VIII ELEMENTARY ELECTRODYNAMICS

8.1 Motion of a charge in an electric field

The force on a particle of charge q in an electric field is given by the electric part of the Lorentz force:

$$\mathbf{F} = q\mathbf{E}\,, \tag{8.1}$$

and from Newton's second law, this is equal to the mass of the particle times its acceleration. Thus

$$m\ddot{\mathbf{r}} = q\mathbf{E}\,, \tag{8.2}$$

so that the acceleration is given by

$$\ddot{\mathbf{r}} = \frac{q}{m}\mathbf{E}\,. \tag{8.3}$$

We will encounter the ratio q/m quite frequently, and one should appreciate why q and m occur together in this way. The equation analogous to Equation (8.2) for the gravitational interaction would be

$$m\ddot{\mathbf{r}} = m\mathbf{g}$$

where **g** is the *gravitational* field. Observe that here we have the mass of the particle m on both sides of the equation. On the left side it represents the *inertial* aspect of mass, while on the right side it is the *gravitational* aspect of mass. It was Newton who first realised that mass had this dual manifestation, although we had to wait for Einstein to elevate this experimentally observed phenomenon to a fundamental principle, the *equivalence principle*.

When we come to the electric force, when Equation (8.2) describes the situation, then the charge q takes the place of the gravitational mass. Then it is the ratio q/m that appears and it no longer cancels; this ratio will be different for different objects.

Since a charged particle in a constant electric field will be subject to a constant acceleration, it is a simple matter to solve the (second order) equation of motion to yield

$$\mathbf{r}(t) = \mathbf{r}(0) + \dot{\mathbf{r}}(0)t + \frac{1}{2}\frac{q}{m}\mathbf{E}t^2 \tag{8.4}$$

This is the way a charged particle would behave in a constant electric field. It is different from the motion of a charge in a conductor, as we studied using the Drude model of Section (5.3). There we were dealing with the flow of charge carriers in matter, and we saw that the collisions of the carriers resulted in a 'terminal velocity' which was proportional to the electric field.

8.2 Motion of a charge in a magnetic field

The force on a charged particle here is the magnetic part of the Lorentz force:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{8.5}$$

In this case the force is *perpendicular* to the velocity, the condition for *circular motion*. Let us write the central force as

$$F = qvB_\perp ,$$

where $B_\perp$ is the component of the magnetic field perpendicular to the velocity.

The equation for circular motion may be written as

$$F = mv^2/r$$

so that

$$qvB_\perp = mv^2/r . \tag{8.6}$$

The radius of the circular motion is then

$$r = mv/qB_\perp. \tag{8.7}$$

Again observe that q and m appear only as a quotient.

The *angular velocity* ω is the time derivative of the (radian) angle. This may be expressed as

$$\omega = v/r$$

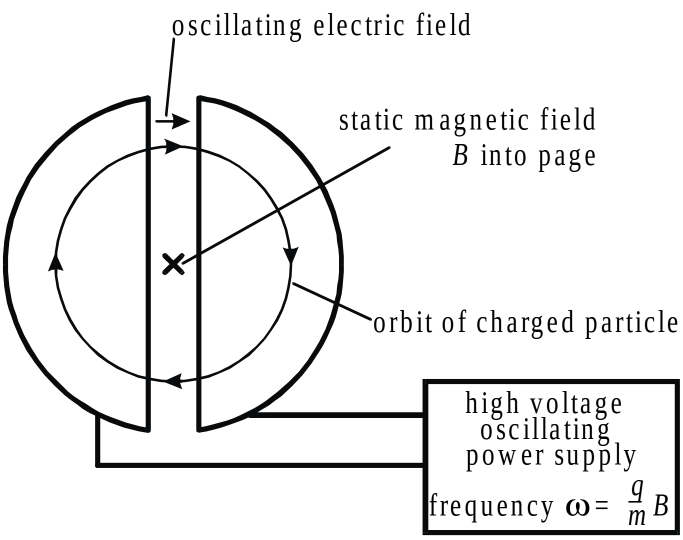
so that we obtain

$$\omega = (q/m)B_\perp, \tag{8.8}$$

the angular velocity is *independent* of the speed of the particle and the radius of the circle. The angular velocity is only proportional to the (perpendicular component) of the magnetic field and the familiar ratio q/m.

8.3 The cyclotron

The cyclotron is a device that exploits the behaviour of charged particles in a magnetic field, to accelerate them to very high speeds. The cyclotron was pioneered by Ernest O. Lawrence. He started by making models that sat on a table, and he progressed to machines the size of buildings. As we shall see, the efficacy of the cyclotron depends on the fact that the angular velocity of the particle is independent of the speed of the particles. Lawrence believed that he could continue to make bigger and better cyclotrons capable of faster and faster speeds. In fact special relativity tells us that at high very speeds, since the mass increases, the angular velocity decreases. This would reduce the efficiency of the cyclotron. Lawrence did not really believe in relativity and he wanted to spend millions of dollars on a gigantic cyclotron. It took all the persuasive powers of J. Robert Oppenheimer to convince Lawrence that it would be a waste of money.



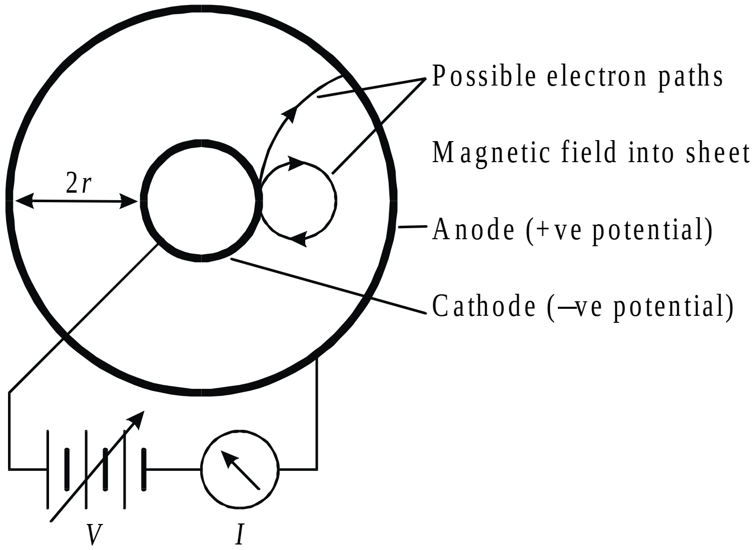
The layout of a cyclotron is shown in the diagram. There is a magnetic field pointing into the sheet and the motion of the charged particles is therefore a circle in the plane. The clever part is the two D-shaped conductors. The high voltage oscillating supply creates an oscillating electric field in the space between the Ds. If the frequency is arranged to be equal to the cyclotron frequency, then as the charges move in the top space they will be accelerated towards the right and when they are in the bottom space they will be accelerated towards the left. Thus the oscillating electric field is feeding energy into the particles, making them move faster. As their speed increases the radius of the circular motion will increase. At a certain radius the magnetic field abruptly stops and the charges continue to move in a straight line at their final velocity. This velocity depends on the exit radius

$$v_{\mathrm{final}} = \omega r_{\mathrm{exit}} = (q/m)\, B_{\perp} r_{\mathrm{exit}} . \tag{8.9}$$

Observe that the exit velocity or energy of the particles depends only on two variables over which one has experimental control: the strength of the magnetic field and the radius of the orbit.

8.4 The magnetron

The magnetron is a thermionic vacuum device for generating microwaves; it is the essential component of microwave cookers. The magnetron utilises both electric and magnetic fields, and it has a cylindrical geometry with concentric anode and cathode. The anode is held at a positive potential with respect to the cathode. So when heated, the electrons emitted from the cathode will travel towards the anode. However the perpendicular magnetic field will also alter the motion.



Qualitatively, it is clear that the trajectory of an electron will be a curve. The larger the magnetic field, the more curved the path. For sufficiently large magnetic fields or sufficiently small electric fields the motion will be essentially circular, with the electrons missing the anode and returning to the cathode. Thus at a critical value of the electric field or the magnetic field no current will flow from anode to cathode. Since, as we have seen, the equation of motion for an electron will depend on the q/m ratio, it follows that the cut-off condition for the current will depend on q/m. Knowing the value of the magnetic field and the electric field thus permits a determination of the q/m ratio of the electron.

Without solving the full equation of motion for an electron in a combined electric and magnetic field, we may obtain an order of magnitude estimation in the following way. The critical condition is when the electron orbit just grazes the surface of the anode, when the radius of the circular motion is r. From Equation (8.7) this means that the speed v of the electron will be

$$v = Br\frac{q}{m}$$

so that its kinetic energy $\frac{1}{2}mv^2$ is

$$\frac{B^2r^2q^2}{2m}.$$

But this energy comes from the electric field, which may be expressed in terms of the potential difference V between the anode and cathode:

$$qV = \frac{B^2r^2q^2}{2m}.$$

Here the extra factor of q cancels, so that the ratio q/m for the electron is

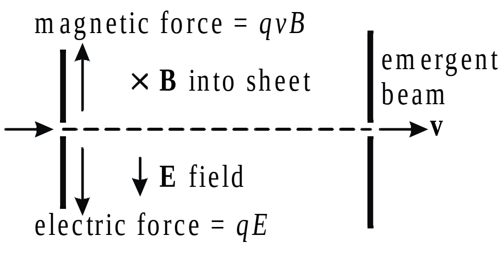
$$\frac{q}{m} = \frac{2V}{B^2r^2}. \qquad (8.10)$$

An exact solution to the electron equation of motion, taking account of the correct variation of the electric field for the cylindrical geometry will lead to an expression similar, but having a different numerical coefficient.

One of the original uses of the magnetron was as a magnetically controlled switch. Although it is used for the measurement of the electronic q/m ratio, the main use is in the generation of microwaves, particularly in microwave cookers.

8.5 Velocity selector

If there is an electric field and a magnetic field oriented perpendicularly, and a charged particle moves perpendicular to both then it will experience a magnetic force and a magnetic force in opposite directions. The magnetic force depends on the velocity of the particle whereas the electric force does not. It therefore follows that for one particular velocity the electric force will exactly balance the magnetic force. And if there is no net force on such a particle then it will move in a straight line.



The forces balance when

$$qE = qvB,$$

in other words for the motion of the particles to be not deflected,

$$v = E/B, \qquad (8.11)$$

independent of the charge (and the mass) of the particle.

This provides a means for selecting those particles moving with a specified velocity. The technique is used in various instruments including mass spectrographs.

8.6 Flux encircled by a trajectory

The path of a charged particle in a magnetic field is a circle. More precisely, the path projected in the plane perpendicular to the magnetic field is a circle. In Equation (8.7) we saw that the radius of the circle was given by

$$r = mv/qB.$$

We shall express this behaviour in terms of the angular momentum of the particle, and then explore the consequence of the conservation of angular momentum.

The angular momentum (along the direction of the field) may be expressed as

$$L = mvr\ ,$$

from which the velocity can be eliminated, so that

$$L = qr^2B. \tag{8.12}$$

This is a suggestive expression. Firstly, note that r^2 is proportional to the area of the circular trajectory, and secondly note that the area multiplied by B is the magnetic flux Φ contained within the orbit:

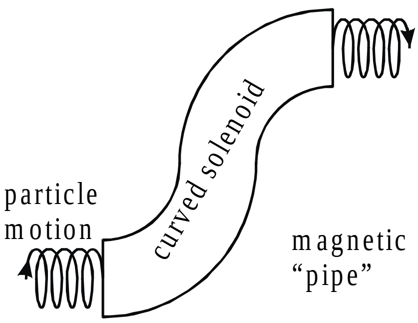
$$\Phi = \pi r^2B.$$

Thus in terms of the angular momentum, the flux is given by

$$\Phi = \pi L/q. \tag{8.13}$$

We see that for a given charge the flux enclosed by the trajectory depends *only* on the angular momentum of the particle. And then the conservation of angular momentum implies that the enclosed flux is constant.

Now let us consider the more general case where the trajectory of the charged particle is a spiral in the magnetic field. So long as the field varies slowly over each turn of the spiral we may apply the above result that the enclosed flux is constant. This means that if the lines of flux follow some curved path, for instance inside a bent solenoid, then the charged particles will continue to spiral around the flux along the path of the solenoid. This provides a means for piping beams of electrons or positrons from one place to another. This is also the explanation of why charged particles emitted from the sun travel along (while spiralling around) the lines of **B** in the magnetosphere.



If we consider the case where the intensity of the magnetic field varies with position then the radius of the spiral will vary with the intensity of the field. The radius will be greater in regions of low field and smaller in regions of high field. When energy conservation is considered this leads to the possibility of confining the particles to a given region of space: the magnetic mirror.

8.7 The magnetic mirror

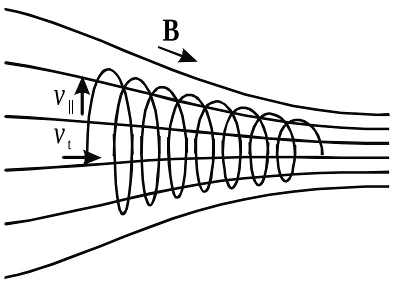
Conservation of angular momentum determines the motion of the charged particles in the plane perpendicular to the direction of the magnetic field. In a larger field, where the radius of the motion is smaller, the tangential speed is thus greater. The total kinetic energy may be expressed as the sum of contributions from the circular motion and the velocity along the lines. And it is the total energy that is conserved. We therefore write

$$E = E_{\mathrm{rot}} + E_{\mathrm{tr}}$$

where the rotational energy is given by

$$E_{\mathrm{rot}} = \frac{1}{2} m v_t^2 = \frac{L^2}{2mr^2} = \frac{LqB}{2m} \tag{8.14}$$

and v_{t} is the tangential speed. The translational energy is given by



$$E_{tr} = \frac{1}{2} m v_{\|}^2 \tag{8.15}$$

where $v_{\|}$ is the velocity along the direction of **B**.

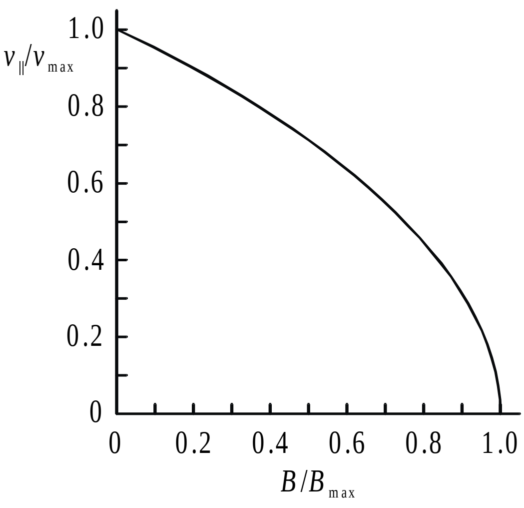
Conservation of energy implies that as the charged particle moves to regions of higher field, where its rotational speed increases, the velocity along the field direction must correspondingly increase.

This means that there will be a maximum field B_{max} for which the parallel velocity is zero. As the particle approaches this field it slows down to zero (parallel) velocity, changes direction, and returns in the opposite direction: as if it has hit a mirror. The velocity along the field is given from the energy expression

$$\frac{1}{2} m v_{\|}^2 = E - \frac{LqB}{2m} . \tag{8.16}$$

The maximum field is then given by $2mE/qB$ and we can then express the velocity as

$$v_{\|} = \frac{\sqrt{Lq}}{m} \sqrt{B_{\mathrm{max}} - B} . \tag{8.17}$$



The accompanying graph shows how the velocity along the field varies with the magnitude of the field. We see that if there are regions of low field surrounded by regions of high field, then charged particles will be confined to regions below the maximum field. In other words, it is as if the charged particles are confined between two mirrors.

When you have completed this chapter you should:

- be able to calculate the motion in static electric and magnetic fields;
- appreciate that a charge cam move in a circular orbit in a uniform magnetic field;
- be able to calculate the properties of this circular motion;
- understand the principles of operation of the cyclotron and appreciate the limitations on its behaviour set by special relativity;
- understand the principles of operation of the magnetron and how it can be used to determine the e/m ratio of the electron;
- understand the principles of operation of the velocity selector and that its behaviour is independent of the charge and mass of the particles;
- be able to calculate the magnetic flux contained within a circular trajectory of a charged particle;
- understand how this may be used to make a ‘magnetic pipe’;
- understand the operation of a ‘magnetic mirror’.