$$B_{\mathrm{n}}^{(1)} = B_{\mathrm{n}}^{(2)}.$$

In other words, the *normal* component of **B** at a boundary is continuous.

11.13 EM waves in matter

When there are no free charges and currents the macroscopic Maxwell equations are given by

$$\operatorname{div}\mathbf{D} = 0$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{div}\mathbf{B} = 0$$

$$\operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}.$$

As it stands these can not be solved since there are too many unknowns in the equations. One needs constitutive relations for the material medium to relate **D** to **E** and **H** to **B**. And if we restrict consideration to Linear Isotropic Homogeneous systems, then

$$\mathbf{D} = \varepsilon\,\mathbf{E} \quad \text{and} \quad \mathbf{H} = \mathbf{B}/\mu$$

giving the set of equations

$$\operatorname{div}\mathbf{E} = 0$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{div}\mathbf{B} = 0$$

$$\operatorname{curl}\mathbf{B} = \varepsilon\mu\frac{\partial \mathbf{E}}{\partial t}.$$

These are similar to Equations (7.11) – (7.14) which applied in the case of a vacuum, except that there we had $\varepsilon_0\mu_0$ while here we have $\varepsilon\mu$. There, in a vacuum, we found that electromagnetic waves could propagate with a speed c given by

$$c^2 = 1/\mu_0\varepsilon_0 .$$

Thus now, in matter, we find that electromagnetic waves propagate with speed v given by

$$v^2 = 1/\mu\varepsilon$$

since we can show that **E** satisfies the wave equation

$$\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

with similar equations for **B**, **D** and **H**.

(We should, at this stage, show that **E** and **B** and the direction of propagation are mutually perpendicular; that is left as a Problem.)

11.14 Macroscopic and microscopic fields

This discussion parallels that for the electric fields, in Section (10.10). *Microscopically* there are just **E** and **B** fields, which vary wildly in matter. *Macroscopically* there are the four fields **E**, **D**, **B** and **H**, which vary smoothly in matter; they arise from taking a spatial average over many atomic dimensions.

When you have completed this chapter you should:

- Know that magnetic properties of matter are due to the motion of its bound electric charges;
- appreciate that most properties can be understood in terms of magnetic dipoles;
- be familiar with the definition of magnetic dipole moment;
- know the definition of magnetisation;
- understand the difference between diamagnetism and paramagnetism, and their different origins;
- know the phenomenon of ferromagnetism;
- understand the idea of bound surface current;
- understand the concept of bound volume current and that it vanishes in a uniformly magnetised specimen;
- be able to calculate the bound current density in terms of the magnetisation;
- know how Ampère's law can be modified in matter, giving the **H** field in terms of the free current;
- know the definition of the **H** field;
- understand that the displacement current in matter is given in terms of the **D** field;
- be able to write down the macroscopic Maxwell equations;
- be familiar with the definitions of magnetic susceptibility and magnetic permeability;
- know the typical orders of magnitude for diamagnetic and paramagnetic susceptibility;
- be familiar with the similarities and differences between the relations between the various electric vectors and the various magnetic vectors;
- be able to calculate the H field in a body from the current in a coil surrounding it and B from the voltage measured across the coil;
- be able to calculate the magnetic energy in a magnetic body;
- be familiar with the characteristic behaviour of a ferromagnet; the hysteresis curve, coercivity and remnance;

- understand the boundary conditions on magnetic fields at the interface between two bodies;

- be able to demonstrate that electromaagnetic fields propagate in matter and express the speed of propagation in terms of the permeability and the permittivity;

- understand the distinction between microscopic and macroscopic fields.