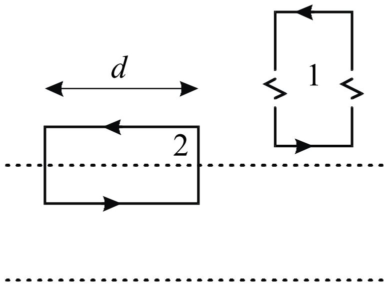
IX INDUCTANCE AND MAGNETIC FIELDS

9.1 Field in a solenoid

A varying current in a conductor will produce a varying magnetic field. And this varying magnetic field then has the capability of inducing an EMF or voltage in the conductor. The combination of these is the phenomenon of *inductance*: a changing current in a conductor will cause an EMF. The effect is enhanced considerably if the conductor is in the form of a coil or solenoid. Firstly the field produced is proportional to the number of turns, since what matters is the total current flowing through an area. And secondly, the voltage is induced in each loop so that the contributions from each turn add together.



We start with the first part of the problem: calculating the magnetic field produced by a current flowing in a solenoid. The picture shows a coil of n turns per unit length, carrying a current of I amperes. Although we do not need it immediately, we shall specify the cross section area of the coil as a.

Calculation of the magnetic field of the solenoid may be done using Ampère's law (exploiting the symmetry of the system). Thus we use

$$\oint_{\text{closed loop}} \mathbf{B} . \mathbf{dl} = \mu_0 I$$

with an appropriately chosen closed path of integration.

We first show that the magnetic field outside a very long solenoid is zero (except, obviously, at the ends). Consider the loop labelled 1. By the symmetry of the system, and since the magnetic field must be perpendicular to the direction of flow of the current, the magnetic field must point parallel to the axis of the coil. Then only the two horizontal elements of the loop make a contribution to **B**. There can be no contribution from the upper element since it may be placed as far away as we like. So the only contribution comes from the lower element. The field along this element multiplied by its length is proportional to the current threading the loop. This is zero as the loop does not intersect the coil. Since the length of the element is arbitrary, it follows that the field must be zero. *Thus there can be no magnetic field outside a long solenoid.*

Turning to the field inside the coil, consider loop 2 in the figure. We have just established that the field outside the solenoid is zero. Thus only the line integral along the internal horizontal element of the loop contributes to the Ampère's law integral. If the length of the element is d and if there are N turns contained within the loop then Ampère's law gives

$$Bd = \mu_0 NI.$$

But since there are n turns per unit length, the number of turns in the loop of length d is nd. Then

$$Bd = \mu_0 ndI$$

or

$$B = \mu_0 nI. \tag{9.1}$$

This is the expression for the field inside a long solenoid. The field is independent of the position along the coil. This is because we are considering a very long coil, which results in translational invariance. Note also, that the field is independent of the radial position in the coil; another consequence of symmetry. We thus conclude that the field in a long solenoid is uniform or *homogeneous*. This is a good way of producing a uniform magnetic field.

9.2 Magnetic vector potential

We have seen that we can obtain the electric field **E** from the scalar potential V as

$$\mathbf{E} = -\mathrm{grad}V$$

in the electro*static* case. Recall that the possibility for doing this relied on the fact that in electrostatics the curl of **E** is zero. Note, however, that in the presence of varying magnetic fields the above equation does not hold since then curl**E** is no longer zero.

In the presence of sources of **B**, that is, electric currents, we *cannot* express **B** as the gradient of a scalar potential since the curl of **B** is then non-zero:

$$\mathrm{curl}\mathbf{B} = \mu_0 \mathbf{j}.$$

The asymmetry between the treatment of electric and magnetic fields is not so surprising since the (static) **E** field is irrotational whereas the **B** field is solenoidal.

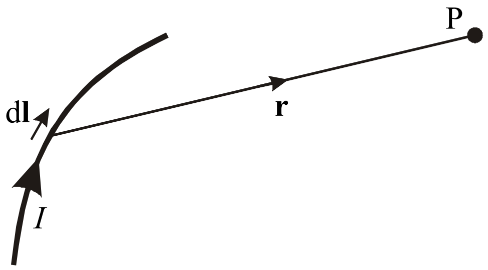
While the **B** field cannot, in general, be expressed as the gradient of a scalar potential, it can be written as the curl of a *vector* potential:

$$\mathbf{B} = \mathrm{curl}\mathbf{A}. \tag{9.2}$$

Here **A** is called the magnetic vector potential. There is no immediately obvious advantage of this quantity since we have simply replaced one vector by another. However we will see that the use of the vector potential will help with a number of issues. Although not of direct relevance to this course, the magnetic vector potential goes together with the scalar electric potential to make a relativistic four-vector. And using the magnetic vector potential will facilitate a correction to $\mathbf{E} = -\mathrm{grad}V$, to make it generally true.

We shall start this discussion of the magnetic vector potential by showing that the magnetic field *can* be expressed as the curl of another field and we will then investigate some of the consequences and related matters. In the interests of simplicity we shall consider the field produced by an *element* of a conducting wire. The field of the entire wire can then be obtained by integration over its path.

The magnetic field at a point **r** due to a current I flowing in an element d**l** of a wire can be written as



$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}\times\mathbf{r}}{r^3}.$$

We now use the result

$$\frac{\mathbf{r}}{r^3} = -\mathrm{grad}\frac{1}{r}$$

to rewrite the magnetic field as

$$\mathrm{d}\mathbf{B} = -\frac{\mu_0 I}{4\pi}\mathbf{dl}\times\mathrm{grad}\frac{1}{r}.$$

Next we make use of the vector calculus identity

$$\mathrm{curl}\, a\mathbf{b} = a\,\mathrm{curl}\,\mathbf{b} - \mathbf{b}\times\mathrm{grad}\, a$$

to write

$$\mathrm{curl}\frac{\mathbf{dl}}{r} = \frac{1}{r}\mathrm{curl}\,\mathbf{dl} - \mathbf{dl}\times\mathrm{grad}\frac{1}{r}.$$

Now $\mathrm{curl}\,\mathbf{dl} = 0$ since this is like differentiating x with respect to x. Thus the expression for d**B** takes the form

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\mathrm{curl}\frac{\mathbf{dl}}{r}.$$

This means that we *can* write **B** as the curl of a vector **A**:

$$\mathbf{B} = \mathrm{curl}\mathbf{A}$$

where the contribution to this vector from current flowing in the element **dl** of the wire is given by

$$\mathrm{d}\mathbf{A} = \frac{\mu_0 I}{4\pi}\frac{\mathbf{dl}}{r}. \tag{9.3}$$

The vector **A** is known as the magnetic vector potential.

To check for consistency, we can take the divergence of **B** expressed in terms of **A**

$$\mathrm{div}\mathbf{B} = \mathrm{div}\,\mathrm{curl}\mathbf{A} = 0$$

since div curl is identically zero. So we see that expressing **B** in this form automatically ensures that div**B** = 0. In electrostatics we saw that the relation curl**E** = 0 implied that **E** could be expressed as the gradient of a scalar potential since curl grad is zero. And in the magnetic case we now see that the relation div**B** = 0 implies that **B** can be expressed as the curl of a vector potential since div curl is zero.

There is an important connection between the magnetic vector potential and magnetic flux. Recall the definition of curl, embodied in Stokes's theorem

$$\iint_{\text{area}} \text{curl}\,\mathbf{A}\,.\,\text{d}\mathbf{a} = \oint_{\text{perimeter}} \mathbf{A}\,.\,\text{d}\mathbf{r}$$

where we have used the vector argument **A**. Since **B** = curl**A**, the left hand side of this equation is simply the magnetic flux through the area. Thus

$$\Phi = \oint_{\text{perimeter}} \mathbf{A}\,.\,\text{d}\mathbf{r}\,; \tag{9.4}$$

the line integral of **A** around a closed loop gives the magnetic flux threading the loop.

⌈In regions of space where there are no currents present it *is* possible to express **B** as the gradient of a scalar potential and that potential will obey a Laplace equation. Then it will be easier to use this potential as an intermediary in the calculation of **B**. The interior of a solenoid would be one such example. ⌋

9.3 Fields from potentials

Now let us look at the problem of incorporating electromagnetic induction into the expression

$$\mathbf{E} = -\text{grad}V\,.$$

As it stands, this expression is incompatible with

$$\text{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

since curl grad of any scalar is zero.

Let us express the curl **E** equation in terms of the magnetic vector potential

$$\text{curl}\mathbf{E} = -\frac{\partial}{\partial t}\text{curl}\mathbf{A}\,.$$

This can be written as

$$\text{curl}\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0\,.$$

Now we know that any vector whose curl vanishes can be written as the gradient of a scalar since curl grad is identically zero. From this it follows that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = \text{grad of a scalar}\,.$$

This equation is clearly true in the time-independent case, where we identify the scalar as minus the electric potential V

$$\mathbf{E} = -\text{grad}V\,.$$

Thus in the general case we now have

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\text{grad}V\,,$$

or

$$\mathbf{E} = -\mathrm{grad}V - \frac{\partial \mathbf{A}}{\partial t}. \tag{9.5}$$

This is the general result, valid in dynamic as well as static cases. Thus we have succeeded in rehabilitating the deficient $\mathbf{E} = -\mathrm{grad}V$. We conclude that the electric and magnetic fields may be written in terms of the scalar and vector potentials as

$$\left.\begin{aligned} \mathbf{E} &= -\mathrm{grad}V - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \mathrm{curl}\mathbf{A}. \end{aligned}\right\} \tag{9.6}$$

9.4 Self inductance

We now proceed to the discussion of inductance. As we stated above, a changing current will produce a varying magnetic field, and the varying magnetic field will induce a voltage. Equation (9.1) tells us how a current in a solenoid will produce a magnetic field. And since the field was seen to be uniform over the area a of the coil, the magnetic flux through a single turn is $\mu_0 nIa$. If the length of the coil is l, then the number of turns is given by nl. And then the total magnetic flux Φ linking the solenoid is

$$\Phi = \mu_0 n^2 Ial.$$

We know from Ampère's law that the flux (linkage) must be proportional to the current. The constant of proportionality depends, essentially, on geometrical properties of the conductor. Let us denote this quantity by L, so that for the long solenoid

$$L = \mu_0 n^2 al \tag{9.7}$$

And in general

$$\Phi = L\,I. \tag{9.8}$$

We shall see that L is what we know as the inductance of the solenoid.

From Faraday's law we know that a varying flux will cause a voltage:

$$V = \frac{\mathrm{d}\Phi}{\mathrm{d}t},$$

ignoring the minus sign since we may choose the direction in which to measure the potential. Therefore if the current in the coil is changing, there will be a voltage across the coil given by

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t}. \tag{9.9}$$

This is the conventional expression for (self) inductance. Recall its magnitude for a long coil:

$$L = \mu_0 n^2 la .$$

Observe that L is proportional to the *volume la* of the coil and the *square* of the number of turns, as hinted at the start of Section (9.1). As you should know, the unit of inductance is the *henry*.

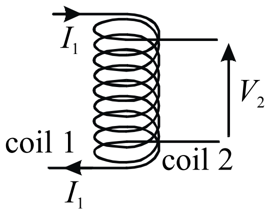
If the solenoid is not "infinitely" long then there is not total flux linkage with all the turns. Then the inductance is somewhat less than that of Equation (9.7). In such cases a 'fudge' factor α (known as Nagaoka's factor) is introduced. We then write

$$L = \mu_0 n^2 la\alpha \tag{9.10}$$

where the factor α is a function only of the *aspect ratio* of the coil: its ratio of radius to length. If the aspect ratio of the solenoid is x then a good approximation to α is $1/(1 + 0.9x)$.

9.5 Mutual inductance

By an extension of the arguments of the previous section, it follows that if two coils are in close proximity then a varying current in one coil will induce a voltage in the other and *vice versa*. This effect is known as *mutual inductance*. And such a coupled assembly of coils is known as a *transformer*.



Let us firstly assume that the two coils occupy the same space so that all the flux which links one coil also links the other. We are thus assuming here that the lengths and areas are the same for both coils. The only difference we permit at this stage is in the number of turns per unit length. We shall specify that coil 1 has n_1 turns per unit length and coil 2 has n_2 turns per unit length.

If there is a current I_1 flowing in coil 1 then this will produce a magnetic field

$$B = \mu_0 I_1 n_1 .$$

Denoting by l the length, and a the area of the coils, the number of turns in coil 2 is given by $n_2 l$. Then the magnetic flux linking the turns of coil 2 is given by

$$\begin{aligned}\Phi_2 &= n_2 lBa \\ &= \mu_0 n_1 n_2 la\, I_1\end{aligned}$$

and the voltage across the second coil is then

$$V_2 = \frac{\mathrm{d}\Phi_2}{\mathrm{d}t} = \mu_0 n_1 n_2 la \frac{\mathrm{d}I_1}{\mathrm{d}t} . \tag{9.11}$$

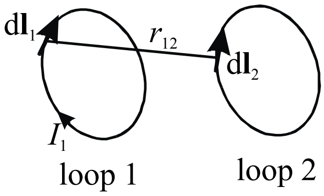
The constant of proportionality between the flux in one coil and the current in the other, or the voltage in one coil and the rate of change of current in the other is known as the *mutual inductance*, denoted by M. Thus we write

$$\left.\begin{aligned}\Phi_2 &= M_{21} I_1 \\ V_2 &= M_{21}\frac{\mathrm{d}I_1}{\mathrm{d}t}\end{aligned}\right\} \tag{9.12}$$

where, in this idealised case M is given by

$$M_{21} = \mu_0 n_1 n_2 la. \tag{9.13}$$

This expression is symmetric in the indices 1 and 2, demonstrated for the idealised case where the coils occupy the same space. This implies that if the current were to flow in coil 2 then the same voltage would be induced in coil 1. This is a very general property, independent of the shape and relative locations of the coils, as we shall now show.



We calculate the vector potential at loop 2 due to the current I_1 flowing in loop 1. The contribution to the vector potential at the position of $\mathbf{dl}_2$ due to the current in the element $\mathbf{dl}_1$ of loop 1 is, Equation (9.3)

$$\mathrm{d}\mathbf{A}_2 = \frac{\mu_0 I_1}{4\pi}\frac{\mathbf{dl}_1}{r_{12}}.$$

Then at this point the total vector potential is given by integrating the expression around the whole of loop 1:

$$\mathbf{A}_2 = \frac{\mu_0 I_1}{4\pi}\oint_{\text{loop 1}}\frac{\mathbf{dl}_1}{r_{12}}.$$

The magnetic flux threading loop 2 is then found by taking the line integral of $\mathbf{A}_2$ around loop 2

$$\Phi_2 = \int_{\text{loop 2}} \mathbf{A}_2 \,.\, \mathbf{dl}_2$$

so in this case

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi}\oint_{\text{loop 2}}\oint_{\text{loop 1}}\frac{\mathbf{dl}_1 \,.\, \mathbf{dl}_2}{r_{12}}.$$

This gives a general expression for calculating the mutual inductance

$$M_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0}{4\pi}\oint_{\text{loop 2}}\oint_{\text{loop 1}}\frac{\mathbf{dl}_1 \,.\, \mathbf{dl}_2}{r_{12}} \tag{9.14}$$

but this is seen to be symmetric in the indices 1 and 2; the flux in loop 2 due to a current in loop 1 is equal to the flux in loop 1 due to the same current in loop 2. And thus we conclude that

$$M_{12} = M_{21}. \tag{9.15}$$

9.6 Coupling coefficient and matrix representation

We see that for two coils, wound in the same place (including the fudge factor α), we have:

$$L_1 = \mu_0 n_1^2 la\alpha$$
$$L_2 = \mu_0 n_2^2 la\alpha \,.$$

If we calculate the mutual inductance between the coils, then clearly the same fudge factor would apply. In other words we would have

$$M = \mu_0 n_1 n_2 la\alpha \,.$$

The mutual inductance is thus seen to be equal to the *geometric mean* of the self inductance of the individual coils.

$$M = \sqrt{L_1 L_2} \ .$$

If there is not complete coupling between the coils then M would be less than this value. For widely separated coils, clearly the mutual inductance would be zero. To treat the general case we introduce a *coupling coefficient* K, and write

$$M = K\sqrt{L_1 L_2} \ . \tag{9.16}$$

We finish this section by writing the equations for the currents and voltages in a transformer in the general case. If there is a current in both coils then the voltage in a given coil depends on the varying current in that coil and the varying current in the other coil:

$$\begin{aligned} V_1 &= L_1 \frac{\mathrm{d}I_1}{\mathrm{d}t} + M \frac{\mathrm{d}I_2}{\mathrm{d}t} \\ V_2 &= M \frac{\mathrm{d}I_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}I_2}{\mathrm{d}t} \end{aligned} \tag{9.17}$$

which may be conveniently expressed in matrix form as

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} . \tag{9.18}$$

9.7 Energy of a magnetic field

When we evaluated the energy of an electric field we did it in an indirect manner, by calculating the work needed to establish a given potential across a capacitor. Then the potential was related to the electric field. The merit of using a capacitor was that within its interior the **E** field is uniform. We now evaluate the energy of a magnetic field in a parallel fashion. We will calculate the work needed to establish a given *current* in a long *inductor*. The merit of using a long inductor is that within its interior the **B** field is uniform.

The defining property of an inductor is Equation (9.9):

$$V = L \frac{\mathrm{d}I}{\mathrm{d}t} .$$

So in a small time dt the current will change by a small amount dI:

$$V\mathrm{d}t \ = \ L\mathrm{d}I.$$

If we multiply both sides of this equation by the current I, giving

$$IV\mathrm{d}t \ = \ L\, I\mathrm{d}I$$

then on the left hand side IV gives the power, so multiplying this by the time interval dt gives the work done during this interval, which we denote by dW. Thus we have

$$dW = L\,I dI,$$

which we may integrate up from an initial current of zero to a final current I:

$$\left.\begin{aligned} W &= L\int_0^I i\,di \\ &= \frac{1}{2}LI^2. \end{aligned}\right\} \qquad (9.19)$$

This gives the energy stored in an inductor of inductance L carrying a current of I amperes.

Now the current is related to the magnetic field through Equation (9.1), which we write as

$$I = \frac{B}{\mu_0 n}.$$

And since L is related to the various geometric factors by Equation (9.10):

$$L = \mu_0 n^2 la,$$

these may be combined to give

$$W = \frac{1}{2\mu_0}B^2 la \qquad (9.20)$$

where we see that the n cancels out. Observe that la is the volume of the region containing the magnetic field, which tells us that there is a magnetic energy *density* U_B given by

$$U_B = \frac{1}{2\mu_0}B^2 \qquad (9.21)$$

So in a combined electric and magnetic field, adding the two contributions to the field energy, we have

$$U = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2. \qquad (9.22)$$

This is the general expression for the energy density of an electromagnetic field. The field energy contained in a region of space is then found by integrating the field energy density over the volume of the region. Recall that in Section (7.6) we considered the question of energy conservation in an electromagnetic field, where we introduced the Poynting vector.

9.8 Finding the potentials

How are the potentials determined by the sources? We know that in the electro*static* case the electric potential obeys the Poisson equation

$$\nabla^2 V = -\rho/\varepsilon_0 .$$

We now ask how the electric scalar potential V and the magnetic vector potential $\mathbf{A}$ are related to the electric and magnetic sources ρ and $\mathbf{j}$ in the general dynamic case.

In the electric case we combine

$$\mathrm{div}\mathbf{E} = \rho / \varepsilon_0 \qquad \text{and} \qquad \mathbf{E} = -\mathrm{grad}V - \partial\mathbf{A}/\partial t$$

to give

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} - \frac{\partial}{\partial t}\mathrm{div}\mathbf{A} \tag{9.23}$$

while in the magnetic case we combine

$$\mathrm{curl}\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \mathrm{curl}\mathbf{A}$$

to give (recall curl curl = grad div – ∇^2)

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} + \mathrm{grad}\left(\mathrm{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right). \tag{9.24}$$

Equations (9.23) and (9.24) are differential equations from which the potentials V and $\mathbf{A}$ may be determined from the sources ρ and $\mathbf{j}$ (and the appropriate boundary conditions). However the equations do look rather complicated. And simplification is possible. It is only the gradient of V that is important so one can add an arbitrary constant to V without changing the observable electric field; this we know already. But similarly, since curl grad ≡ 0, we can add the gradient of an arbitrary scalar field to $\mathbf{A}$ without changing the observable magnetic field. It then follows that we can simplify Equations (9.23) and (9.24) through the imposition of supplementary restrictions on V and $\mathbf{A}$ which have no effect on the observable $\mathbf{E}$ and $\mathbf{B}$.

Helmholtz's theorem tells us that a vector field is determined once its curl and its divergence are specified. We have no choice with the curl of $\mathbf{A}$; this must give the correct value for the $\mathbf{B}$ field. But the divergence of $\mathbf{A}$ is another matter; we can choose this to be whatever we like; it can even depend on time if we wish. The choice in the precise specification of $\mathbf{A}$, for instance stating what its divergence is, is called the choice of *gauge*.

An elegant choice of gauge is to make the bracket in the right hand side of Equation (9.24) to be zero. That is, we choose div $\mathbf{A}$ to be

$$\mathrm{div}\mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}. \tag{9.25}$$

Not only does it remove the grad term in Equation (9.24), it also converts the

$-\partial \operatorname{div} \mathbf{A}/\partial t$ term of Equation (9.23) to $\left(1/c^2\right)\partial^2 V/\partial t^2$

This separates electric and magnetic effects; V is given solely in terms of ρ and **A** is given solely in terms of **j**. We have two inhomogeneous wave equations; electric and magnetic effects are untangled and the vector equation for **A** is equivalent to three scalar equations for the its three components (so long as we are using rectangular Cartesian coordinates).

$$\left.\begin{aligned} \Box^2 V &= -\rho/\varepsilon_0 \\ \Box^2 \mathbf{A} &= -\mu_0 \mathbf{j}. \end{aligned}\right\} \qquad (9.26)$$

This is called the *Lorentz gauge*. Actually this is named after the wrong Lorentz; it was introduced by L. Lorenz, not H. A. Lorentz!

A special case of the Lorentz gauge occurs in the static case: when time derivatives are zero. Then div **A** = 0; we have the *Coulomb gauge*. And we then recover the conventional Poisson's equations

$$\left.\begin{aligned} \nabla^2 V &= -\rho/\varepsilon_0 \\ \nabla^2 \mathbf{A} &= -\mu_0 \mathbf{j}. \end{aligned}\right\} \qquad (9.27)$$

When the magnetic field is uniform there is a special gauge that is particularly convenient to use. The *Landau gauge* is a special case of the Coulomb gauge (which is a special case of the Lorentz gauge.) In the Landau gauge the vector potential is specified by

$$\mathbf{A} = -B_z y \hat{\mathbf{x}}. \qquad (9.28)$$

The **B** field is found by taking the curl of **A**:

$$\begin{aligned} \mathbf{B} = \operatorname{curl} \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -B_z y & 0 & 0 \end{vmatrix} \\ &= B_z \hat{\mathbf{z}}. \end{aligned}$$

In other words the vector potential given by the Lorentz gauge Equation (9.28) results in a uniform magnetic field pointing in the z direction. This proves useful in many practical applications; it is the simplest way of writing a vector potential which gives a uniform magnetic field.

The solution of Equations (9.27) are given by:

$$\left.\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \iiint_{\text{volume}} \frac{\rho(\mathbf{r})}{r} \mathrm{d}v \\ \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint_{\text{volume}} \frac{\mathbf{j}(\mathbf{r})}{r} \mathrm{d}v. \end{aligned}\right\} \qquad (9.29)$$

We shall not consider the solutions in the dynamic case; these are two complex for this course. The components of **A** are expressed in terms of the components of **j** just at the value of V is expressed in terms of the value of ρ (to within a constant factor). This shows that just as V augments **A** to create a four-vector, so the charge density augments the current density to give a four-vector.

It is also worthwhile to point out that while the equation relating **A** and **j** is certainly a vector equation, the structure of the equation is such that A_x is determined by j_x alone, A_y by j_y and A_z by j_z. In other words this vector equation is equivalent to three independent scalar equations (so long as we are using rectangular Cartesian coordinates).

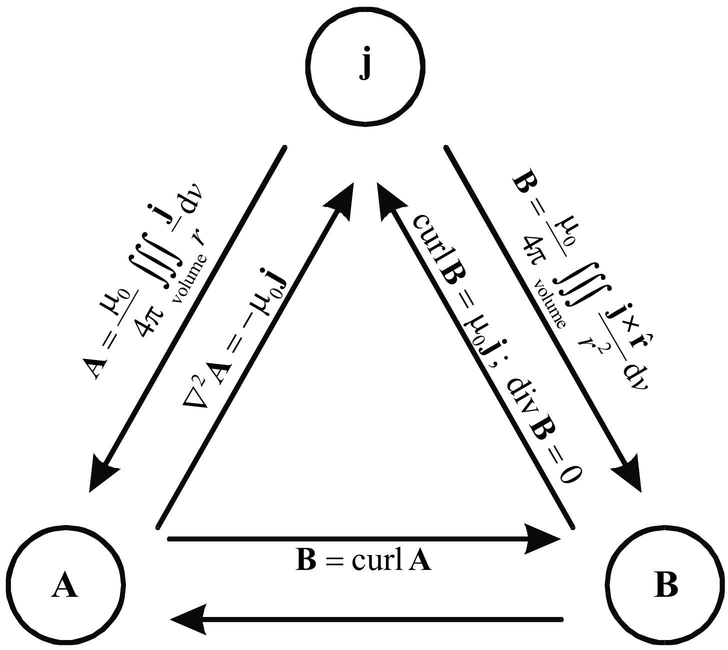
For completeness we give the expressions for **E** and **B** in terms of their sources in the static case

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}} \frac{\rho(\mathbf{r})}{r^2}\hat{\mathbf{r}}\,\mathrm{d}v \\ \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\iiint_{\text{volume}} \frac{\mathbf{j}(\mathbf{r})}{r^2}\times\hat{\mathbf{r}}\,\mathrm{d}v\,.\end{aligned}\right\} \tag{9.30}$$

Observe that the equations for V and **A** are much simpler.

9.9 Summary of magnetostatic results

The magnetostatic relations between the three quantities **B**, **A** and **j** are summarised in the following diagram, borrowed from *Introduction to Electrodynamics* by D. J. Griffiths. This parallels the similar diagram for the electrostatic case given in Section (4.6).



When you have completed this chapter you should:

- be able to calculate **B** inside a long solenoid;
- be familiar with the concept of magnetic vector potential;
- be able to calculate **A** from an arbitrary current distribution;
- understand the connection between **A** and magnetic flux;
- be able to generalise **E** = −grad$V$ to the case of electromagnetic induction, using **A**;
- know the meaning of self inductance and be able to calculate L for a long solenoid;
- understand that L is proportional to the volume of a solenoid and the square of the number of turns;
- know the meaning of mutual inductance and be able to calculate M for co-positioned long solenoids;
- understand why and be able to demonstrate that $M_{12} = M_{21}$;
- be familiar with the idea of coupling coefficient;
- be able to write the $V - I$ relation for a transformer in matrix form;
- interpret the work done in establishing a magnetic field in an inductor in terms of magnetic field energy;
- be able to calculate the **E** and **B** fields from V and **A** in the general case;
- understand the choice of gauge in the specification of **A** and be familiar with the Lorentz gauge and the Coulomb gauge.