VII MAXWELL'S EQUATIONS

7.1 The story so far

In this section we will summarise the understanding of electromagnetism which we have arrived at so far. We know that there are two fields which must be considered, the electric field \mathbf{E} and the magnetic field \mathbf{B} . And we know that in these fields a charge q will experience a force: the Lorentz force:

$$\mathbf{F} = q\{\mathbf{E} + \mathbf{v} \times \mathbf{B}\}. \tag{7.1}$$

The fields are produced by charges; static charges produce \mathbf{E} fields and moving charges also produce \mathbf{B} fields. These effects may be summarised in the expressions for the divergence and the curl of \mathbf{E} and \mathbf{B} :

$$\operatorname{div} \mathbf{E} = \rho/\varepsilon_0, \qquad (7.2)$$

$$\operatorname{curl} \mathbf{E} = 0, \qquad (7.3)$$

$$\operatorname{div}\mathbf{B} = 0, \qquad (7.4)$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{j} \,. \tag{7.5}$$

We may say that Equations (7.2) to (7.5) express the *active* aspects of charge, while Equation (7.1) expresses the *passive* aspect. And as far as the active aspects are concerned, we see that **B** has a curl but no divergence, while **E** has a divergence but (at the moment – as we have not encountered electromagnetic induction) no curl. A field with a curl but no divergence is called a *solenoidal* field, while one with a divergence but no curl is called an *irrotational* field.

Helmholtz's theorem of vector calculus tells us that a field is fully specified in terms of its divergence and curl; *they indicate the full nature of the field*. Thus, for instance, the field lines of a solenoidal field have no end points; they must therefore consist of closed loops. And conversely, there can be no *vortices* in an irrotational field.

As we have hinted a number of times, Equations (7.2) to (7.5) are not complete; there are other ways in which the fields can be produced. We will treat electromagnetic induction in the next section, when we shall see that curl**E** is not necessarily zero; Equation (7.3) will have to be modified. And then we will examine that special step made by Maxwell when he realised that Ampères law, or Equation (7.5) was not quite complete and there *must* be another source of magnetic fields. This involves the introduction of the displacement current. At that stage we will have reached the pinnacle of Maxwell's equations and we can go both forward to explore their consequences and backwards to examine their foundations in some more detail.

7.2 Faraday's law



Michael Faraday made the important observation that a moving magnetic field has the potential for causing an electric current to flow. If a magnetic field is moved relative to a conductor or if a conductor moves in a magnetic field then there will be a force on the charges in the conductor. Although for Faraday this was an experimental observation, we shall see how the effect follows from what we know already; in the appropriate frame of reference this is nothing more than the magnetic part of the Lorentz force. We shall study this by means of the following model system.

Consider the indicated conductor of length l moving along a conducting frame. A charge q in the conductor will feel a force **F** given by

$$\mathbf{F} = q \, \mathbf{v} \times \mathbf{B} \, .$$

The *electromotive force* \mathcal{E} is defined as the work done when a unit charge moves around a closed loop through the action of such a force. Recall that in the electrostatic case this was found to be zero. Now, however, things are different.



No work is done as the charge moves along the fixed frame; the only contribution comes from movement through the length **l**. Thus the work done on a charge q is given by $-q\mathbf{l} \cdot (\mathbf{v} \times \mathbf{B})$, the minus sign occurring because **F** and **l** point in opposite directions. The electromotive force, the work done on unit charge is then

$$\mathcal{E} = -\mathbf{l} \cdot (\mathbf{v} \times \mathbf{B}),$$

which may be transformed, using a triple product identity in Equation (3.11), to

$$\mathcal{E} = -(\mathbf{l} \times \mathbf{v}) \cdot \mathbf{B}.$$

Now $(\mathbf{l} \times \mathbf{v})$ is the rate of change of area in the direction *antiparallel* to **B**. And the dot product of the (correctly oriented area) and the **B** field is the magnetic flux Φ threading the area. So in terms of Φ we may express the induced EMF as

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{7.6}$$

which is referred to as Faraday's law.

If the entire frame remains fixed but the **B** field threading the loop changes, then Faraday's law will still apply. But now there is no *magnetic* force, as the velocity of the conductor is zero. So the EMF must, in this reference frame, be interpreted in terms of an *electric* field. Denoting by **E** the electric field pointing *along* the line **l**, Equation (7.6) can be written

$$\oint_{\substack{\text{closed}\\\text{loop}}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_{\substack{\text{enclosed}\\\text{area}}} \mathbf{B} \cdot d\mathbf{a}$$
$$= -\iint_{\substack{\text{enclosed}\\\text{area}}} \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{a}$$

which, from the definition of the curl, is equivalent to

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$
(7.7)

Physically this is telling us that a varying magnetic field is a source of \mathbf{E} – but it produces a *solenoidal* field and not the usual *irrotational* \mathbf{E} field of an electric charge. In such cases, obviously, \mathbf{E} can not be expressed as the gradient of a potential.

The minus sign in the equations describing electromagnetic induction has an important interpretation. Referring to the diagram of the conducting frame, the force on the charge is in the clockwise direction. The current flowing around the loop, as a consequence of this, will produce a magnetic field pointing *downwards*. In other words, the effect of electromagnetic induction is to try to produce a magnetic field to *oppose* the applied field. This is called Lenz's law.

7.3 Displacement current

Faraday's law is a modification to the elementary results, as summarised in Equations (7.2) to (7.5). Equation (7.3) is now modified and varying magnetic fields can produce electric fields. Maxwell realised that the converse was also the case; varying electric fields can produce magnetic fields; the two go hand in hand. Varying magnetic fields produce electric fields and varying electric fields produce magnetic fields.

We saw that Faraday's law involved no new physics; it followed from the application of the Lorentz force formula, and it resulted in a modification of the curl**E** equation. It dawned on Maxwell that Ampère's law, the curl**B** equation above was not quite right; it was incomplete. Maxwell saw there was a paradox and he resolved this by adding another term to the curl**B** equation.

It is true that a *steady* current cannot flow into a capacitor forever. But a varying current can flow, as the capacitor charges or discharges.



Now consider the application of Ampère's law to the capacitor indicated. The current *I* in the wire will cause a **B** field. The line integral of **B** around the closed loop is given, from Ampère's law, by μ_0 times the current *I*. This follows from considering the current penetrating the flat surface 1. But there is no restriction to a *flat* surface; the closed loop itself may be curved. Clearly *any* surface which is bounded by the closed loop may be used to evaluate the enclosed current. And this must include the deformed surface 2. But here the

penetrating current is zero! This is the paradox that confronted Maxwell. There is no problem in the steady state, but with varying currents something is not quite right.

When the current varies outside the capacitor the \mathbf{E} field varies inside. And since deforming the inside of the surface can have no effect on the line integral about its perimeter, it follows that the varying \mathbf{E} field inside the capacitor corresponding to the varying current outside must also be capable of creating the same \mathbf{B} field. Although there is no real electric current flowing in the capacitor, one imagines a fictitious current called the *displacement current* which also can participate in Ampère's law. At this stage the name displacement current means nothing, and it is unimportant. Later we will see the reason for the name.

Let us calculate the displacement current in the capacitor in terms of what is really there: the electric field. In Equation (2.21) we found the electric field in a capacitor in terms of its charge Q:

$$E = Q/\varepsilon_0 A$$

where *A* is the area of the plates. The current is the dertivative of the electric charge, so when the electric field inside the capacitor is changing, the current flowing is

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon_0 A \frac{\mathrm{d}E}{\mathrm{d}t} \,. \tag{7.8}$$

Inside the capacitor this is the displacement current.

The parallel with Faraday's law is quite striking. Since A times E is the electric flux, we can write the displacement current as

$$I = \varepsilon_0 \frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t}.$$

For Faraday's law we had a voltage given by the derivative of the *magnetic* flux. Here we have a current given by the derivative of the *electric* flux.

It is convenient (and necessary for the perfection of the curl**B** field equation) to talk in terms of the displacement current *density*. Dividing the result of Equation (7.8) by the area, and taking account of the direction we obtain the current density \mathbf{j}_d :

$$\mathbf{j}_{\mathrm{d}} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$
(7.9)

The reason for this discussion is that displacement current, just as real current, is capable of producing a magnetic field. And so the curl**B** equation has to be modified to

$$\operatorname{curl} \mathbf{B} = \mu_0(\mathbf{j} + \mathbf{j}_d),$$

or

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$
 (7.10)

This is the final Maxwell equation. It tells us that \mathbf{B} is produced by both electric currents and by varying electric fields. The set of Maxwell equations will be summarised in the next section.

7.4 Maxwell's equations

We have completed our discussion of the production of electric and magnetic fields by stationary and moving charges in free space. Thus far we have not considered dielectric and magnetic materials; we have not considered fields in matter at all. That will be done in a future section.

The first equation we obtained was that for divE. The source of E was seen to be electric charge and we found

$$\operatorname{div} \mathbf{E} = \rho / \varepsilon_0 , \qquad (M1)$$

a statement of Gauss's law. That equation followed directly from Coulomb's law alone and it remained true through subsequent discussions.

The second equation we found was for the curl of E. In the case of electrostatics we saw the curl to be zero, an indication that E was conservative. However later we encountered electromagnetic induction, and Faraday's law, which caused us to modify the result to

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$
 (M2)

The divergence of **B** was found to be zero, and there were no dynamic effects which caused us to reconsider this. If magnetic monopoles are found to exist then there would be a source term, but that is of no real consequence here. Thus we continue to write $\operatorname{div} \mathbf{B} = 0$. (M3)

Finally we come to the equation for curl**B**. We started with Ampère's law, which arose as a consequence of Ampère's formula for the force between current elements. But then we followed Maxwell's arguments to see that Ampère's law was incomplete and the problem was solved through the incorporation of the displacement current. Thus finally we obtained

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$
 (M4)

The equations have been numbered separately as they are so important, and we will often refer to them.

7.5 EM waves in free space

In free space, in the absence of charges and currents, the Maxwell equations simplify to

$$\operatorname{div} \mathbf{E} = 0 \tag{7.11}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.12}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{7.13}$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$
 (7.14)

In this form the symmetry of the equations is quite apparent.

One of the most important consequences of Maxwell's equations is the occurrence of electromagnetic waves. To show the possibility of electromagnetic waves propagating in free space we take the curl of one of the curl equations (it does not matter which one). Using the \mathbf{E} equation we obtain

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{curl}\left(-\frac{\partial \mathbf{B}}{\partial t}\right).$$

Since one of our vector calculus identities (Equation (3.34)) tells us that

$$\operatorname{curl}\operatorname{curl} = \operatorname{grad}\operatorname{div} - \nabla^2$$

and since the divergence of **E** here is zero, we obtain

$$\nabla^2 \mathbf{E} = \frac{\partial}{\partial t} \operatorname{curl} \mathbf{B} ,$$

on interchanging the space and time differentiation. The curl of **B** is given in Equation (7.14), so that we finally obtain

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
(7.15)

which we recognise as the wave equation for the vector \mathbf{E} , and an identical equation may be obtained for \mathbf{B} .

A wave travelling in the *x* direction may be written as $\cos k(x - vt)$ where *k* is the wave number and *v* is the velocity of propagation. If this is substituted into the wave equation, Equation (7.15), then the velocity of propagation of the waves is found to be $(\mu_0 \varepsilon_0)^{-1/2}$, a combination of the electric constant and the magnetic constant for free space. If one were following the historical development of the subject then the conclusion here is that **E** and **B** can propagate as waves with a velocity given in terms of the two measurable constants of electrostatics and magnetostatics. One would then measure ε_0 , perhaps from the inductance of a carefully constructed capacitor. And μ_0 could be similarly obtained from the inductance of a carefully constructed inductor. Then from these two quantities one would calculate the speed of propagation of the **E** and **B** waves and – surprise, surprise! the speed is the same as the measured speed of light. This is what Maxwell did, and after recovering from the shock, and collecting his thoughts, he suggested that perhaps light was just a form of electromagnetic radiation. He was right!

We should, at this stage, show that **E**, **B** and the direction of propagation are mutually perpendicular. This is taken up as a Problem. Ideas on polarisation follow from this.

7.6 Poynting's theorem – energy transport

We have seen that associated with an electric field **E** there is an electric field energy density U_E given by Equation (2.30)

$$U_E = \frac{\varepsilon_0}{2} E^2.$$

And we will see in Section (9.7) that associated with a magnetic field B there is a magnetic field energy density U_B given by Equation (9.21)

$$U_B = \frac{1}{2\mu_0} B^2.$$

Thus associated with both the E and B field there is a field energy density

$$U=\frac{\varepsilon_0}{2}E^2+\frac{1}{2\mu_0}B^2.$$

We note, for reference, one further energy result. The power dissipated per unit volume, Equation (5.9), when charges move in a field is given by

$$p = \mathbf{j} \cdot \mathbf{E}$$
.

Our aim is to find an expression for the *flow* of energy associated with electric and magnetic fields. We note the vector identity

$$\operatorname{div}(\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl} \mathbf{E} - \mathbf{E} \cdot \operatorname{curl} \mathbf{B}$$

and we use the Maxwell equations for $\mbox{curl} E$ and $\mbox{curl} B$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

so that

div
$$(\mathbf{E} \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mu_0 \mathbf{E} \cdot \mathbf{j}$$

This can be tidied up as

$$\frac{1}{\mu_0} \operatorname{div} \left(\mathbf{E} \times \mathbf{B} \right) + \frac{\partial}{\partial t} \left(\frac{1}{2\mu_0} B^2 + \frac{\varepsilon_0}{2} E^2 \right) + \mathbf{E} \cdot \mathbf{j} = 0.$$
 (7.16)

Observe the similarity with the equation of continuity. This expression can be regarded as a statement of conservation of energy. The middle term is, we know, the rate of change of the field energy density. The third term is the rate of doing work, per unit volume, as charges are moved in the field. And the first term is then identified as the energy carried away per unit volume. Then since, From Gauss's theorem,

$$\iiint_{\text{volume}} \operatorname{div} \mathbf{S} \, \mathrm{d}v = \bigoplus_{\substack{\text{bounding}\\\text{surface}}} \mathbf{S} \cdot \mathrm{d}\mathbf{a}$$
$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$
(7.17)

it follows that the vector

is the energy per unit time per unit area carried by the field. The vector S is called the *Poynting vector*; it is the energy flux density.

7.7 Incompatibility between Maxwell's equations and Newton's laws

There are two important consequences of Maxwell's equations. The first, as we have already seen, is the possibility of electromagnetic waves. The second is the breakdown of Newtonian mechanics, leading to Einstein's relativity theory. In this section we will demonstrate the second; we shall start by showing that Maxwell's equations and Newton's laws are incompatible with each other.

The procedure we use here is to consider electric and magnetic fields in two different reference frames, one frame moving with a constant velocity \mathbf{V} with respect to the other. On the assumption that Maxwell's equations hold in one frame, they are found not to hold in the other – if the transformation between frames is done in accordance with to Newtonian mechanics.



In the original reference frame there is an electric field and a magnetic field satisfying Maxwell's equations. For simplicity we shall consider the \mathbf{E} field to be in the y direction and the \mathbf{B} field in the z direction. And we assume the fields depend on the x co-ordinate only and on time.

The new reference frame is moving with a velocity V in the x direction. Quantities in the moving frame are signified with primes. Thus in the original frame we have fields $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$ obeying Maxwell's equations and we enquire whether the transformed fields $\mathbf{E}'(x',t')$ and $\mathbf{B}'(x',t')$ then satisfy Maxwell's equations or not.

7.7.1 Transformation of co-ordinates and fields.

The moving frame is travelling in the x direction. Thus the y co-ordinate and the z co-ordinate will be the same in both frames. But the x co-ordinates will be diverging. The co-ordinates transform according to

$$x' = x - Vt$$

$$y' = y$$

$$z' = z$$

$$t' = t.$$
(7.18)

In order to see how the electric and magnetic fields transform between frames we make use of the Newtonian principle that the force is the same in frames moving with constant velocity with respect to each other. Here this means the invariance of the Lorentz force; the force on a moving charge is independent of reference frame. A charge q moving with velocity \mathbf{v} in the x direction experiences the force

$$q\{E-vB\}=q\{E'-v'B'\}.$$

The minus signs appear because of the directions of \mathbf{v} and \mathbf{B} . But since the velocity transforms as

$$v' = v - V$$

we see that

$$E'-v'B'=(E'+VB')-vB'.$$

But this is equal to the force per unit charge in the un-primed frame

$$E - vB$$
.

This is a sum of a velocity-independent term and one proportional to v. We can equate the coefficients, giving

$$E = E' + VB'$$

$$B = B'.$$
(7.19)

This is the rule by which the fields transform.

7.7.2 Transformation of a Maxwell equation

We shall investigate the transformation of the curl **B** Maxwell equation (in the absence of any currents). Thus we consider

$$\operatorname{curl} \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

Since we are assuming

$$\mathbf{E} = E(x,t)\mathbf{j}$$
 and $\mathbf{B} = B(x,t)\mathbf{k}$

it follows that

curl
$$\mathbf{B} = -\frac{\partial B}{\partial x}\mathbf{j}$$
 and $\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial E}{\partial t}\mathbf{j}$

so our Maxwell equation, for these fields, is

$$\frac{\partial B}{\partial x} = -\frac{1}{c^2} \frac{\partial E}{\partial t}.$$
(7.20)

This equation will be transformed to the moving frame in two stages. First we shall transform the derivatives, using the chain rule, and then we will transform the fields themselves. The chain rule gives us

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial B}{\partial t'} \frac{\partial t'}{\partial x}$$

and

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial E}{\partial x'} \frac{\partial x'}{\partial t}$$

The co-ordinate derivatives are found from the co-ordinate transformations, Eqs (7.18)

$$\frac{\partial x'}{\partial x} = 1 \qquad \frac{\partial x'}{\partial t} = -V$$
$$\frac{\partial t'}{\partial t} = 1 \qquad \frac{\partial t'}{\partial x} = 0.$$

Thus the field derivatives are

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}$$

and

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'} - V \frac{\partial E}{\partial x'}.$$

Finally we transform the fields on the right hand sides of these equations, according to Eqs (7.19) to give

$$\frac{\partial B}{\partial x} = \frac{\partial B'}{\partial x'}$$

and

$$\frac{\partial E}{\partial t} = \frac{\partial E'}{\partial t'} + V \frac{\partial B'}{\partial t'} - V \frac{\partial E'}{\partial x'} - V^2 \frac{\partial B'}{\partial x'}.$$

From these equations we see that the Maxwell equation

$$\frac{\partial B}{\partial x} = -\frac{1}{c^2} \frac{\partial E}{\partial t}$$
(7.20)

transforms to the new reference frame as

$$\frac{\partial B'}{\partial x'} = -\frac{1}{c^2} \frac{\partial E'}{\partial t'} + \frac{1}{c^2} \left\{ V \frac{\partial E'}{\partial x'} - V \frac{\partial B'}{\partial t'} + V^2 \frac{\partial B'}{\partial x'} \right\}.$$
(7.21)

But this does not have the *form* of the Maxwell equation Eq (7.20); the curly bracket mucks things up.

The transformation to a frame moving with a uniform relative velocity, as above, carried out in accordance with the principles of Newtonian mechanics is known as a Galilean transformation. We thus conclude that Maxwell's equations (certainly Eq (7.20)) are not invariant under Galilean transformations.

7.7.3 The Lorentz transformation

As emphasised by Einstein, the correct transformation between two frames of reference moving with a constant relative velocity (in the x direction) is the Lorentz transformation x' = v(x - Vt)

$$x = \gamma (x - Vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma \left(t - \frac{V}{c^2} x \right),$$

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

(7.22)

where

Although we shall not prove this, the corresponding transformations for the electric and magnetic fields is

$$E = \gamma \{E' + VB'\}$$

$$B = \gamma \left\{B' + \frac{V}{c^2}E'\right\};$$
(7.23)

these are found from the Lorentz transformation for forces.

As before, we transform to the primed reference frame in two stages. First we transform the derivatives, using the chain rule:

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial B}{\partial t'} \frac{\partial t'}{\partial x}$$

and

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial E}{\partial x'} \frac{\partial x'}{\partial t}.$$

The co-ordinate derivatives are now found from the Lorentz transformations, Eqs (7.22)

$$\frac{\partial x'}{\partial x} = \gamma \qquad \frac{\partial x'}{\partial t} = -\gamma V$$
$$\frac{\partial t'}{\partial t} = \gamma \qquad \frac{\partial t'}{\partial x} = -\gamma \frac{V}{c^2}.$$

Thus the field derivatives are

$$\frac{\partial B}{\partial x} = \gamma \frac{\partial B}{\partial x'} - \gamma \frac{V}{c^2} \frac{\partial B}{\partial t'}$$

and

$$\frac{\partial E}{\partial t} = \gamma \frac{\partial E}{\partial t'} - \gamma V \frac{\partial E}{\partial x'}.$$

Finally we transform the fields on the right hand sides of these equations, according to Eqs (7.23) to give

$$\frac{\partial B}{\partial x} = \gamma^2 \frac{\partial B'}{\partial x'} + \gamma^2 \frac{V}{c^2} \frac{\partial E'}{\partial x'} - \gamma^2 \frac{V}{c^2} \frac{\partial B'}{\partial t'} - \gamma^2 \frac{V^2}{c^4} \frac{\partial E'}{\partial t'}$$

and

$$\frac{\partial E}{\partial t} = \gamma^2 \frac{\partial E'}{\partial t'} + \gamma^2 V \frac{\partial B'}{\partial t'} - \gamma^2 V \frac{\partial E'}{\partial x'} - \gamma^2 V^2 \frac{\partial B'}{\partial x'}.$$

From these equations we see that the Maxwell equation

(7.20)
$$\frac{\partial B}{\partial x} = -\frac{1}{c^2} \frac{\partial E}{\partial t}$$

transforms to the moving reference frame as

$$\gamma^{2} \frac{\partial B'}{\partial x'} + \gamma^{2} \frac{V}{c^{2}} \frac{\partial E'}{\partial x'} - \gamma^{2} \frac{V}{c^{2}} \frac{\partial B'}{\partial t'} - \gamma^{2} \frac{V^{2}}{c^{4}} \frac{\partial E'}{\partial t'} = -\gamma^{2} \frac{1}{c^{2}} \frac{\partial E'}{\partial t'} - \gamma^{2} \frac{V}{c^{2}} \frac{\partial B'}{\partial t'} + \gamma^{2} \frac{V}{c^{2}} \frac{\partial E'}{\partial x'} + \gamma^{2} \frac{V^{2}}{c^{2}} \frac{\partial B'}{\partial x'}.$$
Upon concellation and collecting terms we obtain

Upon cancellation and collecting terms we obtain

$$\left\{1 - \frac{V^2}{c^2}\right\}\frac{\partial B'}{\partial x'} = -\frac{1}{c^2}\left\{1 - \frac{V^2}{c^2}\right\}\frac{\partial E'}{\partial t'}$$

or

$$\frac{\partial B'}{\partial x'} = -\frac{1}{c^2} \frac{\partial E'}{\partial t'}$$

Now this does have the form of the Maxwell equation Eq (7.20). We thus find that whereas the Maxwell equations are not invariant under Galilean transformations, they *are* invariant under Lorentz transformations.

In conclusion we see that the incompatibility between Newton's mechanics and Maxwell's electromagnetism is resolved by replacing Newton's mechanics by Einstein's relativity.

When you have completed this chapter you should:

- understand how Faraday's law of electromagnetic induction follows from the transformation properties of the Lorentz force formula;
- appreciate that electromagnetic induction means that the curl of **E** no longer vanishes;
- know the physical content of Lenz's law;
- understand the need for the displacement current and how this modifies Ampère's law;
- appreciate the parallels of the displacement and Faraday's law;
- know how the above are combined into the Maxwell field equations;
- understand how Maxwell's equations lead to electromagnetic waves;
- know how the speed of propagation of EM waves relates to the static electric and magnetic properties of space;
- appreciate that **E**, **B** and the direction of propagation of EM waves are mutually perpendicular (in problem sheet);
- understand the conservation of energy as applied to electromagnetic fields and be familiar with the definition and properties of the Poynting vector;
- know that Maxwell's equations are incompatible with Newton's mechanics and understand how this problem is resolved by replacing Newton's mechanics and its Galilean transformations by Einstein's relativity and its Lorentz transformations.